Midterm 2 Solutions

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1. **True or false?**

   (a) $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ **holds for all** r **in** $[-2, 2]$**.**

   (b) $\sum_{n=10}^{\infty} \frac{1000}{n^n}$ **converges.**

   (c) $\sum_{n=2}^{\infty} \frac{3^n}{n!} = e^3 - 1$**.**

   (d) $\frac{1}{4+x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^{n+1}}$**.**

   (e) Let T_3 **be the** 3**rd Taylor polynomial of** $f(x) = \cos x$ **centered at** 0**. Then** $|\cos(0.1) - T_3(0.1)| > \frac{1}{4!}(0.1)^4$**.**

   Solution.

   (a) False. The interval of convergence of the power series $\sum_{n=0}^{\infty} r^n$ is $(-1, 1)$.

   (b) True. Note that
   $$\left|\frac{1000}{n^n}\right|^{\frac{1}{n}} = \frac{1000^{\frac{1}{n}}}{n} \to 0.$$
   Since $0 < 1$, by the root test, this series converges.

   (c) False. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x, so $e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!}$. Next, note that
   $$\sum_{n=0}^{\infty} \frac{3^n}{n!} = 1 + 3 + \sum_{n=2}^{\infty} \frac{3^n}{n!}.$$
   So
   $$e^3 - 1 = 3 + \sum_{n=2}^{\infty} \frac{3^n}{n!} \neq \sum_{n=2}^{\infty} \frac{3^n}{n!}$$

   (d) True. Note that (for $|x| < 2$)
   $$\frac{1}{4+x^2} = \frac{1}{4} \cdot \frac{1}{1 - \left(-\frac{x^2}{4}\right)} = \frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^{n+1}}.$$

   (e) False. The error bound theorem with $K = 1$ implies that $|\cos(0.1) - T_3(0.1)| \leq \frac{1}{4!}(0.1)^4$.

   □

2. **Determine whether the following infinite series converge absolutely, conditionally, or not at all.**

   (a) $\sum (-1)^n \cos(1/n)$

   (b) $\sum (-1)^n \frac{n^3}{5^n}$

   (c) $\sum (-1)^n \frac{2^n n!}{n^n}$

   Solution.

   (a) Note that $\cos(1/n) \to \cos(0) = 1$ as $n \to \infty$. Thus, the limit of the sequence $(-1)^n \cos(1/n)$ does not exist. In particular, $(-1)^n \cos(1/n)$ does not approach 0. By the divergence test, this series diverges.

(b) Note that

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{(-1)^n n^3}\right| = \lim_{n\to\infty}\left|\frac{-1}{5}\cdot\frac{(n+1)^3}{n^3}\right|$$
$$= \frac{1}{5}.$$

Since $\frac{1}{5} < 1$, by the ratio test, the series converges absolutely.

(c) Note that

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{(-1)^n 2^n n!}\right| = \lim_{n\to\infty} 2\cdot\frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}}$$
$$= \lim_{n\to\infty} 2(n+1)\cdot\frac{n^n}{(n+1)(n+1)^n}$$
$$= \lim_{n\to\infty} 2\cdot\left(\frac{n}{n+1}\right)^n$$
$$= \lim_{n\to\infty}\frac{2}{\left(\frac{n+1}{n}\right)^n}$$
$$= \lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^n}.$$

By the hint, $\left(1+\frac{1}{n}\right)^n \to e$. Thus

$$\lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e}.$$

Since $\frac{2}{e} < 1$, by the ratio test, the series converges absolutely.

□

3. **Find the interval of convergence of the following power series.**

$$\sum_{n=0}^{\infty}\frac{(x-4)^n}{3^n n^{\frac{1}{2}}}.$$

Solution. We first find the radius of convergence and the center of the interval by applying the ratio test. Observe that

$$\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{3^{n+1}(n+1)^{\frac{1}{2}}}\cdot\frac{3^n n^{\frac{1}{2}}}{(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{x-4}{3}\frac{n^{\frac{1}{2}}}{(n+1)^{\frac{1}{2}}}\right|$$
$$= \frac{|x-4|}{3}.$$

By the ratio test, the power series converges if $\frac{|x-4|}{3} < 1$ and diverges if $\frac{|x-4|}{3} > 1$. Since $|x-4| < 3$ is equivalent to $-3 < x-4 < 3$ and hence $1 < x < 7$, the power series converges in $(1,7)$. We still don't know about the endpoints, so we check them individually.

When $x = 1$, the power series is

$$\sum_{n=0}^{\infty}\frac{(-3)^n}{3^n n^{\frac{1}{2}}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^{\frac{1}{2}}}.$$

Since $\frac{1}{n^{\frac{1}{2}}} \to 0$ and is a decreasing sequence, this series converges by the alternating series test.

When $x = 7$, the power series is

$$\sum_{n=0}^{\infty} \frac{(3)^n}{3^n n^{\frac{1}{2}}} = \sum_{n=0}^{\infty} \frac{1}{n^{\frac{1}{2}}}.$$

This series diverges by the p-test, since $\frac{1}{2} < 1$.

Therefore, the interval of convergence is $[1, 7)$. □

4. **(a) Let $g(x) = x^3 e^x$. Compute $g^{(21)}(0)$, using the MacLaurin series of g.**

   (b) Let $f(x) = x^{-\frac{1}{3}}$. Compute $T_3(x)$ with center $a = 8$.

Solution.

(a) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Thus,

$$g(x) = x^3 e^x = x^3 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!}.$$

When $n = 18$, $n + 3 = 21$. Thus, the coefficient in front of $x^2 1$ in the above power series is $\frac{1}{18!}$. By the formula for a MacLaurin series, this implies

$$\frac{1}{18!} = \frac{g^{(21)}(0)}{21!}.$$

Therefore,

$$g^{(21)}(0) = \frac{21!}{18!}.$$

(b) The formula for $T_3(x)$ with center $a = 8$ is:

$$T_3(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2 + \frac{f'''(8)}{6}(x-8)^3.$$

We compute:

$$\begin{aligned}
f(x) &= x^{-\frac{1}{3}} \quad \Rightarrow \quad f(8) = 8^{-\frac{1}{3}} \\
f'(x) &= -\frac{1}{3}x^{-\frac{4}{3}} \quad \Rightarrow \quad f'(8) = -\frac{1}{3}8^{-\frac{4}{3}} \\
f''(x) &= \frac{4}{3^2}x^{-\frac{7}{3}} \quad \Rightarrow \quad f''(8) = \frac{4}{3^2}8^{-\frac{7}{3}} \\
f'''(x) &= -\frac{4 \cdot 7}{3^3}x^{-\frac{10}{3}} \quad \Rightarrow \quad f''(8) = -\frac{4 \cdot 7}{3^3}8^{-\frac{10}{3}}.
\end{aligned}$$

Therefore,

$$T_3(x) = 8^{-\frac{1}{3}} + -\frac{1}{3}8^{-\frac{4}{3}}(x-8) + \frac{\frac{4}{3^2}8^{-\frac{7}{3}}}{2}(x-8)^2 + \frac{-\frac{4\cdot 7}{3^3}8^{-\frac{10}{3}}}{6}(x-8)^3.$$

□