PATH CONNECTEDNESS AND INVERTIBLE MATRICES

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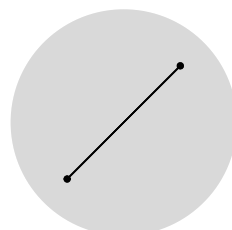
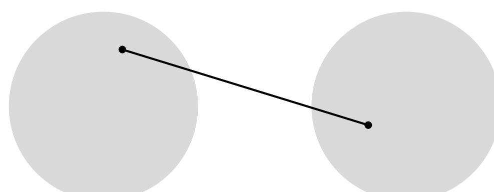
1. Path Connectedness

Given a space,[1] it is often of interest to know whether or not it is path-connected. Informally, a space X is *path-connected* if, given any two points in X, we can draw a path between the points which stays inside X. For example, a disc is path-connected, because any two points inside a disc can be connected with a straight line.



The space which is the disjoint union of two discs is *not* path-connected, because it is impossible to draw a path from a point in one disc to a point in the other disc. Any attempt to do so would result in a path that is not entirely contained in the space:



Though path-connectedness is a very geometric and visual property, math lets us formalize it and use it to gain geometric insight into spaces that we cannot visualize. In these notes, we will consider spaces of matrices, which (in general) we cannot draw as regions in $\mathbb{R}^2$ or $\mathbb{R}^3$. To begin studying these spaces, we first explicitly define the concept of a *path.*

Definition 1.1. A **path** in X is a continuous function $\varphi : [0,1] \to X$.

In other words, to get a path in a space X, we take an interval and stick it inside X in a continuous way:

[1]Formally, a topological space.

Note that we don't actually have to use the interval $[0,1]$; we could continuously map $[1,2]$, $[0,2]$, or *any* closed interval, and the result would be a path in X.

Definition 1.2. A space X is **path-connected** if, for any two points $x, y \in X$, there exists a path $\varphi : [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

This is a mathematical way of saying that a space is path-connected if, given two points, we can always find a path that starts at one point and ends at the other. Let's consider a few examples to see this definition in action.

Example 1.3. Let $X = \mathbb{R}$, the real number line. It is geometrically clear that $\mathbb{R}$ is path-connected, but we can give a rigorous proof using the definition above.

Let a and b be two points in $\mathbb{R}$; i.e., two real numbers. We need a continuous function φ such that $\varphi(0) = a$ and $\varphi(1) = b$. Define $\varphi(t)$ as follows:

$$\varphi(t) = a(1-t) + bt$$

Plugging in $t = 0$ gives us a, and $t = 1$ give us b. Moreover, φ is continuous since it's a degree 1 polynomial in t, and the path φ "lies inside" $\mathbb{R}$ because it always spits out a real number. Therefore, since a and b were arbitrary, it follows that $\mathbb{R}$ is path-connected.

Example 1.4. Let X be the unit circle, as a subset of $\mathbb{R}^2$. Explicitly,

$$X = \left\{ (x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1 \right\}$$

Again, the unit circle is clearly path-connected, because we can "travel around the circle" to reach any point from any other. To formalize that argument, let $\mathbf{a}, \mathbf{b}$ be two points in X. In order to show that X is path connected, we need a continuous function $\varphi : [0,1] \to X$ such that $\varphi(0) = \mathbf{a}$, $\varphi(1) = \mathbf{b}$, and $\varphi(t) \in X$ for all t.

Since X is the unit circle, we can write $\mathbf{a}$ and $\mathbf{b}$ in terms of cosine and sine:

$$\mathbf{a} = (\cos\theta_1, \sin\theta_1) \qquad \mathbf{b} = (\cos\theta_2, \sin\theta_2)$$

Consider the following path:

$$\varphi(t) = \left(\cos\left[\theta_1(1-t) + \theta_2 t\right], \sin\left[\theta_1(1-t) + \theta_2 t\right]\right)$$

Then

$$\varphi(0) = (\cos\theta_1, \sin\theta_1) = \mathbf{a}$$
$$\varphi(1) = (\cos\theta_2, \sin\theta_2) = \mathbf{b}$$

Furthermore, φ is continuous because it is the composition of trig functions with linear polynomials in t. The path lies entirely in the unit circle because $\varphi(t)$ looks like $(\cos(\text{something}), \sin(\text{something}))$, and therefore satisfies $x^2 + y^2 = 1$. Since $\mathbf{a}$ and $\mathbf{b}$ we arbitrary, it follows that X is path connected.

More information about path-connectedness can be found in any introductory book on topology, for example, [1].

2. Some Linear Algebra

As I mentioned above, the spaces we will consider in the rest of these notes are spaces of matrices. Because of this, it will be helpful to review some concepts from linear algebra. For the purposes of this discussion, all of our matrices will be square and have entries in the complex numbers $\mathbb{C}$. In other words, all of our matrices will look like:

$$A = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}$$

where each z_{ij} is a complex number of the form $a + bi$.

Recall that an $n \times n$ matrix A is **invertible** if there exists another matrix (which we denote by A^{-1}) such that the product of the two is the identity matrix:

$$AA^{-1} = A^{-1}A = I := \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

Equivalently, A is invertible when its row-reduced-echelon form is the identity matrix I. Invertible matrices are important for a number of reasons, but at the core they are linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^n$ which can be "reversed" without loss of information; that is, they are *isomorphisms.* Another special class of matrices is the **upper-triangular** matrices. These are matrices whose entries below the main diagonal are all 0, i.e., matrices that look like:

$$\begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where the $*$ represents everything above the diagonal, and can be equal to anything. An example of an upper-triangular matrix is:

$$B = \begin{bmatrix} 1 & 3 & -5+4i \\ 0 & 2+i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper-triangular matrices are nice because **they are invertible precisely when their diagonal entries are nonzero**. This is important! It's so important that I'll say it again: an upper-triangular matrix is invertible exactly when its main-diagonal entries (called the **eigenvalues** of the matrix) are nonzero. For example, the matrix B from above is invertible, because its diagonal entries are $1, 2+i$, and 1, which are all nonzero.

The other reason why upper-triangular matrices are important is that **every matrix is similar to an upper-triangular matrix**. In other words, if A is a matrix, then there is some other invertible matrix P such that:

$$A = P^{-1} \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} P$$

Said differently, we can factor A into a product of matrices such that the middle factor is upper triangular. In this case, we say that the eigenvalues of A are $\lambda_1, \dots, \lambda_n$ (note that this means that A and its upper-triangular counterpart have the same eigenvalues!). This is useful because it allows to loosely identify a general matrix A with its upper-triangular decomposition, and hence its eigenvalues $\lambda_1, \dots, \lambda_n$.

The theory of eigenvalues and matrix decomposition is deep and much more meaningful than presented here, and more information can be found in [2]. For our purposes, the upper-triangular form of a matrix simply gives us a better handle on arbitrary invertible matrices by letting us work with the diagonal entries.

3. Invertible Matrices over $\mathbb{C}$

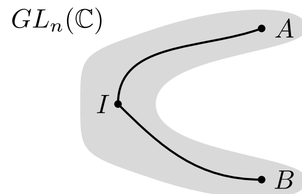
Having discussed path-connectedness and upper-triangular matrix decompositions, we are now ready to consider a new kind of space, namely, the space of invertible matrices. To be a little more explicit, we adopt the following notation:

$$GL_n(\mathbb{C}) := \text{the set of invertible } n \times n \text{ matrices with entries in } \mathbb{C}$$

As odd as it may seem, this is a "space" as much as the unit disc is a space.[2] Even though $GL_n(\mathbb{C})$ is a "space," it isn't something we can visualize. It lives in n^2 dimensions and has a very complicated structure, and hence lacks any immediate geometric intuition. Though understanding and thinking about $GL_n(\mathbb{C})$ is difficult, the machinery we developed above gives us one tool to study the space without needing to visualize it. Using the definition of path-connectedness, we arrive at the main result of these notes.

Theorem 3.1. *The space $GL_n(\mathbb{C})$ is path-connected.*

Proof. In order to show that $GL_n(\mathbb{C})$ is path-connected, we need to show that any two invertible matrices can be connected by a path inside $GL_n(\mathbb{C})$. Note that the identity matrix I is invertible (it is an upper-triangular matrix, and all of its diagonal entries are nonzero). So if we can show that we can connect any invertible matrix to the identity, then any two invertible matrices A and B can be connected via a path which passes through the identity.



So let $A \in GL_n(\mathbb{C})$. In this proof we will build a path using the interval $[0, 2]$. Hence, we will be done after we construct a continuous function $\psi : [0, 2] \to GL_n(\mathbb{C})$ such that $\psi(0) = A$ and $\psi(2) = I$.

It will be helpful to (informally) identify the matrix A with its eigenvalues; i.e., with the diagonal entries of its upper-triangular form. When we deal with "paths of invertible matrices," we will think about moving each eigenvalue along paths in $\mathbb{C}$. To begin this process, let's consider the upper-triangular decomposition of A: $A = P^{-1}TP$, where

$$T = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Here, $\lambda_1, \dots, \lambda_n$ are "eigenvalues" of A and P is some invertible matrix. Note that, since A is invertible, all the eigenvalues $\lambda_1, \dots, \lambda_n$ are nonzero.

We will construct the path connecting A to I using two steps, φ_1 and φ_2.

Let's first think about path-connecting the upper-triangular matrix T to the identity, rather than A. We're pretty close: I is a diagonal matrix with 1's on the diagonal, and T is an "almost diagonal" matrix with $\lambda_1, \dots, \lambda_n$ down the diagonal. We need to do two things: get rid of all the stuff $*$ above the diagonal, and connect each λ_i to 1 via a path in $\mathbb{C}$.

Let's turn the $*$ above the upper diagonal to 0's first. To connect each element $*$ to 0 with a path, we can do the following:

$$*(1-t) \qquad 0 \le t \le 1$$

When $t = 0$, we get $*$, and when $t = 1$, we get 0. Note that this above-diagonal maneuvering does not affect the invertibility of T, because — as I've said a few times now — invertibility of T relies only on the diagonal elements being nonzero. Hence, the $*(1-t)$ paths keep us within $GL_n(\mathbb{C})$.

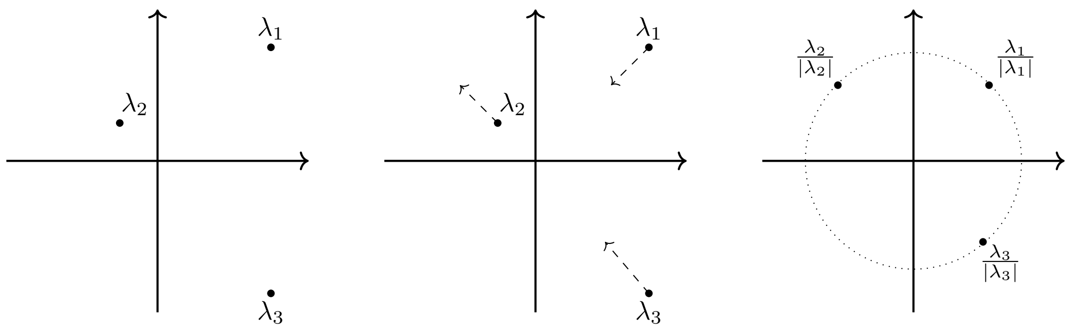
Next, we will connect each λ_j on the diagonal to 1. We have to be careful though — the path in each diagonal entry can never pass through 0, because then the matrix at that point in the path

[2]Formally, $GL_n(\mathbb{C})$ is a metric space under the metric induced by the norm $\|A\| := \sup_{x \neq 0 \in \mathbb{C}^n} \frac{\|Ax\|}{\|x\|}$.

would not be invertible. To be methodical about avoiding 0, the first thing we will do is retract each λ_j onto the unit circle. We can achieve this by doing:

$$\frac{\lambda_j}{(1-t)+t|\lambda_j|} \qquad 0 \leq t \leq 1$$

When $t = 0$, we get λ_j. When $t = 1$, we get $\frac{\lambda_j}{|\lambda_j|}$, which lies on the unit circle (note that $\left|\frac{\lambda_j}{|\lambda_j|}\right| = \frac{|\lambda_j|}{|\lambda_j|} = 1$). Since λ_j is nonzero, this path never passes through 0, and it is continuous because it is a rational function of t. Geometrically, here's what's happening with each eigenvalue in the complex plane:



Let's put this all together to get a "path of matrices." Remember that the matrix T looks like:

$$T = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where the $*$ represents every entry above the upper diagonal, and the 0 represents all of the zero entries below the diagonal. Putting our above-diagonal paths and on-diagonal paths together in a matrix gives us the following path:

$$\varphi_1(t) = \begin{bmatrix} \frac{\lambda_1}{(1-t)+t|\lambda_1|} & & & & *(1-t) \\ & \frac{\lambda_2}{(1-t)+t|\lambda_2|} & & & \\ & & \ddots & & \\ & & & \frac{\lambda_{n-1}}{(1-t)+t|\lambda_{n-1}|} & \\ 0 & & & & \frac{\lambda_n}{(1-t)+t|\lambda_n|} \end{bmatrix} \qquad 0 \leq t \leq 1$$

When $t = 0$, we get $\varphi_1(0) = T$. When $t = 1$, we get the following matrix:

$$\varphi_1(1) = \begin{bmatrix} \frac{\lambda_1}{|\lambda_1|} & & & & 0 \\ & \frac{\lambda_2}{|\lambda_2|} & & & \\ & & \ddots & & \\ & & & \frac{\lambda_{n-1}}{|\lambda_{n-1}|} & \\ 0 & & & & \frac{\lambda_n}{|\lambda_n|} \end{bmatrix}$$

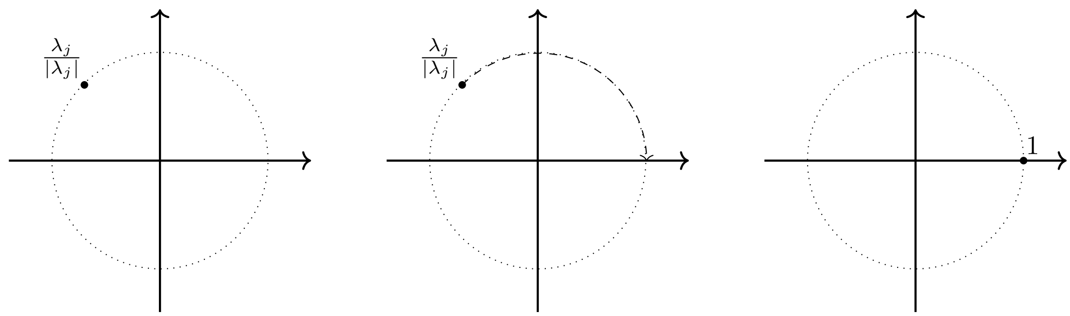
This is a diagonal matrix with every diagonal entry on the unit circle, and is represented by the far-right diagram above. To get the identity matrix, all we have to do is rotate each diagonal entry around the circle to 1. Since we used the interval $0 \leq t \leq 1$ to move the eigenvalues to the unit circle, we will execute the rotation in the interval $1 \leq t \leq 2$. To do this, remember that any complex number on the unit circle can be written as $e^{i\theta}$ for some real number θ. So for each j,

$$\frac{\lambda_j}{|\lambda_j|} = e^{i\theta_j}$$

We can rotate this complex number to 1 as follows:

$$e^{i\theta_j(2-t)} \qquad 1 \le t \le 2$$

When $t = 1$, we get $e^{i\theta_j} = \frac{\lambda_j}{|\lambda_j|}$, and when $t = 2$, we get $e^0 = 1$. Furthermore, the path never passes through 0 (it stays on the unit circle the whole time) and it is continuous because it's a rotation. Here's what's happening with each eigenvalue in the complex plane:



Putting each rotation into a matrix gives us a new path of matrices:

$$\varphi_2(t) = \begin{bmatrix} e^{i\theta_1(2-t)} & & & 0 \\ & e^{i\theta_2(2-t)} & & \\ & & \ddots & \\ 0 & & & e^{i\theta_n(2-t)} \end{bmatrix} \qquad 1 \le t \le 2$$

When $t = 1$, we get the scaled diagonal matrix $\varphi_1(1)$ from above, where each eigenvalue is in the position in the leftmost diagram. When $t = 2$, we get the identity matrix!

What have we just done? We constructed a two step path that starts at T and ends at I. Explicitly,

$$\varphi(t) := \begin{cases} \varphi_1(t) & t \in [0,1] \\ \varphi_2(t) & t \in [1,2] \end{cases}$$

is a continuous path from T to I that is contained in $GL_n(\mathbb{C})$. But our original goal was to connect the matrix A to the identity. Since $A = P^{-1}TP$, all we have to do is conjugate our path by P:

$$P^{-1}\varphi(t)P = \begin{cases} P^{-1}\varphi_1(t)P & t \in [0,1] \\ P^{-1}\varphi_2(t)P & t \in [1,2] \end{cases}$$

When $t = 0$, we get

$$P^{-1}\varphi(0)P = P^{-1}\varphi_1(0)P = P^{-1}TP = A$$

and when $t = 2$ we get

$$P^{-1}\varphi(2)P = P^{-1}\varphi_2(2)P = P^{-1}IP = P^{-1}P = I$$

So $P^{-1}\varphi(t)P$ is a path inside $GL_n(\mathbb{C})$ which connects A to I as t ranges from 0 to 2. Done! □

3.1. **An Alternate Proof.** The proof above used the upper-triangular decomposition of A to connect it to the identity. I find that proof appealing because it deals directly with the eigenvalues of A by moving them around in $\mathbb{C}$, but there are a couple of other ways to show that $GL_n(\mathbb{C})$ is path-connected. One method uses the *polar decomposition* of an invertible matrix.

First, we'll recall some more linear algebra.

Definition 3.2. An $n \times n$ complex matrix U is **unitary** if $U^{-1} = \bar{U}^T$.

In other words, a matrix is unitary if, when you take the transpose (flip the matrix over the main diagonal) and then the complex-conjugate (send every i to $-i$), you get the inverse of the matrix you started with. If we view matrices as generalizations of complex numbers (note that a 1×1 matrix over $\mathbb{C}$ is just a complex number), unitary matrices are the generalization of numbers on the unit circle. Unitary matrices have some other really nice properties:

Theorem 3.3.

(1) *Every unitary matrix is similar to a diagonal matrix.*
(2) *All the eigenvalues of a unitary matrix lie on the unit circle.*

This result is called the *spectral theorem for unitary matrices.* Another important class of matrices is the *positive matrices.*

Definition 3.4. A matrix P is **positive** if all of its eigenvalues are positive.

It turns out that every invertible matrix can be written as the product of a unitary matrix and a positive matrix.

Proposition 3.5. *Let A be an invertible complex matrix. Then there is a unitary matrix U and a positive matrix P such that $A = UP$.*

This is very analogous to the polar decomposition of a complex number: if $z \in \mathbb{C}$, then we can write $z = re^{i\theta}$ where $r = |z|$ is a positive number and $e^{i\theta}$ on the unit circle.

Theorem 3.6. *The space $GL_n(\mathbb{C})$ is path connected.*

Proof. Let $A \in GL_n(\mathbb{C})$. As before, it suffices to show that A can be path-connected to the identity.

By the previous proposition, there is a unitary matrix U and a positive matrix P such that $A = UP$. We will connect U and P to the identity separately.

First, consider U. Since U is unitary, it can be diagonalized: $U = S^{-1}DS$, where the diagonal entries of S lie on the unit circle. As in the proof above, we can path-connect D to I by rotating each eigenvalue around the unit circle to 1. Call this path $\psi(t)$. Then $S^{-1}\psi(t)S$ as $0 \le t \le 1$ is a path which connects U to I and stays within $GL_n(\mathbb{C})$.

Next, consider P. Since P is positive, each eigenvalue is a positive number and can be connected to 1 with a straight line without passing through 0. Hence, the path

$$(1-t)P + tI \qquad 0 \le t \le 1$$

connects P to I while staying inside $GL_n(\mathbb{C})$. Therefore,

$$S^{-1}\psi(t)S\,[(1-t)P + tI] \qquad 0 \le t \le 1$$

connects A to I. □

These two proofs, as I've presented them, skirt around many of the details (for example, rigorously proving continuity of matrix-valued functions). A good reference for these connectivity results is the first chapter of [3], which covers the theory matrix Lie groups.

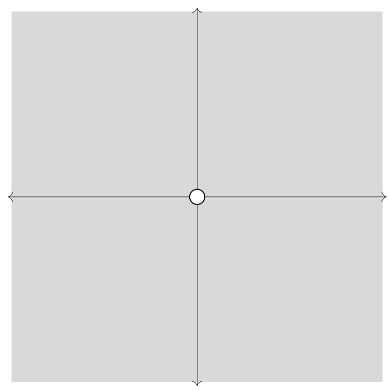
4. Related Results

Having shown that $GL_n(\mathbb{C})$ is path-connected, there are a few natural questions to ask. The answers to these related questions requires more sophisticated background knowledge, so keep that in mind while reading this section.

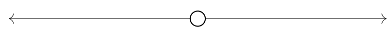
4.1. **Invertible Matrices over $\mathbb{R}$.**

In the previous section, we considered matrices with complex entries. What happens when we consider invertible matrices with real entries? In other words, is $GL_n(\mathbb{R})$ path-connected? It turns out that the answer is *no.* Intuitively, there is more room to "move around" in the complex plane than on the real line.

This become evident when we consider $n = 1$. In this case, $GL_1(\mathbb{C})$ is the space of invertible 1×1 matrices. But since a 1×1 matrix is just a number, $GL_1(\mathbb{C})$ is the space of invertible complex numbers. All complex numbers except for 0 are invertible ($z^{-1} = \frac{1}{z}$ when $z \neq 0$), so $GL_1(\mathbb{C}) = \mathbb{C} \setminus 0$. This looks like:



Clearly, $GL_1(\mathbb{C}) = \mathbb{C} \setminus 0$ is path-connected. On the other hand, $GL_1(\mathbb{R})$ is the set of all invertible *real* numbers, which is $\mathbb{R} \setminus 0$:



It is easy to see that $GL_1(\mathbb{R}) = \mathbb{R} \setminus 0$ is *not* path-connected, because there is no way to travel from the negative numbers to the positive numbers without passing through 0.

The same is true for any n. Formally, we can see that $GL_n(\mathbb{R})$ is not path-connected for any n by using the determinant.

Proposition 4.1. *The space $GL_n(\mathbb{R})$ is not path-connected.*

Proof. Recall that a matrix is invertible if and only if its determinant is nonzero. Furthermore, the determinant of a matrix with real entries is a real number. Another fact is that the map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a continuous function. Next, recall the intermediate value theorem, which says that if X is a path-connected space and $f : X \to Y$ is continuous, then $f(X)$ is a path-connected space. Since $\det(GL_n(\mathbb{R})) = \mathbb{R} \setminus 0$, which is not path-connected, it follows that $GL_n(\mathbb{R})$ is not path-connected. □

While this result may seem disappointing, it allows us to ask a different question: how many *connected components* does $GL_n(\mathbb{R})$ have? The case $n = 1$ is easy to see. There are two connected components, because all of the negative numbers are path-connected and all of the positive numbers are path-connected. It turns out that in the more general case, the answer is the same.

Theorem 4.2. *The space $GL_n(\mathbb{R})$ has two connected components: matrices with positive determinant, and matrices with negative determinant.*

The proof of this theorem, as well as many more details on the connectedness of matrices over $\mathbb{R}$, can be found in [3].

4.2. **Infinite Dimensions.** The discussion up to this point has dealt with finite matrices. When we discussed $GL_n(\mathbb{C})$ and $GL_n(\mathbb{R})$, all of our work relied on the fact that n was a finite number.

What happens when $n = \infty$? We have to be careful, because defining an "invertible infinite-dimensional matrix" as we desire requires knowledge of functional analysis, and in particular, of Hilbert spaces. A friendly introduction to the subject can be found in [4].

A more precise version of the infinite-dimensional question is as follows. Let $\mathcal{H}$ be a (separable) complex infinite-dimensional Hilbert space. Roughly, $\mathcal{H}$ is like an infinite version of $\mathbb{C}^n$. Let $GL_{\mathbb{C}}(\mathcal{H})$ be the space of invertible continuous linear operators on $\mathcal{H}$ (informally, the space of invertible infinite dimensional complex matrices). Is $GL_{\mathbb{C}}(\mathcal{H})$ path-connected? The answer is yes.

Theorem 4.3. *The space $GL_{\mathbb{C}}(\mathcal{H})$ is path-connected.*

This theorem can be proven using the more sophisticated tools analogous to those used in the alternate proof from Section 3. Any invertible linear operator can be written as the product of a unitary operator and a positive operator, and a path to the identity can be constructed in almost the same way. The difficult part comes in generalizing the idea of diagonalization of a unitary matrix; the relevant result is the *spectral theorem for unitary operators.*

The previous theorem addressed invertible operators on a complex Hilbert space, which was our generalization of $GL_n(\mathbb{C})$. What about generalizing $GL_n(\mathbb{R})$? Explicitly, let $\mathcal{H}$ be an infinite dimensional separable *real* Hilbert space (informally, an infinite version of $\mathbb{R}^n$). We saw that in finite dimensions, the invertible real matrices were not path-connected. One might guess that the same is true of $GL_{\mathbb{R}}(\mathcal{H})$, the space of invertible continuous operators on the real Hilbert space $\mathcal{H}$. However, we have the following:

Theorem 4.4. *The space $GL_{\mathbb{R}}(\mathcal{H})$ is path-connected.*

The infinite-dimensionality of $\mathcal{H}$ gives us more room to connect things, so even though $GL_n(\mathbb{R})$ is not path-connected, its infinite-dimensional counterpart is.

4.3. **Kuiper's Theorem.** In fact, a much stronger result is true, which we can express in the language of homotopy groups. Recall that the "0th homotopy group" of a topological space X, denoted $\pi_0(X)$, is the set of connected components of the space. If X is path-connected, then $\pi_0(X)$ has one element, and we say $\pi_0(X) = 0$.

The results from section 4.2 say that $\pi_0(GL_{\mathbb{C}}(\mathcal{H})) = 0$ and $\pi_0(GL_{\mathbb{R}}(\mathcal{H})) = 0$. It turns out that each space is not only path-connected, but also *simply connected.* Explicitly, the fundamental group π_1 of each space is trivial:

$$\pi_1(GL_{\mathbb{C}}(\mathcal{H})) = \pi_1(GL_{\mathbb{R}}(\mathcal{H})) = 0$$

Amazingly, *every* homotopy group is trivial.

Theorem 4.5 (Kuiper's Theorem)**.** *For $n = 0, 1, 2, \ldots$*

$$\pi_n(GL_{\mathbb{C}}(\mathcal{H})) = \pi_n(GL_{\mathbb{R}}(\mathcal{H})) = 0$$

Using algebraic topology, one can show that this implies the following:

Corollary 4.6. *The spaces $GL_{\mathbb{C}}(\mathcal{H})$ and $GL_{\mathbb{R}}(\mathcal{H})$ are contractible to a point.*

This is just one fascinating example of how differently things behave in infinite dimensions, and these results have deep consequences in many fields of math. More details and proofs can be found in [5], Nicolaas Kuiper's original paper.

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