MINIMIZATION, MAXIMIZATION, AND LAGRANGE MULTIPLIER PROBLEMS

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Problems

1. *Find and classify all critical points of the function* $f(x, y) = (x^2 + y^2)e^{y^2 - x^2}$.
2. *Find and classify all critical points of the function* $f(x, y) = y \cos x$.
3. *Find the absolute minimum and absolute maximum value of the function* $f(x, y) = 2x^3 + y^4$ *on the region* $D = \left\{ (x, y) \ : \ x^2 + y^2 \leq 1 \right\}$.
4. *Find the absolute minimum and absolute maximum value of the function* $f(x, y) = x^4 + y^4 - 4xy + 2$ *on the region* $D = \{ (x, y) \ : \ 0 \leq x \leq 3, 0 \leq y \leq 2 \}$.
5. *Suppose that the length of the diagonal of a rectangular box is* L. *What is the maximum volume of the box?*
6. *Use Lagrange multipliers to find the minimum and maximum value of* $f(x, y) = e^{xy}$ *on the curve* $x^3 + y^3 = 16$.
7. *Use Lagrange multipliers to find the minimum and maximum value of* $f(x, y, z) = xyz$ *on the surface* $x^2 + 2y^2 + 3z^2 = 6$.

Solutions

1. *Find and classify all critical points of the function* $f(x, y) = (x^2 + y^2)e^{y^2 - x^2}$.

   Solution. The question asks us to find and classify all the critical points of f. We know that we get critical points when the "derivative is 0," which, in the context of 230, means that the gradient vector is identically equal to the zero vector:

   $$\nabla f(x, y) = \mathbf{0}$$

   Explicitly, we need the following two equations to be true *at the same time* to get a critical point:

   $$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0$$

   So let's calculate the partial derivatives of f, using the product rule:

   $$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x}(x^2 + y^2)e^{y^2 - x^2} \\ &= (2x)e^{y^2 - x^2} + (x^2 + y^2)e^{y^2 - x^2}(-2x) \end{aligned}$$

Similarly, for y,

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}(x^2+y^2)e^{y^2-x^2} \\ &= (2y)e^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(2y)\end{aligned}$$

So in order to find the critical points of f, we need to find all solutions to the following system of equations:

$$\begin{aligned}(2x)e^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(-2x) &= 0 \\ (2y)e^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(2y) &= 0\end{aligned}$$

This is where things get tricky. For systems of equations like this,[1] there is *no general process* for finding solutions; you kind of have to tinker around with stuff until you get what you want.

So consider the system of equations above. In particular, let's look at the first equation:

$$(2x)e^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(-2x) = 0$$

The first thing that stands out to me is that both terms have an $e^{y^2-x^2}$. Furthermore, e^{anything} is never 0. So I can divide both sides by $e^{y^2-x^2}$ and be done with it!

$$2x + (x^2+y^2)(-2x) = 0$$

I guess I can also divide both sides by 2, since 2 is never 0. With some slight rearrangement, this gives me:

$$x - x(x^2+y^2) = 0$$

Note that I *cannot* divide both sides by x, because x might be 0! What I can do is pull the x out front like this:

$$x\left[1-(x^2+y^2)\right] = 0$$

If I want this equation to be true, then I have two possibilities: either $x=0$, or $1-(x^2+y^2)=0$. We'll consider both of these cases separately.

First, suppose that $x=0$. What does this imply about y? Well, let's check out the second equation, plug in $x=0$, and see what happens:

$$(2y)e^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(2y) = 0$$

turns into

$$2ye^{y^2} + y^2e^{y^2}(2y) = 0$$

Again, e^{y^2} is never 0, so forget about it. I'll also divide both sides by 2:

$$y + y^2(y) = 0 \quad \Rightarrow \quad y + y^3 = 0 \quad \Rightarrow \quad y(1+y^2) = 0$$

[1]Unless the system of equations is linear, then there is a general process!

So either $y = 0$ or $1 + y^2 = 0$. But this second condition will never happen in the real numbers (the solutions of that are $y = \pm i$), so this means $y = 0$. So what we found out is that if $x = 0$, then $y = 0$. This gives us one of our critical points: $(0, 0)$.

Now we can consider the other case: $1 - (x^2 + y^2) = 0$, which means that $x^2 + y^2 = 1$. What will this imply? Again, we can consider the second equation, plug in $x^2 + y^2 = 1$, and see what happens:

$$(2y)e^{y^2-x^2} + (x^2 + y^2)e^{y^2-x^2}(2y) = 0$$

turns into

$$(2y)e^{y^2-x^2} + e^{y^2-x^2}(2y) = 0 \quad \Rightarrow \quad 4ye^{y^2-x^2} = 0$$

Once again, since $4e^{y^2-x^2} \neq 0$, we see that $y = 0$. So what this tells us is that if $x^2 + y^2 = 1$, then $y = 0$. But this means that $x^2 = 1$, which means that $x = -1$ or $x = 1$. This gives us two more critical points; $(-1, 0)$ and $(1, 0)$.

Sweet! Now we have all the critical points that we have to classify: $(-1, 0), (1, 0)$, and $(0, 0)$. In order to classify them, we use the second derivative test. We need to calculate the following matrix:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

To make the computations a little easier, I'm going to rewrite the partial derivatives by factoring some stuff out:

$$\frac{\partial f}{\partial x} = 2xe^{y^2-x^2}\left(1 - x^2 - y^2\right)$$
$$\frac{\partial f}{\partial y} = 2ye^{y^2-x^2}\left(1 + x^2 + y^2\right)$$

Using the product rule a couple times, we can calculate the second partial derivatives:

$$f_{xx} = \left[2e^{y^2-x^2} + 2xe^{y^2-x^2}(-2x)\right](1 - x^2 - y^2) + 2xe^{y^2-x^2}(-2x)$$
$$f_{yy} = \left[2e^{y^2-x^2} + 2ye^{y^2-x^2}(2y)\right](1 + x^2 + y^2) + 2ye^{y^2-x^2}(2y)$$
$$f_{xy} = f_{yx} = 2ye^{y^2-x^2}(-2x)(1 + x^2 + y^2) + 2ye^{y^2-x^2}(2x)$$

Next, let's check our critical point $(0, 0)$ by calculating the matrix above. Fortunately, by plugging in 0 for x and y, lots of stuff goes to 0:

$$\begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The determinant of this matrix is $(2)(2) - (0)(0) = 4$. Since $4 > 0$, by the second derivative test, $(0, 0)$ is either a local minimum or a local maximum. Since $f_{xx}(0, 0) = 2$ is positive, the function is concave up, and so $(0, 0)$ is a local minimum.

Let's do the same thing with the other critical points. If we plug in the point $(-1, 0)$, we get:

$$\begin{bmatrix} f_{xx}(-1,0) & f_{xy}(-1,0) \\ f_{yx}(-1,0) & f_{yy}(-1,0) \end{bmatrix} = \begin{bmatrix} -4e^{-1} & 0 \\ 0 & 4e^{-1} \end{bmatrix}$$

Again, plugging in the number isn't as bad as it seems, because lots of stuff goes to 0. The determinant of that matrix is $-16e^{-2}$, which is negative. This means $(-1, 0)$ is a saddle point.

If we plug in the point $(1, 0)$, we get:

$$\begin{bmatrix} f_{xx}(1,0) & f_{xy}(1,0) \\ f_{yx}(1,0) & f_{yy}(1,0) \end{bmatrix} = \begin{bmatrix} -4e^{-1} & 0 \\ 0 & 4e^{-1} \end{bmatrix}$$

which is the same matrix as before. Thus, $(1, 0)$ is also a saddle point. And we're done! □

2. *Find and classify all critical points of the function $f(x, y) = y \cos x$.*

Solution. As in the previous problem, in order to find all the critical points, we have to solve the following system of equations:

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0$$

Let's calculate the partial derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(y \cos x) = -y \sin x \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \cos x) = \cos x \end{aligned}$$

So, after dividing by -1, we need to solve the system of equations:

$$\begin{aligned} y \sin x &= 0 \\ \cos x &= 0 \end{aligned}$$

Since the second equation is simpler, let's look at that one first. In order for $\cos x$ to be 0, x could be $\frac{\pi}{2}$, $\frac{3\pi}{2}$, ... etc. In fact $x = \frac{(2k+1)\pi}{2}$ would work for any integer k. What does this imply about y? Well, let's look at the first equation:

$$y \sin x = 0$$

Note that if $x = \frac{(2k+1)\pi}{2}$, for any integer k, $\sin x$ is never 0. It's always going to be 1 or -1, since $\cos x$ and $\sin x$ are never 0 at the same time. This means that we have to have $y = 0$. So we have a whole bunch of critical points:

$$\ldots, \left(\frac{-3\pi}{2}, 0\right), \left(\frac{-\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right), \ldots$$

Any point of the form $\left(\frac{(2k+1)\pi}{2}, 0\right)$ is a critical point. That seems daunting, but maybe we can classify all of them using the second derivative test.

So we need to calculate the matrix of second derivatives. Fortunately, this is a little easier than before:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -y\cos x & -\sin x \\ -\sin x & 0 \end{bmatrix}$$

The determinant of this matrix is $0 - (\sin x)^2 = -\sin^2 x$. Remember how I mentioned that $\sin x$ will either be -1 or 1 if $x = \frac{(2k+1)\pi}{2}$? This means that the determinant of the second derivative matrix will always be $-(\pm 1)^2 = -1$. This is negative, which means that every critical point is a saddle point. Done! □

3. *Find the absolute minimum and absolute maximum value of the function* $f(x, y) = 2x^3 + y^4$ *on the region* $D = \{ (x, y) \ : \ x^2 + y^2 \leq 1 \}$.

Solution. In order to find the absolute maximum and absolute minimum values of a function on a bounded domain D, we not only have to find and consider all critical points, but we have to check along the boundaries of the region.

We'll worry about the boundary in a minute. First, let's do what we did in the first couple of problems and find all the critical points. The parital derivatives of f are:

$$f_x(x, y) = 6x^2$$
$$f_y(x, y) = 4y^3$$

To find critical points, we need to solve the system of equations

$$6x^2 = 0$$
$$4y^3 = 0$$

This system of equations is pretty easy to solve. From the first equation, we know that $x = 0$ and from the second equation we know that $y = 0$. Hence, the only critical point we have is $(0, 0)$. The one thing we have to check is to see if $(0, 0)$ is in our region D, because if it isn't then we don't care about it. But since $0^2 + 0^2 \leq 1$, $(0, 0)$ is in our region, so it's a point we care about.

Next, we have to check the boundary of our region. In general, the way to handle this is to find the equation of the curve that defines the boundary, and somehow use this equation to convert the original function $f(x, y)$ into a function of one variable. Having done that, we can do a single variable min/max problem with this new function.

Here, it's a little tricky: the boundary of the region D is achieved by making the inequality in $x^2 + y^2 \leq 1$ an equality:

$$x^2 + y^2 = 1$$

So our boundary is a circle of radius 1. It's not clear how we can use the equation $x^2 + y^2 = 1$ to turn the function $f(x, y) = 2x^3 + y^4$ into a function of one variable, though. Here's the deal with circles: parametrize!

$$x(t) = \cos t$$
$$y(t) = \sin t$$

where $0 \le t \le 2\pi$. Now we can plug in our expressions for x and y into the original function and get a function solely in terms of t:

$$f(t) = f(x(t), y(t)) = 2\cos^3 t + \sin^4 t$$

We can find the minimum and maximum values of $f(t)$ by using single variable calculus. The derivative of f with respect to t is:

$$f'(t) = -6\cos^2 t \sin t + 4\sin^3 t \cos t$$

Now we just have to find any point where $f'(t) = 0$. There are a couple possibilities to consider. If $\sin t = 0$, then $f'(t) = 0$. Similarly, if $\cos t = 0$, then $f'(t) = 0$. That means that t could be $0, \frac{\pi}{2}, \pi$, or $\frac{3\pi}{2}$. Why didn't I include 2π? It'll be the same as plugging in 0! Are there any other points? Well, suppose that both $\sin t$ and $\cos t$ are nonzero. Then we can divide both sides by $\sin t$ and $\cos t$ to get

$$-6\cos t + 4\sin^2 t = 0$$

There are probably a number of ways to solve this, but the first thing I thought of doing is replacing $\sin^2 t$ with $1 - \cos^2 t$. After dividing by 2, this gives us

$$-3\cos t + 2 - 2\cos^2 t = 0 \quad \Rightarrow \quad 2(\cos t)^2 + 3(\cos t) - 2 = 0$$

This is a quadratic equation in $\cos t$, so we can use the quadratic formula to solve for $\cos t$:

$$\cos t = \frac{-3 \pm \sqrt{9 - 4(2)(-2)}}{2} = \frac{3 \pm \sqrt{25}}{2} = -1, 4$$

But $\cos t$ can never equal 4, so the only possibility is that $\cos t = -1$. But that only happens when $t = \pi$, and we've already accounted for that. So, for the boundary, we only have to evaluate $f(t)$ when $t = 0, \frac{\pi}{2}, \pi$, or $\frac{3\pi}{2}$. If we also compute $f(0, 0)$, which was our only interior critical point, we get:

$$f(x = 0, y = 0) = 0$$
$$f(t = 0) = 2$$
$$f\left(t = \frac{\pi}{2}\right) = 1$$
$$f(t = \pi) = -2$$
$$f\left(t = \frac{3\pi}{2}\right) = 1$$

The smallest value in that list is -2, and the biggest value is 2. Therefore, the absolute minimum value of f is -2, and the absolute maximum value of f is 2. □

4. *Find the absolute minimum and absolute maximum value of the function* $f(x, y) = x^4 + y^4 - 4xy + 2$ *on the region* $D = \{ (x, y) \ : \ 0 \le x \le 3, 0 \le y \le 2 \}$.

Solution. We proceed as before. First, we have to find all possible interior critical points. After doing that, we have to find any critical points on the boundaries. We'll check the function on all of those points and get the absolute min and max from there.

Let's do the interior critical points first. As usual, we need to find points where both partial derivatives of f are 0. Computing gives us:

$$f_x(x, y) = 4x^3 - 4y$$
$$f_y(x, y) = 4y^3 - 4x$$

Any critical point will satisfy the system of equations:

$$4x^3 - 4y = 0$$
$$4y^3 - 4x = 0$$

Consider the first equation. We can solve for y in terms of x to get

$$y = x^3$$

Take this expression and plug it into the second equation. After dividing by 4, this gives us:

$$(x^3)^3 - x = 0 \quad \Rightarrow \quad x^9 - x = 0$$

Let's factor the equation as follows:

$$x\left(x^8 - 1\right) = 0$$

So either $x = 0$, or $x^8 - 1 = 0$. This gives us three possibilities in total: $x = 0$, $x = 1$, $x = -1$.

If $x = 0$, then since $y = x^3$, $y = 0$. This gives us the critical point $(0, 0)$.

If $x = -1$, then $y = (-1)^3 = -1$. This gives us the critical point $(-1, -1)$.

If $x = 1$, then $y = 1^3 = 1$, which gives us the point $(1, 1)$.

But notice that our region is defined by $0 \le x \le 3$ and $0 \le y \le 2$. The point $(-1, -1)$ is not in our domain! So forget about it. We get two critical points that we care about: $(0, 0)$ and $(1, 1)$.

Next, we have to check along the boundaries. We can describe the boundary curves of our region by making all the inequalities equalities in the description of D. This gives us four curves:

$$x = 0 \quad x = 3 \quad y = 0 \quad y = 3$$

We have to find any interesting points on all of these curves.

Let's start with $x = 0$: when $x = 0$, the function f becomes:

$$f(y) = f(0, y) = 0^4 + y^4 - 4(0)y + 2 = y^4 + 2$$

This is a function of one variable, so we can minimize/maximize it using single variable calculus techniques. Computing the derivative with respect to y and setting it equal to 0 gives us:

$$4y^3 = 0$$

This means that $y = 0$, which gives us the point $(0, 0)$. We already had this point from earlier, but that's okay. The other thing we need to remember to check is the endpoints of the interval: since $0 \leq y \leq 2$, we need to check when $y = 0$ and $y = 2$. When $y = 0$, we get the point $(0, 0)$ (again), and when $y = 2$, we get the point $(0, 2)$. So far, we have collected the following points:

$$(0, 0), (1, 1), (0, 2)$$

Let's keep going to see if we get anymore.

Next, we'll look at the boundary curve $x = 3$. On this curve the function turns into

$$f(y) = f(3, y) = 3^4 + y^4 - 4(3)y + 2 = y^4 - 12y + 83$$

To find any critical points on this boundary, take the derivative and set it equal to 0:

$$f'(y) = 4y^3 - 12 = 0$$

So $y = \sqrt[3]{3}$, which gives us the point $(3, \sqrt[3]{3})$ (which *is* in our domain, since $\sqrt[3]{3}$ is between 0 and 2). Again, we also have to check at the ends of the intervals $0 \leq y \leq 2$. When $y = 0$ and $y = 2$, we get the points $(3, 0)$ and $(3, 2)$. Now our list is:

$$(0, 0), (1, 1), (0, 2), (3, \sqrt[3]{3}), (3, 0), (3, 2)$$

Next, we'll check $y = 0$. The function f becomes:

$$f(x) = f(x, 0) = x^4 + 0^4 - 4x(0) + 2 = x^4 + 2$$

Taking the derivative and setting it equal to 0 gives us $4x^3 = 0$, and so $x = 0$. This gives us the point $(0, 0)$, again. Since $0 \leq x \leq 3$, we have to check when $x = 0$ and when $x = 3$, but if you notice, we've already checked all of those! So we don't get any new points that we didn't already have by checking $y = 0$.

The last boundary curve to check is $y = 2$. Then

$$f(x) = f(x, 2) = x^4 - 8x + 2$$

Setting the derivative equal to zero yields $4x^3 - 8 = 0$, which mean $x = \sqrt[3]{2}$. This gives us the point $(\sqrt[3]{2}, 2)$, which is in our domain. Checking when $x = 0$ and $x = 3$ gives us points we already have. Thus, our final list is:

$$(0,0), (1,1), (0,2), (3, \sqrt[3]{3}), (3,0), (3,2), (\sqrt[3]{2}, 2)$$

All we have to do now is plug all of these values into the function f and pick out the biggest and smallest:

$$\begin{aligned} f(0,0) &= 2 \\ f(1,1) &= 0 \\ f(0,2) &= 18 \\ f(3, \sqrt[3]{3}) &= 83 - 9\sqrt[3]{3} \approx 70 \\ f(3,0) &= 83 \\ f(3,2) &= 75 \\ f(\sqrt[3]{2}, 2) &= 18 - 6\sqrt[3]{2} \approx 10.5 \end{aligned}$$

Given that I calculated all of that without making any mistakes, we can see that the smallest value f takes on is 0, and the biggest value is 83. □

5. *Suppose that the length of the diagonal of a rectangular box is L. What is the maximum volume of the box?*

Solution. Here is your goal for any word problem: strip away all the words and nonsense and translate the question into a mathematical statement. For this question, they tell us that we have a rectangular box. Since they don't say anything else, we can assume without loss of generality that one corner of the box is sitting at the origin. Let's define the length of the sides of the rectangular box by x, y, and z. They tell us that the main diagonal is of length L. In other words, the distance from one corner to the opposite corner (the origin) is L. Algebraically, this means that

$$\sqrt{x^2 + y^2 + z^2} = L$$

You can think about this either in terms of the distance formula, or in terms of the Pythagorean theorem (same thing, really). What the problem wants us to do is find the maximum possible volume. The volume of the box is given by xyz, so we can think of volume as a function:

$$V(x,y,z) = xyz$$

Given all of this, we can rephrase the question as follows: *maximize the function $V(x,y,z) = xyz$ given $\sqrt{x^2 + y^2 + z^2} = L$.*

We could do this with Lagrange multipliers, but what I'll do here is maximize the regular way that we've been doing above. In order to do this, we need to turn the function V into a function of 2 variables, since we don't know how to handle three variable functions. We can do this by solving our constraint equation for one of the variables and plugging in into our function. It doesn't really matter which variable we pick, so let's do z:

$$\sqrt{x^2+y^2+z^2} = L \quad \Rightarrow \quad z = \sqrt{L^2 - x^2 - y^2}$$

Our function V then becomes:

$$V(x,y,z) = xyz \quad \Rightarrow \quad V(x,y) = xy\sqrt{L^2 - x^2 - y^2}$$

What we have to do is find the maximum value of that function.

Taking the partial derivatives of that thing with the square root will be pretty annoying, so one trick that we can employ is to maximize volume-squared. The reason we can do this is because squaring the volume doesn't change *where* the maximum occurs; it just changes what that maximum value is. So for the purposes of finding critical points, let's consider the function

$$V^2(x,y) = (V(x,y))^2 = \left(xy\sqrt{L^2 - x^2 - y^2}\right)^2 = x^2y^2\left(L^2 - x^2 - y^2\right) = L^2x^2y^2 - x^4y^2 - x^2y^4$$

To find the critical points of this function, we compute the partial derivatives with respect to x and y and set them equal to 0:

$$2L^2xy^2 - 4x^3y^2 - 2xy^4 = 0$$
$$2L^2x^2y - 2x^4y - 4x^2y^3 = 0$$

Because this is a word problem, we can assume things from context. We know that x and y will both be nonzero, because if either of them equaled 0 then our volume would be 0 and that's stupid. So what we can do is divide everything in the first equation by xy^2 and divide everything in the second equation by x^2y. We can also divide by 2 across the board. This gives us:

$$L^2 - 2x^2 - y^2 = 0$$
$$L^2 - x^2 - 2y^2 = 0$$

Probably the best thing to do from here is to solve for y^2 in the first equation, and plug that into the second equation.

$$L^2 - 2x^2 - y^2 = 0 \quad \Rightarrow \quad y^2 = L^2 - 2x^2$$

which gives us

$$L^2 - x^2 - 2(L^2 - 2x^2) = 0 \quad \Rightarrow \quad -L^2 + 3x^2 = 0$$

Solving for x gives us $x = \frac{L}{\sqrt{3}}$. Since $y^2 = L^2 - 2x^2$, this means that

$$y^2 = L^2 - 2\left(\frac{L}{\sqrt{3}}\right)^2 = \frac{L^2}{3}$$

and so $y = \frac{L}{\sqrt{3}}$. Cool! So we know that the maximum volume occurs when $(x, y) = \left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right)$. Why didn't we have to do the second derivative test? Well we only got one critical point, and by the construction of the problem, we know this has to be the max! But if you want to, you could always use the second derivative test to verify that it is indeed a local maximum.

Let's compute the volume:

$$\begin{aligned} V\left(\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}\right) &= \left(\frac{L}{\sqrt{3}}\right)\left(\frac{L}{\sqrt{3}}\right)\sqrt{L^2 - \left(\frac{L}{\sqrt{3}}\right)^2 - \left(\frac{L}{\sqrt{3}}\right)^2} \\ &= \frac{L^2}{3}\sqrt{\frac{L^2}{3}} \\ &= \frac{L^3}{3\sqrt{3}} \end{aligned}$$

□

6. *Use Lagrange multipliers to find the minimum and maximum value of* $f(x, y) = e^{xy}$ *on the curve* $x^3 + y^3 = 16$.

Solution. The principle of Lagrange multipliers says that if we are trying maximize/minimize a function $f(x, y)$ given a constraint $g(x, y) = c$, then interesting things will happen when

$$\nabla f = \lambda \nabla g$$

for some number λ. For us, we are trying to maximize and minimize $f(x, y) = e^{xy}$ given the constraint that $x^3 + y^3 = 16$. So $f(x, y) = e^{xy}$, $g(x, y) = x^3 + y^3$, and $c = 16$. Let's compute the gradients of f and g:

$$\begin{aligned} \nabla f(x, y) &= \langle f_x,\, f_y\rangle = \langle ye^{xy},\, xe^{xy}\rangle \\ \nabla g(x, y) &= \langle g_x,\, g_y\rangle = \left\langle 3x^2,\, 3y^2\right\rangle \end{aligned}$$

So we need to solve the following system of equations:

$$\begin{aligned} ye^{xy} &= \lambda 3x^2 \\ xe^{xy} &= \lambda 3y^2 \end{aligned}$$

I mentioned this in problem 1, but this is where things get tricky; there just isn't a general process for solving these things. Here's my thought process:

The first thing I want to consider is the possibility that x or y, or both, are 0. The reason I want to do this first is because if I consider those cases, then I can assume they're *not* zero and start dividing things. So let's suppose that $x = 0$ and we'll see what happens. In the first equation, if $x = 0$, then $ye^0 = 0$, and so $y = 0$ as well. Let's check to see if $y = x = 0$ works in the second equation: if x and y are both 0, then we get $0 = 0$, and so the point $(0, 0)$ is definitely a possibility. But wait! I just realized that it's actually *not* a possibility.[2] Why? There is a third equation in our system of equations that I forgot to write down - our original constraint equation:

$$x^3 + y^3 = 16$$

If $x = y = 0$, then we get $0 = 16$, which is not true. So $(0, 0)$ does not work.

Now that we took care of the trivial case (or rather proved that there is no trivial case), we can assume that both x and y are nonzero. The first thing I think of trying is solving for λ in both equations and setting them equal to each other:

$$\begin{aligned} ye^{xy} = \lambda 3x^2 \quad &\Rightarrow \quad \lambda = \frac{ye^{xy}}{3x^2} \\ xe^{xy} = \lambda 3y^2 \quad &\Rightarrow \quad \lambda = \frac{xe^{xy}}{3y^2} \end{aligned}$$

This gives us:

$$\frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2}$$

After cross multiplying, we can divide by 3 on both sides, and - since e^{xy} is never 0 - we can divide by that as well. Our equation turns into:

$$y^3 = x^3$$

Now, we can plug this into our constraint equation:

$$x^3 + y^3 = 16 \quad \Rightarrow \quad x^3 + x^3 = 16 \quad \Rightarrow \quad x^3 = 8$$

and so $x = 2$. This implies that $y = 2$. And that's it! The only critical point we have is $(2, 2)$. Therefore, the maximum value of the function f on $x^3 + y^3 = 16$ is:

$$f(2, 2) = e^4$$

Lesson learned: use the constraint equation to your advantage when solving Lagrange multiplier problems. It's every bit as important as your λ equations. □

7. *Use Lagrange multipliers to find the minimum and maximum value of $f(x, y, z) = xyz$ on the surface $x^2 + 2y^2 + 3z^2 = 6$.*

   Solution. With three variables, the principle of Lagrange multipliers is the same, just with an extra equation. We know that interesting things will happen when

$$\nabla f = \lambda \nabla g$$

[2] You really are reading my mind work in real time

where $f(x, y, z) = xyz$ is the function we're trying to maximize, and $g(x, y, z) = x^2 + 2y^2 + 3z^2$ is part of our constraint equation. Let's compute the gradients:

$$\nabla f(x, y, z) = \langle f_x,\, f_y,\, f_z \rangle = \langle yz,\, xz,\, xy \rangle$$
$$\nabla g(x, y, z) = \langle g_x,\, g_y,\, g_z \rangle = \langle 2x,\, 4y,\, 6z \rangle$$

Therefore, we get a system of equations:

$$yz = \lambda 2x$$
$$xz = \lambda 4y$$
$$xy = \lambda 6z$$
$$x^2 + 2y^2 + 3z^2 = 6$$

Observe the following: we know that x, y, and z cannot *all* be 0. Otherwise, our constraint equation would give us $0 = 6$, which is false. We now consider all the other possibilities.

First, we'll consider the possibilities where only one of the variables is nonzero.

x nonzero, y zero, z zero: Suppose that $x \neq 0$. If both y and z are 0, then we can just use the constraint equation and solve for x:

$$x^2 + 2(0)^2 + 3(0)^2 = 6$$

and so $x = \pm\sqrt{6}$. This gives us two points: $(\sqrt{6}, 0, 0)$ and $(-\sqrt{6}, 0, 0)$.

x zero, y nonzero, z zero: Suppose that $y \neq 0$ and both x and z are 0. Then we can just use the constraint equation and solve for y:

$$(0)^2 + 2y^2 + 3(0)^2 = 6$$

and so $y = \pm\sqrt{3}$. This gives us two points: $(0, \sqrt{3}, 0)$ and $(0, -\sqrt{3}, 0)$.

x zero, y zero, z nonzero: Suppose that $z \neq 0$ and both x and y are 0. Then we can just use the constraint equation and solve for z:

$$(0)^2 + 2(0)^2 + 3z^2 = 6$$

and so $z = \pm\sqrt{2}$. This gives us two points: $(0, 0, \sqrt{2})$ and $(0, 0, -\sqrt{2})$.

Next, we consider the case when two of the variables are nonzero and one of them is zero.

x nonzero, y nonzero, z zero: Here, we might try to solve for λ somehow. Since $x \neq 0$,

$$yz = \lambda 2x \quad \Rightarrow \quad \lambda = \frac{yz}{2x}$$

Plugging this into the equation with xy on the left gives us:

$$xy = \lambda 6z \quad \Rightarrow \quad xy = \frac{3yz^2}{x}$$

Since y is nonzero, we can divide both sides by y. Then moving the x on the bottom to the left gives us $x^2 = 3z^2$. But we're assuming that $z = 0$, which means that $x = 0$. This is a contradiction! So we can't have $z = 0$ if both x and y are nonzero. By the symmetry of all these equations, it is easy to see that in general, it is not possible to have two nonzero variables and one zero variable.

All that's left is to consider the case when all three are nonzero.

x nonzero, y nonzero, z nonzero: We'll proceed as we did in the previous part by trying to solve for λ. Since $x \neq 0$,

$$yz = \lambda 2x \quad \Rightarrow \quad \lambda = \frac{yz}{2x}$$

Plugging this into the equation with xy on the left gives us:

$$xy = \lambda 6z \quad \Rightarrow \quad xy = \frac{3yz^2}{x}$$

Since y is nonzero, we can divide both sides by y. Then, moving the x on the bottom to the left gives us $x^2 = 3z^2$. Similarly, plugging the λ expression into the equation with xz gives us:

$$xz = \lambda 4y \quad \Rightarrow \quad xz = \frac{2y^2 z}{x}$$

Since $z \neq 0$, we get $x^2 = 2y^2$. We can use this new information (that $x^2 = 2y^2$ and $x^2 = 3z^2$) to change our constraint equation as follows:

$$x^2 + 2y^2 + 3z^2 = 6 \quad \Rightarrow \quad x^2 + x^2 + x^2 = 6$$

So $3x^2 = 6$, which means that $x = \pm\sqrt{2}$. From this, we get that $(\pm\sqrt{2})^2 = 2y^2$, and so $y = \pm 1$. Similarly, $(\pm\sqrt{2})^2 = 3z^2$, and so $z = \pm\sqrt{\frac{2}{3}}$. From all of this, we get eight critical points to consider:

$$\left(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}}\right)$$

This brings our total list of points to check to:

$$(\pm\sqrt{6}, 0, 0), (0, \pm\sqrt{3}, 0), (0, 0, \pm\sqrt{2}), \left(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}}\right)$$

That's fourteen different points to check! Before you freak out, let's remember what our function is: $f(x, y, z) = xyz$. If we evaluate f at $(\pm\sqrt{6}, 0, 0), (0, \pm\sqrt{3}, 0)$, or $(0, 0, \pm\sqrt{2})$, we'll get 0. So now we've taken care of those points. The only other points left are of the form $\left(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}}\right)$. Even though there are eight different points like that, the only difference between any of them is the sign. So if we evaluate f at any of those points, we'll get

$$f\left(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}}\right) = \pm\sqrt{2}(1)\sqrt{\frac{2}{3}} = \pm\frac{2}{\sqrt{3}}$$

where the sign depends on how many negative signs appear in the point $\left(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}}\right)$. If there are 1 or 3 negative signs, then f will evaluate to $-\frac{2}{\sqrt{3}}$. If there are 0 or 2 negative signs, then f will evaluate to $\frac{2}{\sqrt{3}}$. And that's it! Our possible extreme values for f are 0 and $\pm\frac{2}{\sqrt{3}}$. Therefore, the maximum value of f on the surface $x^2 + x^2 + x^2 = 6$ is $\frac{2}{\sqrt{3}}$, and the minimum value is $-\frac{2}{\sqrt{3}}$.

Here's the moral of this problem: one tactic that you'll use in solving systems of equations in Lagrange multiplier problems is to routinely consider all possibilities for the variables being zero/nonzero. You'll be able to account for some special cases when many of the variables are 0, and work your way to the more complicated situations as you assume that everything is nonzero.

$\square$