MIDTERM 1 SOLUTIONS

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1. **True or false?**

   (a) $\sum \frac{1}{n^{7/8}}$ **converges.**

   (b) $\sum_{n=3}^{\infty} \frac{1}{n(n+1)} = 1$**.**

   (c) $\sum_{n=1}^{\infty} (\frac{2}{5})^n = \frac{2}{3}$

   (d) **Let** a_n **and** b_n **be sequences such that** $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ **and** $\sum a_n$ **converges. Then** $\sum b_n$ **converges.**

   (e) $\sum (-1)^n \frac{3}{n^{1/5}}$ **converges.**

Solution.

(a) False.

(b) False. Note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus

$$S_N = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) = \frac{1}{3} - \frac{1}{N+1}.$$

So $S_N \to \frac{1}{3}$, hence $\sum_{n=3}^{\infty} \frac{1}{n(n+1)} = \frac{1}{3}$.

(c) True. By the geometric series formula,

$$\sum_{n=1}^{\infty} (\frac{2}{5})^n = \frac{2/5}{1-2/5} = \frac{2/5}{3/5} = \frac{2}{3}.$$

(d) False. Take $a_n = 0$ and $b_n = 1$.

(e) True. Follows from the alternating series test.

□

2. **Compute the derivatives in (a) and (b), and the anti-derivatives in (c) and (d).**

   (a) $x^{x \ln x}$

   (b) $\cos(x^{1+\sin x})$

   (c) $\int x^x \ln x + x^x \, dx$

   (d) $\int \frac{1}{x \ln x} + \frac{1}{x} \, dx$

Solution.

(a) There are variables in the exponent and base, so we need to introduce the logarithm. Let $f(x) = x^{x \ln x}$. Then

$$\ln(f(x)) = \ln(x^{x \ln x}) = x \ln(x) \cdot \ln(x) = x \ln(x)^2.$$

Taking the derivative of both sides gives

$$\frac{1}{f(x)} \cdot f'(x) = \ln(x)^2 + 2x \ln(x) \cdot \frac{1}{x} = \ln(x)^2 + 2 \ln(x).$$

Thus,

$$f'(x) = f(x) \cdot \left(\ln(x)^2 + 2\ln(x)\right) = x^{x \ln x} \left(\ln(x)^2 + 2 \ln(x)\right).$$

(b) By the chain rule,

$$\frac{d}{dx} \cos(x^{1+\sin x}) = \sin(x^{1+\sin x}) \cdot \frac{d}{dx}\left(x^{1+\sin x}\right).$$

So it remains to compute $\frac{d}{dx}\left(x^{1+\sin x}\right)$. Let $g(x) = x^{1+\sin x}$. As above, we need to use the logarithm to compute the derivative. We have

$$\ln(g(x)) = (1 + \sin x) \ln x$$

so

$$\frac{g'(x)}{g(x)} = \cos x \ln x + \frac{1 + \sin x}{x}$$

and thus

$$g'(x) = x^{1+\sin x} \left(\cos x \ln x + \frac{1 + \sin x}{x}\right).$$

Putting this all together, we have

$$\frac{d}{dx} \cos(x^{1+\sin x}) = \sin(x^{1+\sin x}) \cdot \frac{d}{dx}\left(x^{1+\sin x}\right) = \sin(x^{1+\sin x}) \cdot x^{1+\sin x} \left(\cos x \ln x + \frac{1 + \sin x}{x}\right)$$

(c) Note that

$$x^x \ln x + x^x = x^x(\ln x + 1) = e^{x \ln x}(\ln x + 1)$$

so that

$$\int x^x \ln x + x^x \, dx = \int e^{x \ln x}(\ln x + 1) \, dx.$$

Let $u = x \ln x$. Then $du = (\ln x + 1) \, dx$. Therefore,

$$\int e^{x \ln x}(\ln x + 1) \, dx = \int e^u \, du = e^u + C = e^{x \ln x} + C = x^x + C.$$

(d) We have

$$\int \frac{1}{x \ln x} + \frac{1}{x} \, dx = \int \frac{1}{x \ln x} \, dx + \int \frac{1}{x} \, dx = \frac{1}{x \ln x} \, dx + \ln |x| + C.$$

For the first integral, let $u = \ln x$. Then $du = \frac{1}{x} \, dx$ and so

$$\frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\ln x| + C.$$

Thus,

$$\int \frac{1}{x \ln x} + \frac{1}{x} \, dx = \ln |\ln x| + \ln |x| + C.$$

□

3. **Determine whether the following infinite series converge or not.**

   (a) $\sum \ln\left(\frac{n}{n+1}\right)$

   (b) $\sum \frac{(\ln(n))^2}{n^{\frac{3}{2}}}$

Solution.

(a) First, note that $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$. Computing some partial sums, we have

$$\begin{aligned}
S_1 &= \ln(1) - \ln(2) = -\ln(2).\\
S_2 &= S_1 + \ln(2) - \ln(3) = -\ln(2) + \ln(2) - \ln(3) = -\ln(3).\\
S_3 &= S_2 + + \ln(3) - \ln(4) = -\ln(3) + \ln(3) - \ln(4) = -\ln(4).
\end{aligned}$$

In general,

$$\begin{aligned}
S_N &= [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + \cdots + [\ln(N) - \ln(N+1)]\\
&= -\ln(N+1).
\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} -\ln(N+1) = -\infty.$$

Thus, the series diverges.

(b) *(Solution 1 via the hint given during the exam.)*

We'll run the limit comparison test with the sequence $\frac{1}{n^{5/4}}$. Note that

$$\lim_{n\to\infty} \frac{\frac{(\ln(n))^2}{n^{\frac{3}{2}}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{(\ln(n))^2 n^{5/4}}{n^{3/2}} = \lim_{n\to\infty} \frac{(\ln(n))^2}{n^{1/4}} = 0.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges, since this is a p-series with $p = 5/4 > 1$. Since the above limit is 0, and $\frac{1}{n^{5/4}}$ is in the denominator, by the limit comparison test we can also conclude that the given series converges.

(Solution 2 via the direct comparison test, similar to assigned homework problems.)

Recall that $\ln(n) \leq n^a$ for any positive number a, at least when n is sufficiently large. In particular, $\ln(n) \leq n^{1/8}$ for n sufficiently large. This implies that

$$0 \leq \frac{(\ln(n))^2}{n^{3/2}} \leq \frac{(n^{1/8})^2}{n^{3/2}} = \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges, since this is a p-series with $p = 5/4 > 1$. By the above inequality, we may conclude by the direct comparison test that the given series converges.

□

4. **Find the limits of the following sequences.**

   (a) $b_n = \sqrt{9 + \frac{2}{n^2}}$

   (b) $a_n = \frac{n^7 + \sin(n^2)}{n^7 - n^3}$

(c) $c_n = \frac{e^{n^2}}{n^n}$

(d) $d_n = \frac{1}{n^{\sin(1/n)}}$

Solution.

(a)
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sqrt{9 + \lim_{n\to\infty} \frac{2}{n^2}} = \sqrt{9} = 3.$$

(b) Note that
$$\frac{n^7 + \sin(n^2)}{n^7 - n^3} = \frac{1 + \frac{\sin(n^2)}{n^7}}{1 - \frac{1}{n^4}}.$$

Since $0 \le \left|\frac{\sin(n^2)}{n^7}\right| \le \frac{1}{n^7}$, $\frac{\sin(n^2)}{n^7} \to 0$. Thus,
$$\lim_{n\to\infty} a_n = \frac{1 + \lim_{n\to\infty} \frac{\sin(n^2)}{n^7}}{1 - \lim_{n\to\infty} \frac{1}{n^7}} = \frac{1+0}{1-0} = 1.$$

(c) Note that
$$\frac{e^{n^2}}{n^n} = \left(\frac{e^n}{n}\right)^n.$$

Since $\frac{e^n}{n} \to \infty$ as $n \to \infty$ (e.g. by L'Hopital's rule), $\left(\frac{e^n}{n}\right)^n \to \infty$ as well.

(d) There are variables in the base and exponent of an expression, so we need to involve the logarithm. Note that
$$\ln(d_n) = \ln\left(\frac{1}{n^{\sin(1/n)}}\right) = \ln\left(n^{-\sin(1/n)}\right) = -\sin(1/n)\ln(n) = \frac{-\ln(n)}{\frac{1}{\sin(1/n)}}.$$

By L'Hopital's rule,
$$\begin{aligned}
\lim_{x\to\infty} \frac{-\ln(x)}{\frac{1}{\sin(1/x)}} &= \lim_{x\to\infty} \frac{-\frac{1}{x}}{-\frac{\cos(1/x)\cdot -x^{-2}}{\sin^2(1/x)}} = \lim_{x\to\infty} \frac{\sin^2(1/x)}{x\cos(1/x)\cdot -x^{-2}} \\
&= \lim_{x\to\infty} -\frac{\sin(1/x)}{1/x} \cdot \frac{\sin(1/x)}{\cos(1/x)} \\
&= -1 \cdot 0 \\
&= 0.
\end{aligned}$$

This implies that
$$\lim_{n\to\infty} \ln(d_n) = \lim_{n\to\infty} \frac{-\ln(n)}{\frac{1}{\sin(1/n)}} = 0.$$

Thus $\lim_{n\to\infty} d_n = e^0 = 1$.

□