MODEL SOLUTIONS FOR 31B PROBLEMS

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Here is a collection of **well-written solutions** to most standard exercises in 31B. This document should be used not as a reference for *how to do* each type of problem, but rather *how to write up a solution* in a clear and effective manner. Everything from the general structure to the language and correct symbol usage is important!

The blue links below are shortcuts to each problem.

Contents

1 A problem where you show a function is invertible and compute the derivative of the inverse.

Let $f(x) = x^9 + 2x + 1$. Show that f is invertible, and compute $(f^{-1})'(4)$.

Solution. To show that f is invertible on its domain, we will show that it is strictly increasing. Strictly increasing functions are one-to-one, and thus invertible on their domain. Note that

$$f'(x) = 9x^8 + 2 \geq 2 > 0$$

for all x. Since $f' > 0$, f is strictly increasing and thus invertible by first remark.

Next, we know that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))}.$$

Note that $f(1) = (1)^9 + 2(1) + 1 = 4$. This implies that $f^{-1}(4) = 1$, and so

$$(f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{9(1)^8 + 2} = \frac{1}{11}.$$

□

2 A problem where you use a L'Hopital to evaluate an exponential indeterminate form.

Evaluate the following limit.

$$\lim_{x \to 0} (\sec x)^{\frac{1}{x^2}}.$$

Solution. Note that

$$\lim_{x \to 0} \sec x = \frac{1}{\cos 0} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Thus, the above limit is indeterminate of type 1^∞. Let $L = \lim_{x \to 0} (\sec x)^{\frac{1}{x^2}}$. Then

$$\ln L = \lim_{x \to 0} \frac{1}{x^2} \ln(\sec x) = \lim_{x \to 0} \frac{\ln(\sec x)}{x^2}.$$

Applying L'Hopital's rule, we have

$$\ln L = \lim_{x \to 0} \frac{\frac{1}{\sec x} \cdot \sec x \tan x}{2x} = \lim_{x \to 0} \frac{1}{2\cos x} \cdot \frac{\sin x}{x}.$$

Note that

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

by L'Hopital's rule. Thus,

$$\ln L = \left(\lim_{x \to 0} \frac{1}{2\cos x} \right) \cdot \left(\lim_{x \to 0} \frac{\sin x}{x} \right) = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Therefore, $L = e^{\ln L} = e^{\frac{1}{2}}$. □

3 A problem where you use a trigonometric substitution to evaluate an antiderivative.

Evaluate the following antiderivative.

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\, dx.$$

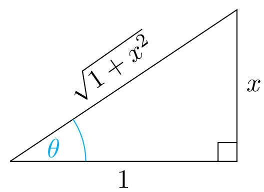
Solution. Let $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$, and we have

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\, dx = \int \frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}}\, \sec^2\theta\, d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}\, d\theta = \int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta\, d\theta.$$

Thus,

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\, dx = \int \cos\theta\, d\theta = \sin\theta + C.$$

Next, we draw a right triangle with angle θ such that $\tan\theta = x$.



This picture implies that $\sin\theta = \frac{x}{\sqrt{1+x^2}}$. Thus,

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\, dx = \frac{x}{\sqrt{1+x^2}} + C.$$

□

4 A problem where you use partial fractions and complete the square to evaluate an antiderivative.

Evaluate the following antiderivative.

$$\int \frac{2x^3+10x^2+15x+6}{(x^2+x+1)(x^2+4x+5)}\, dx.$$

Solution. We begin by decomposing the integrand via partial fractions. Note that both quadratic factors in the denominator are irreducible because $1^2 - 4 < 0$ and $4^2 - 4 * 5 < 0$. Thus, we seek a decomposition of the form

$$\frac{2x^3+10x^2+15x+6}{(x^2+x+1)(x^2+4x+5)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+4x+5}.$$

Multiplying through by the denominator of the left hand side gives

$$2x^3+10x^2+15x+6 = (Ax+B)(x^2+4x+5) + (Cx+D)(x^2+x+1).$$

Comparing x^3 coefficients gives the following equation:

$$2 = A + C.$$

Comparing x^2 coefficients gives the following equation:

$$10 = 4A + B + C + D.$$

Comparing x coefficients gives the following equation:

$$15 = 5A + 4B + C + D.$$

Comparing the constant terms gives the following equation:

$$6 = 5B + D.$$

Subtracting the x equation from the constant term equation gives

$$-9 = -5A + B - C.$$

Plugging in $C = 2 - A$ into this equation gives $-9 = -5A + B - (2 - A)$ and thus

$$-9 = -4A + B - 2 \quad \Rightarrow \quad 4A - B = 7.$$

Subtracting the x^2 equation from the x equation gives

$$5 = A + 3B \quad \Rightarrow \quad A = 5 - 3B.$$

Thus,

$$4(5 - 3B) - B = 7 \quad \Rightarrow \quad 20 - 13B = 7 \quad B = 1.$$

Thus, $A = 5 - 4 = 2$. Since $C = 2 - A$, we have $C = 0$. Finally since $D = 6 - 5B$ we have $D = 1$. (Whew.) Therefore,

$$\int \frac{2x^3 + 10x^2 + 15x + 6}{(x^2 + x + 1)(x^2 + 4x + 5)}\,dx = \int \frac{2x + 1}{x^2 + x + 1} + \frac{1}{x^2 + 4x + 5}\,dx.$$

To evaluate the first antiderivative, let $u = x^2 + x + 1$. Then $du = (2x + 1)\,dx$ and we have

$$\int \frac{2x + 1}{x^2 + x + 1}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln(x^2 + x + 1) + C.$$

For the second term, we complete the square. Note that

$$x^2 + 4x + 5 = x^2 + 4x + 4 - 4 + 5 = (x + 2)^2 + 1.$$

Letting $u = x + 2$ then gives

$$\int \frac{1}{x^2 + 4x + 5}\,dx = \int \frac{1}{(x + 2)^2 + 1}\,dx = \int \frac{1}{u^2 + 1}\,du = \arctan(u) + C = \arctan(x + 2) + C.$$

Therefore,

$$\int \frac{2x^3 + 10x^2 + 15x + 6}{(x^2 + x + 1)(x^2 + 4x + 5)}\,dx = \ln(x^2 + x + 1) + \arctan(x + 2) + C.$$

□

5 A problem where you use a substitution to show that an improper integral converges.

Does the following integral converge? If so, evaluate it.

$$\int_{8675309}^{\infty} \frac{e^x}{1+(e^x)^2}\,dx.$$

Solution. First, the integral is improper at ∞ and nowhere else. Thus,

$$\int_{8675309}^{\infty} \frac{e^x}{1+(e^x)^2}\,dx = \lim_{R\to\infty}\int_{8675309}^{R} \frac{e^x}{1+(e^x)^2}\,dx.$$

Next, let $u = e^x$. Then $du = e^x\,dx$. We have

$$\lim_{R\to\infty}\int_{e^{8675309}}^{e^R} \frac{1}{1+u^2}\,du = \lim_{R\to\infty} \arctan(u)\Big|_{e^{8675309}}^{e^R} = \lim_{R\to\infty} \arctan(e^R) - \arctan(e^{8675309})$$
$$= \frac{\pi}{2} - \arctan(e^{8675309}).$$

Since this number is finite, it follows that the integral converges, and converges to

$$\frac{\pi}{2} - \arctan(e^{8675309}).$$

□

6 A problem where you use comparison to show that an improper integral diverges.

Does the following integral converge? If so, evaluate it.

$$\int_{2}^{\infty} \frac{7+x^{\frac{1}{4}}}{\sqrt{x-1}}\,dx.$$

Solution. Note that

$$\frac{7+x^{\frac{1}{4}}}{\sqrt{x-1}} \geq \frac{7}{\sqrt{x}}$$

as the denominator of the left side is smaller and the numerator of the left side is bigger. By the p-test for integrals, the integral

$$\int_{2}^{\infty} \frac{7}{\sqrt{x}}\,dx$$

diverges because $p = 1/2 \leq 1$. Thus, by the comparison test for integrals, the integral

$$\int_{2}^{\infty} \frac{7+x^{\frac{1}{4}}}{\sqrt{x-1}}\,dx$$

diverges as well. □

7 A problem where you split an improper integral into multiple pieces.

Does the following integral converge?

$$\int_0^\infty \frac{1}{x^7 + \sqrt{x}}\,dx.$$

Solution. First, note that the integral is improper for two reasons: at ∞, and also at 0 since the denominator of the integrand is 0 when $x = 0$. Thus, we split the integral into two pieces

$$\int_0^\infty \frac{1}{x^7 + \sqrt{x}}\,dx = \int_0^1 \frac{1}{x^7 + \sqrt{x}}\,dx. + \int_1^\infty \frac{1}{x^7 + \sqrt{x}}\,dx.$$

We consider each piece separately.

First, note that

$$0 \le \frac{1}{x^7 + \sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

Since the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges by the p-test for integrals ($p < 1$), by integral comparison it follows that the integral $\int_0^1 \frac{1}{x^7+\sqrt{x}}\,dx$ converges.

Next, note that

$$0 \le \frac{1}{x^7 + \sqrt{x}} \le \frac{1}{x^7}.$$

Since the integral $\int_1^\infty \frac{1}{x^7}\,dx$ converges by the p-test for integrals ($p > 1$), by integral comparison it follows that the integral $\int_1^\infty \frac{1}{x^7+\sqrt{x}}\,dx$ converges.

Because both pieces of the initial integral converge, it follows that

$$\int_0^\infty \frac{1}{x^7 + \sqrt{x}}\,dx$$

converges.

□

8 A problem where you use L'Hopital to evaluate the limit of a sequence.

Evaluate the following limit.

$$\lim_{n\to\infty} \tan\left(\frac{1}{n}\right) 3^n.$$

Solution. Since $\lim_{n\to\infty} \tan(1/n) = \tan 0 = 0$ and $\lim_{n\to\infty} 3^n = \infty$, this is an indeterminate form of type $0 \cdot \infty$. By L'Hopital's rule,

$$\begin{aligned}
\lim_{n\to\infty} \tan\left(\frac{1}{n}\right) 3^n &= \lim_{x\to\infty} \tan\left(\frac{1}{x}\right) 3^x = \lim_{x\to\infty} \frac{\tan(1/x)}{3^{-x}} \\
(L'H) &= \lim_{x\to\infty} \frac{\sec^2(1/x)\cdot\frac{-1}{x^2}}{-3^{-x}\cdot\ln 3} = \lim_{x\to\infty} \frac{\sec^2(1/x)}{\ln 3}\cdot\frac{3^x}{x^2} \\
(L'H) &= \frac{\sec^2(0)}{\ln 3}\lim_{x\to\infty} \frac{3^x \ln 3}{2x} \\
(L'H) &= \frac{1}{\ln 3}\lim_{x\to\infty} \frac{3^x(\ln 3)^2}{2} = \infty.
\end{aligned}$$

□

9 A problem where you use partial fractions to compute the partial sums of a telescoping series.

Determine whether the following series converges or diverges. If so, evaluate it.

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 - 3n - 2}.$$

Solution. First, we decompose the summand via partial fractions: we seek A and B such that

$$\frac{1}{9n^2 - 3n - 2} = \frac{1}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}.$$

Multiplying through by the denominator gives

$$1 = A(3n+1) + B(3n-2) = 3(A+B)n + (A - 2B).$$

This implies that $A + B = 0$ and $A - 2B = 1$, thus $A = \frac{1}{3}$ and $B = -\frac{1}{3}$. So

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 - 3n - 2} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{3n-2} - \frac{1}{3n+1} \right).$$

Next, we compute a formula for S_N, the Nth partial sum. Note that

$$\begin{aligned}
S_1 &= \frac{1}{3}\left[\left(\frac{1}{1} - \frac{1}{4}\right)\right] = \frac{1}{3}\left(1 - \frac{1}{4}\right) \\
S_2 &= \frac{1}{3}\left[\left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right)\right] = \frac{1}{3}\left(1 - \frac{1}{7}\right) \\
S_3 &= \frac{1}{3}\left[\left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right)\right] = \frac{1}{3}\left(1 - \frac{1}{10}\right).
\end{aligned}$$

Observing the pattern, we have

$$S_N = \frac{1}{3}\left(1 - \frac{1}{3N+1}\right).$$

By definition of the value of an infinite series,

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 - 3n - 2} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{3}\left(1 - \frac{1}{3N+1}\right) = \frac{1}{3}.$$

Thus, the series converges to $\frac{1}{3}$. □

10 A problem where you use the direct comparison test.

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3 - 1}}{2n^7 + n + 1}.$$

Solution. Note that

$$0 \leq \frac{\sqrt{n^3 - 1}}{2n^7 + n + 1} \leq \frac{\sqrt{n^3}}{2n^7} = \frac{1}{2n^{\frac{11}{2}}}.$$

Since the series

$$\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{11}{2}}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{11}{2}}}$$

converges by the p-test ($p = \frac{11}{2} > 1$), the direct comparison test implies that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3 - 1}}{2n^7 + n + 1}$$

converges. □

11 A problem where you use the alternating series test.

Determine whether the following series converges or diverges.

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{\ln(\ln n)}.$$

Solution. Note that $\ln(\ln n) > 0$ if $n \geq 4$. Indeed, $\ln 4 > \ln e = 1$, so that $\ln(\ln n) \geq \ln(\ln 4) > \ln 1 = 0$. This implies that $\frac{1}{\ln(\ln n))} > 0$.

Next, we show that $\frac{1}{\ln(\ln n))}$ is decreasing. Note that

$$\frac{d}{dx} \ln(\ln x) = \frac{1}{x \ln x}.$$

Since $\ln x > 0$ for $x \geq 4$, this shows that $\ln(\ln x)$ has positive derivative and thus is increasing if $x \geq 4$. Consequently, $\frac{1}{\ln(\ln x)}$ is decreasing, and so the sequence $\frac{1}{\ln(\ln n))}$ is decreasing.

Finally, note that

$$\lim_{n\to\infty} \frac{1}{\ln(\ln n)} = 0.$$

Thus, by the alternating series test, the series

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{\ln(\ln n)}$$

converges. □

12 A problem where you use the limit comparison test and the ratio test.

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n + \sin(n)}{2^n - 1}.$$

Solution. First, observe that $n + \sin n \geq 0$ and so the summand above is positive. Next,

$$\lim_{n\to\infty} \frac{\frac{n+\sin(n)}{2^n-1}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{1 + \frac{\sin n}{n}}{1 - \frac{1}{2^n}} = \frac{1+0}{1+0} = 1$$

since $\frac{1}{2^n} \to 0$ as $n \to \infty$ and $0 \le \left|\frac{\sin n}{n}\right| \le \frac{1}{n} \to 0$ as $n \to \infty$. Since $0 < 1 < \infty$, by the limit comparison test, the series

$$\sum_{n=1}^{\infty} \frac{n + \sin(n)}{2^n - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n}{2^n}$$

either both converge or both diverge.

So consider the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

We use the ratio test. Note that

$$\lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2}.$$

Since $\frac{1}{2} < 1$, by the ratio test, the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges. By the preceding remark, the limit comparison test then implies that the series

$$\sum_{n=1}^{\infty} \frac{n + \sin(n)}{2^n - 1}$$

converges. □

13 A problem where you test for absolute convergence, conditional convergence, or divergence.

Determine whether the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(\ln n)^{\frac{1}{4}}}.$$

Solution. First, we consider the series

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\,(\ln n)^{\frac{1}{4}}} \right| = \sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^{\frac{1}{4}}}.$$

We use the integral test. Let $f(x) = \frac{1}{x\,(\ln x)^{\frac{1}{4}}}$. Note that f is positive and continuous for $x \ge 2$, since $\ln x \ge \ln 2 > \ln 1 = 0$. Next, we show that f is decreasing. Note that

$$f'(x) = \frac{-\left((\ln x)^{\frac{1}{4}} + x \cdot \frac{1}{4}(\ln x)^{-\frac{3}{4}} \cdot \frac{1}{x}\right)}{x^2 (\ln x)^{\frac{1}{2}}} = -\frac{1 + \frac{1}{4}(\ln x)^{-1}}{x^2 (\ln x)^{\frac{1}{4}}}.$$

Because $\ln x > 0$ for $x > 1$, we know $1 + \frac{1}{4}(\ln x)^{-1} > 0$ for $x > 1$. Thus,

$$f'(x) = -\frac{1 + \frac{1}{4}(\ln x)^{-1}}{x^2 (\ln x)^{\frac{1}{4}}} < 0.$$

for $x \ge 2$. This implies that f is decreasing on the interval $[2, \infty)$. Thus, the integral test applies. Consider the improper integral

$$\int_2^{\infty} \frac{1}{x\,(\ln x)^{\frac{1}{4}}}\, dx.$$

Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$. Since $\ln x \to \infty$ as $x \to \infty$, the above integral transforms into

$$\int_2^\infty \frac{1}{x\,(\ln x)^{\frac{1}{4}}}\,dx = \int_{\ln 2}^\infty \frac{1}{u^{\frac{1}{4}}}\,du.$$

Since $\ln 2 > 0$, by the p-test for integrals, the above integral diverges. Thus, by the integral test, the series

$$\sum_{n=2}^\infty \frac{1}{n\,(\ln n)^{\frac{1}{4}}}$$

diverges.

Next, we consider the original series

$$\sum_{n=2}^\infty \frac{(-1)^n}{n\,(\ln n)^{\frac{1}{4}}}.$$

This is an alternating series, so we'll try the alternating series test. Because the function $f(x) = \frac{1}{x\,(\ln x)^{\frac{1}{4}}}$ is positive and decreasing by the previous computations, this implies that the sequence $\frac{1}{n\,(\ln n)^{\frac{1}{4}}}$ is positive and decreasing. Also, note that

$$\lim_{n\to\infty} \frac{1}{n\,(\ln n)^{\frac{1}{4}}} = 0.$$

Thus, by the alternating series test, the series

$$\sum_{n=2}^\infty \frac{(-1)^n}{n\,(\ln n)^{\frac{1}{4}}}$$

converges.

Therefore, the series

$$\sum_{n=2}^\infty \frac{(-1)^n}{n\,(\ln n)^{\frac{1}{4}}}$$

converges conditionally. ☐

14 A problem where you find the interval of convergence of a power series.

Find the interval of convergence of the power series

$$\sum_{n=1}^\infty \frac{(x+1)^n}{3^n\sqrt{n+1}}.$$

Solution. First, let's use the ratio test. Note that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{3^{n+1}\sqrt{n+2}} \cdot \frac{3^n\sqrt{n+1}}{(x+1)^n}\right| &= \lim_{n\to\infty}\left|\frac{x+1}{3}\cdot\frac{\sqrt{n+2}}{\sqrt{n+1}}\right| \\ &= \left|\frac{x+1}{3}\right|\end{aligned}$$

since $\frac{\sqrt{n+2}}{\sqrt{n+1}} = \sqrt{\frac{1+\frac{2}{n}}{1+\frac{1}{n}}} \to 1$ as $n \to \infty$. So by the ratio test, the power series converges when

$$\left|\frac{x+1}{3}\right| < 1$$

and diverges when

$$\left|\frac{x+1}{3}\right| > 1.$$

Rewriting the first condition as an interval, we know the power series converges when

$$|x+1| < 3 \quad \rightarrow \quad -3 < x+1 < 3 \quad \rightarrow \quad -4 < x < 2.$$

The ratio test in inconclusive when $\left|\frac{x+1}{3}\right| = 1$, so we need to check the endpoints -4 and 2 individually.

Let $x = 2$. The resulting series is

$$\sum_{n=1}^{\infty} \frac{(2+1)^n}{3^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}.$$

Note that

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1.$$

Since $0 < 1 < \infty$, by the limit comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ behaves the same as the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Because the latter is a divergent p-series ($p = 1/2$), the former diverges as well.

Let $x = -4$. The resulting series is

$$\sum_{n=1}^{\infty} \frac{(-4+1)^n}{3^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

Note that $\frac{1}{\sqrt{n+1}} > 0$, $\frac{1}{\sqrt{n+1}} \to 0$, and $\frac{1}{\sqrt{n+1}}$ is decreasing, since the numerator is constant and the denominator is increasing:

$$\sqrt{(n+1)+1} = \sqrt{n+2} \geq \sqrt{n+1}.$$

Thus, by the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges.

Therefore, the interval of convergence is $[-4, 2)$. □

15 A problem where you expand a function into a power series using the geometric series formula.

Expand the following function into a power series and give its interval of convergence:

$$f(x) = \frac{x}{2+3x}.$$

Solution. Recall that if $|u| < 1$, $\sum_{n=0}^{\infty} u^n = \frac{1}{1-u}$. Motivated by this, note that

$$f(x) = \frac{x}{2+3x} = \frac{x}{2} \cdot \frac{1}{1+\frac{3}{2}x} = \frac{x}{2} \cdot \frac{1}{1-\left(-\frac{3}{2}x\right)}.$$

When $\left|-\frac{3}{2}x\right| < 1$, we then have

$$f(x) = \frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{3}{2}x\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^{n+1}}x^{n+1}.$$

Because this expansion is valid precisely when $\left|-\frac{3}{2}x\right| < 1$, the interval of convergence is

$$-\frac{2}{3} < x < \frac{2}{3}.$$

□

16 A problem where you find a power series solution to a differential equation.

Find a power series solution to the differential equation

$$y'' = xy' + y$$

with initial conditions $y(0) = 1$ **and** $y'(0) = 0$**.**

Solution. Let

$$y = P(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

be a power series solution of the above equation. Then

$$P''(x) = xP'(x) + P(x). \tag{1}$$

Moreover, the initial conditions imply that $a_0 = P(0) = 1$ and $a_1 = P'(0) = 0$.

Next, note that

$$P'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

and

$$P''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

The right hand side of (1) is then

$$xP'(x) + P(x) = x \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_n x^n.$$

Thus, (1) says that

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1) a_n x^n.$$

Reindexing the left hand sum (so that the powers of x match up), we have

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=0}^{\infty} (n+1) a_n x^n.$$

Equating coefficients, this implies that

$$(n+2)(n+1) a_{n+2} = (n+1) a_n.$$

In other words,

$$a_{n+2} = \frac{a_n}{n+2}.$$

Reindexing again gives us:

$$a_n = \frac{a_{n-2}}{n}. \tag{2}$$

We want to recursively use this condition to get a formula for a_n, but because the index decreases by 2 each time, it makes sense to break this into even and odd cases. The recursive relation (2) gives

$$a_{2k} = \frac{a_{2k-2}}{2k} \qquad \text{and} \qquad a_{2k+1} = \frac{a_{2k-1}}{2k+1}. \tag{3}$$

Then

$$a_{2k} = \frac{a_{2k-2}}{2k} = \frac{1}{2k}\frac{a_{2k-4}}{2k-2} = \frac{1}{2k}\frac{1}{2k-2}\frac{a_{2k-6}}{2k-4} = \cdots = \frac{1}{(2k)(2k-2)(2k-4)\cdots 4\cdot 2}a_0$$
$$= \frac{1}{2^k(k)(k-1)(k-2)\cdots 2\cdot 1}a_0.$$

Since $a_0 = 1$, this implies that

$$a_{2k} = \frac{1}{2^k\,k!}.$$

Next we consider the odd equations. Similar to the above computation,

$$a_{2k+1} = \frac{a_{2k-1}}{2k+1} = \frac{1}{2k+1}\frac{a_{2k-3}}{2k-1} = \frac{1}{(2k+1)(2k-1)(2k-3)\cdots 1}a_1.$$

Since $a_1 = 0$, this implies that $a_{2k+1} = 0$. In other words, all odd coefficients are 0.

Putting this all together, it follows that

$$P(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k\,k!}$$

is a power series solution to the stated differential equation.

□

17 A problem where you use the Taylor polynomial error bound theorem.

Let $f(x) = \cos(2x)$ and let T_n be the nth Taylor polynomial of f centered at 1. Find an n such that

$$|\cos(2.5) - T_n(1.25)| \leq 10^{-3}.$$

Solution. By the Taylor polynomial error bound theorem

$$|\cos(2(1.25)) - T_n(1.25)| \leq \frac{K|1.25-1|^{n+1}}{(n+1)!} = \frac{K}{4^{n+1}(n+1)!}$$

where K is a number satisfying $|f^{n+1}(x)| \leq K$ on the interval $[1, 1.25]$.

Note that

$$f(x) = \cos(2x)$$
$$f'(x) = -2\sin(2x)$$
$$f''(x) = -2^2\cos(2x)$$
$$f^{(3)} = 2^3\sin(2x)$$

and so $|f^{n+1}(x)| = 2^{n+1}|\sin(2x)|$ or $|f^{n+1}(x)| = 2^{n+1}|\cos(2x)|$ depending on the parity of n. In either case, since sine and cosine are uniformly bounded in absolute value by 1, $|f^{n+1}(x)| \leq 2^{n+1}$ and so we can take $K = 2^{n+1}$. Thus,

$$|\cos(2(1.25)) - T_n(1.25)| \leq \frac{2^{n+1}}{4^{n+1}(n+1)!} = \frac{1}{2^{n+1}(n+1)!}.$$

Thus, it suffices to find an n such that

$$\frac{1}{2^{n+1}(n+1)!} \leq \frac{1}{10^3} = \frac{1}{1000}.$$

Note that if $n = 1000$ then
$$\frac{1}{2^{1001}1001!} \leq \frac{1}{1001!} \leq \frac{1}{1001} \leq \frac{1}{1000}.$$
So $n = 1000$ works.

For a much sharper answer, note that

$$2^5 \cdot 5! = 32 * 120 \geq 30 * 100 = 3000 \geq 1000.$$

So
$$\frac{1}{2^5 \cdot 5!} \leq \frac{1}{1000}$$
and so $n = 4$ also works.

□

18 A problem where you compute a Taylor series by hand.

Let $f(x) = \ln x$. Compute the Taylor series of f centered at 3 and give the interval of convergence.

Solution. We need to compute
$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n.$$

Note that

$$\begin{aligned}
f(x) &= \ln x \\
f'(x) &= \frac{1}{x} \\
f''(x) &= \frac{-1}{x^2} \\
f^{(3)}(x) &= \frac{2}{x^3} \\
f^{(4)}(x) &= \frac{-3 \cdot 2}{x^4} \\
f^{(5)}(x) &= \frac{4 \cdot 3 \cdot 2}{x^5} \\
&\vdots \\
f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n}.
\end{aligned}$$

Thus, $f(3) = \ln 3$ and $f^{(n)}(3) = \frac{(-1)^{n+1}(n-1)!}{3^n}$ for $n \geq 1$. Therefore, the desired Taylor series is

$$T(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{3^n \cdot n!}(x-3)^n = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n \cdot n}(x-3)^n.$$

Next, we find the interval of convergence. We begin with the ratio test. Note that

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-3)^{n+1}}{3^{n+1}(n+1)} \cdot \frac{3^n \cdot n}{(-1)^{n+1}(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{x-3}{3} \right| \cdot \frac{n}{n+1} = \left| \frac{x-3}{3} \right|.$$

Thus, the series converges if $|x-3| < 3$ and diverges if $|x-3| > 3$. We need to individually check the points corresponding to $|x-3| = 3$, which are 0 and 6.

First, let $x = 6$. The series we get is

$$\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Since $\frac{1}{n}$ is positive, decreasing ($n+1 \geq n$), and $\frac{1}{n} \to 0$, by the alternating series test, this series converges.

Next, let $x = 0$. The series we get is

$$\ln 3 - \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is a divergent p-series. Thus, the interval of convergence is $(0, 6]$. □

19 A problem where you compute a MacLaurin series by integrating a known MacLaurin series.

Let $f(x) = \arctan(4x^2)$. Compute the MacLaurin series of f using the MacLaurin series for $\frac{1}{1-x}$ and give the interval of convergence.

Solution. First, note that

$$f'(x) = \frac{8x}{1+16x^4} = 8x \cdot \frac{1}{1-(-16x^4)}.$$

Next, recall that the MacLaurin series

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

is valid for $|u| < 1$. Thus, if $16x^4 < 1$, we have

$$f'(x) = 8x \cdot \frac{1}{1-(-16x^4)} = 8x \sum_{n=0}^{\infty} \left(-16x^4\right)^n = \sum_{n=0}^{\infty} 8(-1)^n 16^n x^{4n+1}.$$

Thus, if $16x^4 < 1$,

$$\begin{aligned} f(x) - f(0) &= \int_0^x f'(t)\,dt = \int_0^x \sum_{n=0}^{\infty} 8(-1)^n 16^n t^{4n+1}\,dt \\ &= \sum_{n=0}^{\infty} 8(-1)^n 16^n \int_0^t t^{4n+1}\,dt \\ &= \sum_{n=0}^{\infty} 8(-1)^n 16^n \frac{x^{4n+2}}{4n+2}. \end{aligned}$$

Since $f(0) = \arctan(0) = 0$, it follows that if $16x^4 < 1$ and thus $-\frac{1}{2} < x < \frac{1}{2}$,

$$f(x) = \sum_{n=0}^{\infty} 4(-1)^n 16^n \frac{x^{4n+2}}{2n+1}.$$

Finally, note that if $x = -1/2$ then

$$\sum_{n=0}^{\infty} 4(-1)^n 16^n \frac{(-1/2)^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Since $\frac{1}{2n+1} \to 0$ and is positive and decreasing ($2(n+1)+1 = 2n+3 \geq 2n+1$), by the alternating series test, this series converges. If $x = 1/2$, then

$$\sum_{n=0}^{\infty} 4(-1)^n 16^n \frac{(1/2)^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

and by the previous remark this series converges as well.

Therefore, the interval of convergence of the above MacLaurin series is actually $[-1/2, 1/2]$. □