Here is a collection of well-written solutions to most standard exercises in 31B. This document should be used not as a reference for how to do each type of problem, but rather how to write up a solution in a clear and effective manner. Everything from the general structure to the language and correct symbol usage is important!

The blue links below are shortcuts to each problem.

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1 A problem where you show a function is invertible and compute the derivative of the inverse.

Let \( f(x) = x^9 + 2x + 1 \). Show that \( f \) is invertible, and compute \( (f^{-1})'(4) \).

**Solution.** To show that \( f \) is invertible on its domain, we will show that it is strictly increasing. Strictly increasing functions are one-to-one, and thus invertible on their domain. Note that

\[
f'(x) = 9x^8 + 2 \geq 2 > 0
\]

for all \( x \). Since \( f' > 0 \), \( f \) is strictly increasing and thus invertible by first remark.

Next, we know that

\[
(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))}.
\]

Note that \( f(1) = (1)^9 + 2(1) + 1 = 4 \). This implies that \( f^{-1}(4) = 1 \), and so

\[
(f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{9(1)^8 + 2} = \frac{1}{11}.
\]

\(\Box\)

2 A problem where you use a L’Hopital to evaluate an exponential indeterminate form.

Evaluate the following limit.

\[
\lim_{x \to 0} \frac{\sec x}{x^2}.
\]

**Solution.** Note that

\[
\lim_{x \to 0} \sec x = \frac{1}{\cos 0} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1}{x^2} = \infty.
\]

Thus, the above limit is indeterminate of type \( 1^\infty \). Let \( L = \lim_{x \to 0} (\sec x)^{\frac{1}{x^2}} \). Then

\[
\ln L = \lim_{x \to 0} \frac{1}{x^2} \ln(\sec x) = \lim_{x \to 0} \frac{\ln(\sec x)}{x^2}.
\]

Applying L’Hopital’s rule, we have

\[
\ln L = \lim_{x \to 0} \frac{1}{x^2} \cdot \sec x \tan x = \lim_{x \to 0} \frac{1}{2 \cos x} \cdot \sin x.
\]

Note that

\[
\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1
\]

by L’Hopital’s rule. Thus,

\[
\ln L = \left( \lim_{x \to 0} \frac{1}{2 \cos x} \right) \cdot \left( \lim_{x \to 0} \frac{\sin x}{x} \right) = \frac{1}{2} \cdot 1 = \frac{1}{2}.
\]

Therefore, \( L = e^{\ln L} = e^{\frac{1}{2}} \).

\(\Box\)
3 A problem where you use a trigonometric substitution to evaluate an antiderivative.

Evaluate the following antiderivative.
\[ \int \frac{1}{(1 + x^2)^{3/2}} \, dx. \]

Solution. Let \( x = \tan \theta \). Then \( dx = \sec^2 \theta \, d\theta \), and we have
\[ \int \frac{1}{(1 + x^2)^{3/2}} \, dx = \int \frac{1}{(1 + \tan^2 \theta)^{3/2}} \sec^2 \theta \, d\theta = \int \frac{\sec^2 \theta}{(\sec^2 \theta)^{3/2}} \, d\theta = \int \frac{1}{\sec \theta} \, d\theta = \int \cos \theta \, d\theta. \]
Thus,
\[ \int \frac{1}{(1 + x^2)^{3/2}} \, dx = \int \cos \theta \, d\theta = \sin \theta + C. \]
Next, we draw a right triangle with angle \( \theta \) such that \( \tan \theta = x \).

This picture implies that \( \sin \theta = \frac{x}{\sqrt{1 + x^2}} \). Thus,
\[ \int \frac{1}{(1 + x^2)^{3/2}} \, dx = \frac{x}{\sqrt{1 + x^2}} + C. \]

4 A problem where you use partial fractions and complete the square to evaluate an antiderivative.

Evaluate the following antiderivative.
\[ \int \frac{2x^3 + 10x^2 + 15x + 6}{(x^2 + x + 1)(x^2 + 4x + 5)} \, dx. \]

Solution. We begin by decomposing the integrand via partial fractions. Note that both quadratic factors in the denominator are irreducible because \( 1^2 - 4 < 0 \) and \( 4^2 - 4 \times 5 < 0 \). Thus, we seek a decomposition of the form
\[ \frac{2x^3 + 10x^2 + 15x + 6}{(x^2 + x + 1)(x^2 + 4x + 5)} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 + 4x + 5}. \]
Multiplying through by the denominator of the left hand side gives
\[ 2x^3 + 10x^2 + 15x + 6 = (Ax + B)(x^2 + 4x + 5) + (Cx + D)(x^2 + x + 1). \]
Comparing \( x^3 \) coefficients gives the following equation:
\[ 2 = A + C. \]
Comparing $x^2$ coefficients gives the following equation:

$$10 = 4A + B + C + D.$$ 

Comparing $x$ coefficients gives the following equation:

$$15 = 5A + 4B + C + D.$$ 

Comparing the constant terms gives the following equation:

$$6 = 5B + D.$$ 

Subtracting the $x$ equation from the constant term equation gives

$$-9 = -5A + B - C.$$ 

Plugging in $C = 2 - A$ into this equation gives $-9 = -5A + B - (2 - A)$ and thus

$$-9 = -4A + B - 2 \quad \Rightarrow \quad 4A - B = 7.$$ 

Subtracting the $x^2$ equation from the $x$ equation gives

$$5 = A + 3B \quad \Rightarrow \quad A = 5 - 3B.$$ 

Thus,

$$4(5 - 3B) - B = 7 \quad \Rightarrow \quad 20 - 13B = 7 \quad B = 1.$$ 

Thus, $A = 5 - 4 = 2$. Since $C = 2 - A$, we have $C = 0$. Finally since $D = 6 - 5B$ we have $D = 1$. (Whew.) 

Therefore,

$$\int \frac{2x^3 + 10x^2 + 15x + 6}{(x^2 + x + 1)(x^2 + 4x + 5)} \, dx = \int \frac{2x + 1}{x^2 + x + 1} + \frac{1}{x^2 + 4x + 5} \, dx.$$ 

To evaluate the first antiderivative, let $u = x^2 + x + 1$. Then $du = (2x + 1) \, dx$ and we have

$$\int \frac{2x + 1}{x^2 + x + 1} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln(x^2 + x + 1) + C.$$ 

For the second term, we complete the square. Note that

$$x^2 + 4x + 5 = x^2 + 4x + 4 - 4 + 5 = (x + 2)^2 + 1.$$ 

Letting $u = x + 2$ then gives

$$\int \frac{1}{x^2 + 4x + 5} \, dx = \int \frac{1}{(x + 2)^2 + 1} \, dx = \int \frac{1}{u^2 + 1} \, du = \arctan(u) + C = \arctan(x + 2) + C.$$ 

Therefore,

$$\int \frac{2x^3 + 10x^2 + 15x + 6}{(x^2 + x + 1)(x^2 + 4x + 5)} \, dx = \ln(x^2 + x + 1) + \arctan(x + 2) + C.$$ 

\[\square\]
5  A problem where you use a substitution to show that an improper integral converges.

Does the following integral converge? If so, evaluate it.

\[ \int_{\infty}^{\infty} \frac{e^x}{1 + (e^x)^2} \, dx. \]

**Solution.** First, the integral is improper at \( \infty \) and nowhere else. Thus,

\[ \int_{\infty}^{\infty} \frac{e^x}{1 + (e^x)^2} \, dx = \lim_{R \to \infty} \int_{8675309}^{R} \frac{e^x}{1 + (e^x)^2} \, dx. \]

Next, let \( u = e^x \). Then \( du = e^x \, dx \). We have

\[ \lim_{R \to \infty} \int_{e^{8675309}}^{e^R} \frac{1}{1 + u^2} \, du = \lim_{R \to \infty} \arctan(u) \bigg|^{e^R}_{e^{8675309}} = \lim_{R \to \infty} \arctan(e^R) - \arctan(e^{8675309}) = \frac{\pi}{2} - \arctan(e^{8675309}). \]

Since this number is finite, it follows that the integral converges, and converges to

\[ \frac{\pi}{2} - \arctan(e^{8675309}). \]

\[ \square \]

6  A problem where you use comparison to show that an improper integral diverges.

Does the following integral converge? If so, evaluate it.

\[ \int_{2}^{\infty} \frac{7 + x^\frac{3}{4}}{\sqrt{x} - 1} \, dx. \]

**Solution.** Note that

\[ \frac{7 + x^\frac{3}{4}}{\sqrt{x} - 1} \geq \frac{7}{\sqrt{x}} \]

as the denominator of the left side is smaller and the numerator of the left side is bigger. By the \( p \)-test for integrals, the integral

\[ \int_{2}^{\infty} \frac{7}{\sqrt{x}} \, dx \]

diverges because \( p = 1/2 \leq 1 \). Thus, by the comparison test for integrals, the integral

\[ \int_{2}^{\infty} \frac{7 + x^\frac{3}{4}}{\sqrt{x} - 1} \, dx \]

diverges as well. \[ \square \]
7 A problem where you split an improper integral into multiple pieces.

Does the following integral converge?

\[ \int_0^\infty \frac{1}{x^7 + \sqrt{x}} \, dx. \]

Solution. First, note that the integral is improper for two reasons: at \( \infty \), and also at 0 since the denominator of the integrand is 0 when \( x = 0 \). Thus, we split the integral into two pieces

\[ \int_0^\infty \frac{1}{x^7 + \sqrt{x}} \, dx = \int_0^1 \frac{1}{x^7 + \sqrt{x}} \, dx + \int_1^\infty \frac{1}{x^7 + \sqrt{x}} \, dx. \]

We consider each piece separately.

First, note that

\[ 0 \leq \frac{1}{x^7 + \sqrt{x}} \leq \frac{1}{\sqrt{x}}. \]

Since the integral \( \int_0^1 \frac{1}{\sqrt{x}} \, dx \) converges by the \( p \)-test for integrals (\( p < 1 \)), by integral comparison it follows that the integral \( \int_0^1 \frac{1}{x^7 + \sqrt{x}} \, dx \) converges.

Next, note that

\[ 0 \leq \frac{1}{x^7 + \sqrt{x}} \leq \frac{1}{x^7}. \]

Since the integral \( \int_1^\infty \frac{1}{x^7} \, dx \) converges by the \( p \)-test for integrals (\( p > 1 \)), by integral comparison it follows that the integral \( \int_1^\infty \frac{1}{x^7 + \sqrt{x}} \, dx \) converges.

Because both pieces of the initial integral converge, it follows that

\[ \int_0^\infty \frac{1}{x^7 + \sqrt{x}} \, dx \]

converges.

8 A problem where you use L’Hopital to evaluate the limit of a sequence.

Evaluate the following limit.

\[ \lim_{n \to \infty} \tan \left( \frac{1}{n} \right) 3^n. \]

Solution. Since \( \lim_{n \to \infty} \tan(1/n) = \tan 0 = 0 \) and \( \lim_{n \to \infty} 3^n = \infty \), this is an indeterminate form of type \( 0 \cdot \infty \). By L’Hopital’s rule,

\[ \lim_{n \to \infty} \tan \left( \frac{1}{n} \right) 3^n = \lim_{x \to \infty} \tan \left( \frac{1}{x} \right) 3^x = \lim_{x \to \infty} \frac{\tan(1/x)}{3^{-x}} \]

\[ (L’H) = \lim_{x \to \infty} \frac{\sec^2(1/x) \cdot \frac{-1}{x}}{\ln 3} = \lim_{x \to \infty} \frac{\sec^2(1/x) \cdot 3^x}{x^2} \]

\[ (L’H) = \frac{\sec^2(0)}{\ln 3} \lim_{x \to \infty} \frac{3^x \ln 3}{2x} \]

\[ (L’H) = \frac{1}{\ln 3} \lim_{x \to \infty} \frac{3^x (\ln 3)^2}{2} = \infty. \]
9 A problem where you use partial fractions to compute the partial sums of a telescoping series.

Determine whether the following series converges or diverges. If so, evaluate it.

\[
\sum_{n=1}^{\infty} \frac{1}{9n^2 - 3n - 2}.
\]

**Solution.** First, we decompose the summand via partial fractions: we seek \(A\) and \(B\) such that

\[
\frac{1}{9n^2 - 3n - 2} = \frac{1}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}.
\]

Multiplying through by the denominator gives

\[
1 = A(3n + 1) + B(3n - 2) = 3(A + B)n + (A - 2B).
\]

This implies that \(A + B = 0\) and \(A - 2B = 1\), thus \(A = \frac{1}{3}\) and \(B = -\frac{1}{3}\). So

\[
\sum_{n=1}^{\infty} \frac{1}{9n^2 - 3n - 2} = \frac{1}{3} \sum_{n=1}^{\infty} \left( \frac{1}{3n-2} - \frac{1}{3n+1} \right).
\]

Next, we compute a formula for \(S_N\), the \(N\)th partial sum. Note that

\[
S_1 = \frac{1}{3} \left[ \left( \frac{1}{1} - \frac{1}{4} \right) \right] = \frac{1}{3} \left( 1 - \frac{1}{4} \right),
\]

\[
S_2 = \frac{1}{3} \left[ \left( \frac{1}{1} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) \right] = \frac{1}{3} \left( 1 - \frac{1}{7} \right),
\]

\[
S_3 = \frac{1}{3} \left[ \left( \frac{1}{1} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) + \left( \frac{1}{7} - \frac{1}{10} \right) \right] = \frac{1}{3} \left( 1 - \frac{1}{10} \right).
\]

Observing the pattern, we have

\[
S_N = \frac{1}{3} \left( 1 - \frac{1}{3N+1} \right).
\]

By definition of the value of an infinite series,

\[
\sum_{n=1}^{\infty} \frac{1}{9n^2 - 3n - 2} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{1}{3} \left( 1 - \frac{1}{3N+1} \right) = \frac{1}{3}.
\]

Thus, the series converges to \(\frac{1}{3}\). \(\square\)

10 A problem where you use the direct comparison test.

Determine whether the following series converges or diverges.

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n^3 - 1}}{2n^2 + n + 1}.
\]

**Solution.** Note that

\[
0 \leq \frac{\sqrt{n^3 - 1}}{2n^2 + n + 1} \leq \frac{\sqrt{n^3}}{2n^2} = \frac{1}{2n^{\frac{1}{2}}},
\]

Thus, the series converges to \(\frac{1}{3}\).
Since the series
\[ \sum_{n=1}^{\infty} \frac{1}{2n+1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \]
converges by the \( p \)-test \( (p = \frac{11}{2} > 1) \), the direct comparison test implies that
\[ \sum_{n=1}^{\infty} \frac{\sqrt{n^3 - 1}}{2n^2 + n + 1} \]
converges.

11 A problem where you use the alternating series test.

Determine whether the following series converges or diverges.
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(\ln n)}. \]

Solution. Note that \( \ln(\ln n) > 0 \) if \( n \geq 4 \). Indeed, \( \ln 4 > \ln e = 1 \), so that \( \ln(\ln n) \geq \ln(\ln 4) > \ln 1 = 0 \). This implies that \( \frac{1}{\ln(\ln n)} > 0 \).

Next, we show that \( \frac{1}{\ln(\ln x)} \) is decreasing. Note that
\[ \frac{d}{dx} \ln(\ln x) = \frac{1}{x \ln x}. \]

Since \( \ln x > 0 \) for \( x \geq 4 \), this shows that \( \ln(\ln x) \) has positive derivative and thus is increasing if \( x \geq 4 \). Consequently, \( \frac{1}{\ln(\ln x)} \) is decreasing, and so the sequence \( \frac{1}{\ln(\ln n)} \) is decreasing.

Finally, note that
\[ \lim_{n \to \infty} \frac{1}{\ln(\ln n)} = 0. \]

Thus, by the alternating series test, the series
\[ \sum_{n=4}^{\infty} \frac{(-1)^n}{\ln(\ln n)} \]
converges.

12 A problem where you use the limit comparison test and the ratio test.

Determine whether the following series converges or diverges.
\[ \sum_{n=1}^{\infty} \frac{n + \sin(n)}{2^n - 1}. \]

Solution. First, observe that \( n + \sin n \geq 0 \) and so the summand above is positive. Next,
\[ \lim_{n \to \infty} \frac{n + \sin(n)}{2^n} = \lim_{n \to \infty} \frac{1 + \frac{\sin n}{n}}{1 - \frac{1}{2^n}} = \frac{1 + 0}{1 + 0} = 1 \]
since $\frac{1}{2^n} \to 0$ as $n \to \infty$ and $0 \leq \left| \frac{\sin n}{n} \right| \leq \frac{1}{n} \to 0$ as $n \to \infty$. Since $0 < 1 < \infty$, by the limit comparison test, the series
\[ \sum_{n=1}^{\infty} \frac{n + \sin(n)}{2^n - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n}{2^n} \]
either both converge or both diverge.

So consider the series
\[ \sum_{n=1}^{\infty} \frac{n}{2^n} . \]

We use the ratio test. Note that
\[ \lim_{n \to \infty} \left| \frac{n + 1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n \to \infty} \frac{n + 1}{2n} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2} = \frac{1}{2} . \]

Since $\frac{1}{2} < 1$, by the ratio test, the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges. By the preceding remark, the limit comparison test then implies that the series
\[ \sum_{n=1}^{\infty} \frac{n + \sin(n)}{2^n - 1} \]
converges.

13 A problem where you test for absolute convergence, conditional convergence, or divergence.

Determine whether the following series converges absolutely, converges conditionally, or diverges.
\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n (\ln n)^{\frac{3}{2}}} . \]

Solution. First, we consider the series
\[ \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n (\ln n)^{\frac{3}{2}}} \right| = \sum_{n=2}^{\infty} \frac{1}{n (\ln n)^{\frac{3}{2}}} . \]

We use the integral test. Let $f(x) = \frac{1}{x (\ln x)^{\frac{3}{2}}}$. Note that $f$ is positive and continuous for $x \geq 2$, since $\ln x \geq \ln 2 > \ln 1 = 0$. Next, we show that $f$ is decreasing. Note that
\[ f'(x) = -\frac{\left( \ln x \right)^{\frac{3}{2}} + x \cdot \frac{3}{2} (\ln x)^{-\frac{1}{2}} \cdot \frac{1}{2} }{x^2 (\ln x)^{\frac{1}{2}}} = -\frac{1 + \frac{3}{4} (\ln x)^{-1}}{x^2 (\ln x)^{\frac{1}{2}}} . \]

Because $\ln x > 0$ for $x > 1$, we know $1 + \frac{3}{4} (\ln x)^{-1} > 0$ for $x > 1$. Thus,
\[ f'(x) = -\frac{1 + \frac{3}{4} (\ln x)^{-1}}{x^2 (\ln x)^{\frac{1}{2}}} < 0 . \]

for $x \geq 2$. This implies that $f$ is decreasing on the interval $[2, \infty)$. Thus, the integral test applies. Consider the improper integral
\[ \int_{2}^{\infty} \frac{1}{x (\ln x)^{\frac{3}{2}}} \, dx . \]
Let $u = \ln x$. Then $du = \frac{1}{x} \, dx$. Since $\ln x \to \infty$ as $x \to \infty$, the above integral transforms into
\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \int_{\ln 2}^{\infty} \frac{1}{u^\frac{3}{4}} \, du.
\]
Since $\ln 2 > 0$, by the $p$-test for integrals, the above integral diverges. Thus, by the integral test, the series
\[
\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^\frac{3}{4}}
\]
diverges.

Next, we consider the original series
\[
\sum_{n=2}^{\infty} (-1)^n \frac{1}{n (\ln n)^\frac{3}{4}}.
\]
This is an alternating series, so we’ll try the alternating series test. Because the function $f(x) = \frac{1}{x (\ln x)^\frac{3}{4}}$ is positive and decreasing by the previous computations, this implies that the sequence $\frac{1}{n (\ln n)^\frac{3}{4}}$ is positive and decreasing. Also, note that
\[
\lim_{n \to \infty} \frac{1}{n (\ln n)^\frac{3}{4}} = 0.
\]
Thus, by the alternating series test, the series
\[
\sum_{n=2}^{\infty} (-1)^n \frac{1}{n (\ln n)^\frac{3}{4}}
\]
converges.

Therefore, the series
\[
\sum_{n=2}^{\infty} (-1)^n \frac{1}{n (\ln n)^\frac{3}{4}}
\]
converges conditionally.

\[\square\]

**14 A problem where you find the interval of convergence of a power series.**

Find the interval of convergence of the power series
\[
\sum_{n=1}^{\infty} \frac{(x + 1)^n}{3^n \sqrt{n + 1}}
\]

**Solution.** First, let’s use the ratio test. Note that
\[
\lim_{n \to \infty} \left| \frac{(x + 1)^{n+1}}{3^{n+1} \sqrt{n + 2}} \cdot \frac{3^n \sqrt{n + 1}}{(x + 1)^n} \right| = \lim_{n \to \infty} \left| \frac{x + 1}{3} \cdot \sqrt{\frac{n + 2}{n + 1}} \right| = \left| \frac{x + 1}{3} \right|
\]
since $\sqrt{\frac{n + 2}{n + 1}} = \sqrt{1 + \frac{2}{n + 1}} \to 1$ as $n \to \infty$. So by the ratio test, the power series converges when
\[
\left| \frac{x + 1}{3} \right| < 1
\]
and diverges when
\[ \left| \frac{x+1}{3} \right| > 1. \]
Rewriting the first condition as an interval, we know the power series converges when
\[ |x+1| < 3 \quad \rightarrow \quad -3 < x + 1 < 3 \quad \rightarrow \quad -4 < x < 2. \]
The ratio test is inconclusive when \( \left| \frac{x+1}{3} \right| = 1 \), so we need to check the endpoints \(-4\) and \(2\) individually.
Let \( x = 2 \). The resulting series is
\[
\sum_{n=1}^{\infty} \frac{(2+1)^n}{3^n \sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}.
\]
Note that
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1.
\]
Since \( 0 < 1 < \infty \), by the limit comparison test, the series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \) behaves the same as the series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \). Because the latter is a divergent \( p \)-series \( (p = 1/2) \), the former diverges as well.
Let \( x = -4 \). The resulting series is
\[
\sum_{n=1}^{\infty} \frac{(-4+1)^n}{3^n \sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.
\]
Note that \( \frac{1}{\sqrt{n+1}} > 0 \), \( \frac{1}{\sqrt{n+1}} \to 0 \), and \( \frac{1}{\sqrt{n+1}} \) is decreasing, since the numerator is constant and the denominator is increasing:
\[
\sqrt{(n+1)+1} = \sqrt{n+2} \geq \sqrt{n+1}.
\]
Thus, by the alternating series test, the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \) converges.
Therefore, the interval of convergence is \([-4, 2)\).

15 A problem where you expand a function into a power series using the geometric series formula.

Expand the following function into a power series and give its interval of convergence:
\[ f(x) = \frac{x}{2 + 3x}. \]

Solution. Recall that if \( |u| < 1 \), \( \sum_{n=0}^{\infty} u^n = \frac{1}{1-u} \). Motivated by this, note that
\[
f(x) = \frac{x}{2 + 3x} = \frac{x}{2} \cdot \frac{1}{1 + \frac{3}{2}x} = \frac{x}{2} \cdot \frac{1}{1 - \left(-\frac{3}{2}x\right)}.
\]
When \( \left| -\frac{3}{2}x \right| < 1 \), we then have
\[
f(x) = \frac{x}{2} \sum_{n=0}^{\infty} \left(-\frac{3}{2}x\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{n+1}} x^{n+1}.
\]
Because this expansion is valid precisely when \( \left| -\frac{3}{2}x \right| < 1 \), the interval of convergence is
\[ -\frac{2}{3} < x < \frac{2}{3}. \]
16  A problem where you find a power series solution to a differential equation.

Find a power series solution to the differential equation

\[ y'' = xy' + y \]

with initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \).

Solution. Let

\[ y = P(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \]

be a power series solution of the above equation. Then

\[ P''(x) = x P'(x) + P(x). \]  (1)

Moreover, the initial conditions imply that \( a_0 = P(0) = 1 \) and \( a_1 = P'(0) = 0 \).

Next, note that

\[ P'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \]

and

\[ P''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \]

The right hand side of (1) is then

\[ x P'(x) + P(x) = x \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_n x^n. \]

Thus, (1) says that

\[ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1) a_n x^n. \]

Reindexing the left hand sum (so that the powers of \( x \) match up), we have

\[ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=0}^{\infty} (n+1) a_n x^n. \]

Equating coefficients, this implies that

\[ (n+2)(n+1)a_{n+2} = (n+1)a_n. \]

In other words,

\[ a_{n+2} = \frac{a_n}{n+2}. \]

Reindexing again gives us:

\[ a_n = \frac{a_{n-2}}{n}. \]  (2)

We want to recursively use this condition to get a formula for \( a_n \), but because the index decreases by 2 each time, it makes sense to break this into even and odd cases. The recursive relation (2) gives

\[ a_{2k} = \frac{a_{2k-2}}{2k} \quad \text{and} \quad a_{2k+1} = \frac{a_{2k-1}}{2k+1}. \]  (3)
Then
\[
a_{2k} = \frac{a_{2k-2}}{2k} = \frac{1}{2k} \frac{a_{2k-4}}{2k-2} = \frac{1}{2k} \frac{1}{2k-2} \frac{a_{2k-6}}{2k-4} = \ldots = \frac{1}{(2k)(2k-2)(2k-4) \cdots 2} a_0 \\
= \frac{1}{2^k(k-1)(k-2) \cdots 2} a_0.
\]

Since \( a_0 = 1 \), this implies that
\[
a_{2k} = \frac{1}{2^k k!}.
\]

Next we consider the odd equations. Similar to the above computation,
\[
a_{2k+1} = \frac{a_{2k-1}}{2k+1} = \frac{1}{2k+1} \frac{a_{2k-3}}{2k-1} = \frac{1}{(2k+1)(2k-1)(2k-3) \cdots 1} a_1.
\]

Since \( a_1 = 0 \), this implies that \( a_{2k+1} = 0 \). In other words, all odd coefficients are 0.

Putting this all together, it follows that
\[
P(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}
\]
is a power series solution to the stated differential equation.

\[
\Box
\]

17 A problem where you use the Taylor polynomial error bound theorem.

Let \( f(x) = \cos(2x) \) and let \( T_n \) be the \( n \)th Taylor polynomial of \( f \) centered at 1. Find an \( n \) such that
\[
|\cos(2.5) - T_n(1.25)| \leq 10^{-3}.
\]

Solution. By the Taylor polynomial error bound theorem
\[
|\cos(2(1.25)) - T_n(1.25)| \leq \frac{K|1.25 - 1|^{n+1}}{(n+1)!} = \frac{K}{4^{n+1}(n+1)!}
\]
where \( K \) is a number satisfying \( |f^{n+1}(x)| \leq K \) on the interval \([1, 1.25] \).

Note that
\[
f(x) = \cos(2x) \\
f'(x) = -2 \sin(2x) \\
f''(x) = -2^2 \cos(2x) \\
f^{(3)}(x) = 2^3 \sin(2x)
\]
and so \( |f^{n+1}(x)| = 2^{n+1} |\sin(2x)| \) or \( |f^{n+1}(x)| = 2^{n+1} |\cos(2x)| \) depending on the parity of \( n \). In either case, since sine and cosine are uniformly bounded in absolute value by 1, \( |f^{n+1}(x)| \leq 2^{n+1} \) and so we can take \( K = 2^{n+1} \). Thus,
\[
|\cos(2(1.25)) - T_n(1.25)| \leq \frac{2^{n+1}}{4^{n+1}(n+1)!} = \frac{1}{2^{n+1}(n+1)!}.
\]

Thus, it suffices to find an \( n \) such that
\[
\frac{1}{2^{n+1}(n+1)!} \leq \frac{1}{10^3} = \frac{1}{1000}.
\]

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Note that if \( n = 1000 \) then
\[
\frac{1}{2^{1001}1001!} \leq \frac{1}{1001!} \leq \frac{1}{1001} \leq \frac{1}{1000}.
\]
So \( n = 1000 \) works.

For a much sharper answer, note that
\[
2^5 \cdot 5! = 32 \cdot 120 \geq 30 \cdot 100 = 3000 \geq 1000.
\]
So
\[
\frac{1}{2^5 \cdot 5!} \leq \frac{1}{1000}
\]
and so \( n = 4 \) also works.

18 A problem where you compute a Taylor series by hand.

Let \( f(x) = \ln x \). Compute the Taylor series of \( f \) centered at 3 and give the interval of convergence.

Solution. We need to compute
\[
T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (x-3)^n.
\]

Note that
\[
\begin{align*}
f(x) &= \ln x \\
f'(x) &= \frac{1}{x} \\
f''(x) &= -\frac{1}{x^2} \\
f^{(3)}(x) &= \frac{2}{x^3} \\
f^{(4)}(x) &= -\frac{3 \cdot 2}{x^4} \\
f^{(5)}(x) &= \frac{4 \cdot 3 \cdot 2}{x^5} \\
&\vdots \\
f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n}.
\end{align*}
\]
Thus, \( f(3) = \ln 3 \) and \( f^{(n)}(3) = \frac{(-1)^{n+1}(n-1)!}{3^n} \) for \( n \geq 1 \). Therefore, the desired Taylor series is
\[
T(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{3^n \cdot n!} (x-3)^n = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n \cdot n} (x-3)^n.
\]

Next, we find the interval of convergence. We begin with the ratio test. Note that
\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+2}(x-3)^{n+1}}{3^{n+1}(n+1)} \cdot \frac{3^n \cdot n}{(-1)^{n+1}(x-3)^n} \right| = \lim_{n \to \infty} \left| \frac{x-3}{3} \cdot \frac{n}{n+1} \right| = \left| \frac{x-3}{3} \right|.
\]
Thus, the series converges if \( |x-3| < 3 \) and diverges if \( |x-3| > 3 \). We need to individually check the points corresponding to \( |x-3| = 3 \), which are 0 and 6.
First, let \( x = 6 \). The series we get is

\[ \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}. \]

Since \( \frac{1}{n} \) is positive, decreasing \( (n + 1 \geq n) \), and \( \frac{1}{n} \to 0 \), by the alternating series test, this series converges.

Next, let \( x = 0 \). The series we get is

\[ \ln 3 - \sum_{n=1}^{\infty} \frac{1}{n}. \]

This is a divergent \( p \)-series. Thus, the interval of convergence is \((0, 6]\).

19 A problem where you compute a MacLaurin series by integrating a known MacLaurin series.

Let \( f(x) = \arctan(4x^2) \). Compute the MacLaurin series of \( f \) using the MacLaurin series for \( \frac{1}{1-u} \) and give the interval of convergence.

**Solution.** First, note that

\[ f'(x) = \frac{8x}{1 + 16x^4} = 8x \cdot \frac{1}{1 - (-16x^4)}. \]

Next, recall that the MacLaurin series

\[ \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \]

is valid for \(|u| < 1\). Thus, if \( 16x^4 < 1 \), we have

\[ f'(x) = 8x \cdot \frac{1}{1 - (-16x^4)} = 8x \sum_{n=0}^{\infty} (-16x^4)^n = 8 \sum_{n=0}^{\infty} (-1)^n 16^n x^{4n+1}. \]

Thus, if \( 16x^4 < 1 \),

\[ f(x) = \sum_{n=0}^{\infty} 4(-1)^n 16^n x^{4n+2} \int_0^x t^{4n+1} dt = \sum_{n=0}^{\infty} 4(-1)^n 16^n x^{4n+2} \frac{1}{4n+2}. \]

Since \( f(0) = \arctan(0) = 0 \), it follows that if \( 16x^4 < 1 \) and thus \( -\frac{1}{2} < x < \frac{1}{2} \),

\[ f(x) = \sum_{n=0}^{\infty} 4(-1)^n 16^n x^{4n+2} \frac{1}{2n + 1}. \]

Finally, note that if \( x = -1/2 \)

\[ \sum_{n=0}^{\infty} 4(-1)^n 16^n \frac{(-1/2)^{4n+2}}{2n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}. \]
Since \( \frac{1}{2n+1} \to 0 \) and is positive and decreasing \((2(n+1) + 1 = 2n + 3 \geq 2n + 1)\), by the alternating series test, this series converges. If \( x = 1/2 \), then

\[
\sum_{n=0}^{\infty} 4(-1)^n 16^n \frac{(1/2)^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]

and by the previous remark this series converges as well.

Therefore, the interval of convergence of the above MacLaurin series is actually \([-1/2, 1/2]\). \(\square\)