Math 234 Practice Problems Solutions

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1. *Evaluate the line integral* $\int_C xy\,dx + y^2\,dy + yz\,dz$*, where* C *is the line segment from the point* $(1, 0, -1)$ *to* $(3, 4, 2)$*.*

   Solution. In order to evaluate this line integral directly (we could try to use the fundamental theorem of line integrals, but if you compute the curl of the vector field $\langle xy,\, y^2,\, yz\rangle$, you'll see that the vector field is not conservative), the first thing we need to do is parametrize the curve C. One way to parametrize the line segment between two points is to do the following: first, find the direction vector for the line segment. We can do this by subtracting the two points:

$$\mathbf{v} := (3, 4, 2) - (1, 0, -1) = \langle 2,\, 4,\, 3\rangle$$

   In general, to find the vector equation of a line, we need a point $\mathbf{x}_0$ on the line and a direction vector $\mathbf{v}$, and then compute $\mathbf{x}_0 + t\mathbf{v}$. Let's use the direction vector we just calculated, and use $(1, 0, -1)$ as our point:

$$\begin{aligned}\mathbf{r}(t) &= \langle 1,\, 0,\, -1\rangle + t\,\langle 2,\, 4,\, 3\rangle \\ &= \langle 1 + 2t,\, 4t,\, -1 + 3t\rangle\end{aligned}$$

   Note that $\mathbf{r}(0) = (1, 0, -1)$ and $\mathbf{r}(1) = (3, 4, 2)$. So as t ranges from 0 to 1, we move from the point $(1, 0, -1)$ to the point $(3, 4, 2)$. That's exactly the curve we want! Therefore, our parametrization is

$$\mathbf{r}(t) = \langle 1 + 2t,\, 4t,\, -1 + 3t\rangle \qquad 0 \le t \le 1$$

   Next, we can calculate dx, dy, and dz.

$$\begin{aligned}x = 1 + 2t \quad &\Rightarrow \quad dx = 2\,dt \\ y = 4t \quad &\Rightarrow \quad dy = 4\,dt \\ z = -1 + 3t \quad &\Rightarrow \quad dz = 3\,dt\end{aligned}$$

   Now we can convert the line integral into an integral in terms of t:

$$\begin{aligned}\int_C xy\,dx + y^2\,dy + yz\,dz &= \int_0^1 (1 + 2t)(4t)(2\,dt) + (4t)^2(4\,dt) + (4t)(-1 + 3t)(3\,dt) \\ &= \int_0^1 8t + 16t^2 + 64t^2 - 12t + 36t^2\,dt \\ &= \frac{110}{3}\end{aligned}$$

□

2. *Evaluate the following integral by making a change of variables:*

$$\iint_R \frac{x-2y}{3x-y}\, dA$$

where R is the parallelogram bounded by $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$.

Solution. The problem tells us that we need to make a change of variables, but it doesn't give us the substitutions we should use. Thus, the first step in solving this problem is to determine a change of variables that would make this thing easier to solve.

The first thing I would do in order to figure out the appropriate substitution is **look at the bounds.** All too often, the reason a variable substitution is useful is because it makes the region easier to describe. So let's write down the equations of the parallelogram down again:

$$x-2y=0$$
$$x-2y=4$$
$$3x-y=1$$
$$3x-y=8$$

Observe that we have two expressions: $x-2y$ and $3x-y$. At the extremes, $x-2y$ equals 0 and 4, and $3x-y$ equals 1 and 8. In other words,

$$0 \le x-2y \le 4 \qquad 1 \le 3x-y \le 8$$

It's pretty clear that if we call our variables $u = x-2y$ and $v = 3x-y$, then we'll have a nice region (a rectangle!) in the uv-plane. And there we go - that's our substitution!

The next thing we need is the Jacobian of the transformation. Since our goal is to do the integral in terms of u and v, we want the Jacobian of x and y in terms of u and v:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

The thing we have to be careful about is the fact that our substitution is u and v in terms of x and y, the reverse of what we want. There are two ways to handle this:

- *Solve for x and y in terms of u and v:* solving $u = x-2y$ for x gives us $x = u+2y$. Then, plugging this into the other equation gives us

  $$v = 3(u+2y) - y = 3u + 5y$$

  Hence,

  $$y = \frac{v-3u}{5}$$

Plugging this into the equation $v = 3x - y$ gives us:

$$v = 3x - \frac{v - 3u}{5} \quad \Rightarrow \quad x = \frac{2v - u}{5}$$

Now we can take the partial derivatives:

$$\begin{aligned}
\left|\frac{\partial(x,y)}{\partial(u,v)}\right| &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\
&= \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} \\
&= \left| -\frac{1}{5} \cdot \frac{1}{5} + \frac{2}{5} \cdot \frac{3}{5} \right| \\
&= \frac{1}{5}
\end{aligned}$$

- *Calculate the inverse Jacobian:* since our variable substitution is linear[1], we can just calculate $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$ and flip it! Explicitly,

$$\begin{aligned}
\left|\frac{\partial(u,v)}{\partial(x,y)}\right| &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\
&= 5
\end{aligned}$$

So

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{5}$$

Either way you do it, we see that $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{5}$.

Now we can change all the variables in our double integral, change the bounds, add in the Jacobian, and we're done.

$$\begin{aligned}
\iint_R \frac{x - 2y}{3x - y}\, dA &= \int_0^4 \int_1^8 \frac{u}{v} \left(\frac{1}{5}\right) dv\, du \\
&= \frac{24 \ln 2}{5}
\end{aligned}$$

□

3. *Use Green's Theorem to evaluate the the line integral*

$$\int_C \sqrt{1 + x^3}\, dx + 2xy\, dy$$

where C is boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$, oriented counterclockwise.

[1]Note: this only works when the transformation is linear.

Solution. Green's Theorem gives us the following equality:

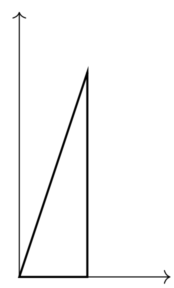
$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_E \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

where E is the region contained inside the closed curve C. Since the boundary of the triangle is a closed curve, and the region is a simply connected region (there are no holes or anything), we can apply Green's Theorem to solve the line integral in question by evaluating the double integral instead.

The first thing we need to do is calculate $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. For this problem $P = \sqrt{1+x^3}$ and $Q = 2xy$. Hence,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 0 = 2y$$

The next thing we need is to describe the bounds of our region E. The region is the inside of this triangle:



The equation of the line for the hypotenuse of the triangle is $y = 3x$. Therefore, the inside of the triangle can be described by $0 \leq x \leq 1$, $0 \leq y \leq 3x$. Thus, applying Green's Theorem, we get

$$\begin{aligned}\int_C \sqrt{1+x^3}\,dx + 2xy\,dy &= \int_0^1 \int_0^{3x} 2y\,dy\,dx \\ &= \int_0^1 y^2 \Big|_0^{3x}\,dx \\ &= \int_0^1 9x^2\,dx \\ &= 3\end{aligned}$$

□

4. *Find the surface area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.*

Solution. The surface area of any surface S is the scalar surface integral of the function $f(x,y,z) = 1$:

$$SA = \iint_S 1\,dS$$

The first thing we need to do in order to evaluate this surface integral is parametrize the surface S. The surface that we're trying to parametrize is the surface $x = y^2 + z^2$. Since we can solve for x as a function of the other two variables we can parametrize by using y and z and our parameters. This means that our parametrization of S will be

$$\mathbf{r}(y, z) = \langle y^2 + z^2, y, z\rangle$$

Then we can use the cylinder $y^2 + z^2 = 9$ to figure out the bounds for our parameters. Since our parameters are y and z and we lie *inside* the cylinder, our bounds can be described by $y^2 + z^2 \leq 9$.

Next, we need to calculate dS.

$$\begin{aligned}
dS &= \|\mathbf{r}_y \times \mathbf{r}_z\|\, dy\, dz \\
&= \left\| \begin{pmatrix} 2y \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2z \\ 0 \\ 1 \end{pmatrix} \right\| dy\, dz \\
&= \left\| \begin{pmatrix} 1 \\ -2y \\ 2z \end{pmatrix} \right\| dy\, dz \\
&= \sqrt{1 + 4y^2 + 4z^2}\, dy\, dz
\end{aligned}$$

Now we can compute the surface integral.

$$\begin{aligned}
SA &= \iint_S 1\, dS \\
&= \iint_{y^2+z^2\leq 9} \sqrt{1 + 4y^2 + 4z^2}\, dy\, dz
\end{aligned}$$

Since the region our parameters y and z lie in is a circle, we can convert this integral to polar coordinates.

$$= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

We can make a u-substitution with $u = 1 + 4r^2$. Then $du = 8r\, du$, and

$$\begin{aligned}
&= \frac{1}{8} \int_0^{2\pi} \int_1^{37} \sqrt{u}\, du\, d\theta \\
&= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_1^{37}\, d\theta \\
&= \frac{(37^{3/2} - 1)}{12} \int_0^{2\pi} d\theta \\
&= \frac{(37^{3/2} - 1)}{12} 2\pi
\end{aligned}$$

□

5. *Let C be a simple, closed curve that lies in the plane $x + y + z = 1$. Show that the line integral*

$$\int_C z\, dx - 2x\, dy + 3y\, dz$$

only depends on the area of the region enclosed by C and not on the shape of C or its position in the plane.

Solution. Note that this problem is taking place in three dimensions, on a slanted plane. Let D be the region in the plane enclosed by the curve C. Somehow, we have to show that the line integral above is related to the area of D. We know that the area of D can be calculated by evaluating the surface integral of the function 1 over D:

$$\text{Area of } D = \iint_D dS$$

Recall the statement of Stokes' Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

So we might be able to use Stokes' Theorem to relate the line integral to an expression giving the area of D.

Since C is a simpled, closed curve, we can use Stokes' Theorem! So let's calculate the surface integral on the right. First, the curl of $\mathbf{F} = \langle z, -2x, 3y\rangle$ is:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} z \\ -2x \\ 3y \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}\end{aligned}$$

Next, what is $\mathbf{n}$? Well, it's a unit vector normal to our surface D. D lies on the plane $x+y+z=1$, so the normal vector will be in the same direction as the normal vector of that plane. The normal vector of the plane is $\langle 1, 1, 1\rangle$, so a *unit* normal vector in that direction is

$$\mathbf{n} = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$$

Hence,

$$\begin{aligned}(\nabla \times \mathbf{F}) \cdot \mathbf{n} &= \langle 3, 1, -2\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle \\ &= \frac{1}{\sqrt{3}}(3+1-2) \\ &= \frac{2}{\sqrt{3}}\end{aligned}$$

Therefore, by Stokes' Theorem,

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \iint_D \frac{2}{\sqrt{3}}\,dS = \frac{2}{\sqrt{3}}\iint_D dS = \frac{2}{\sqrt{3}}(\text{Area of } D)$$

Hence, the line integral only depends on the area of the region D. It doesn't matter where the curve C is, or what it looks like — we can compute the line integral by just knowing the area of the region it encloses. $\square$

6. *Let* $\mathbf{F}(x, y, z) = \langle xy,\, 3y,\, 5y\rangle$*, and let* C *be the positively oriented curve of intersection of the plane* $x + z = 5$ *and the cylinder* $x^2 + y^2 = 81$*. Compute* $\oint_C \mathbf{F} \cdot d\mathbf{r}$*.*

Solution. Notice that the curve C is a closed curve (it's some sort of elliptical curve sitting in space). Thus, we can use Stokes' Theorem! Stokes' Theorem says we can evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by evaluating the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where S is some surface whose boundary is the curve C. For this problem, since the curve C is lying in the plane $x + z = 5$, we can use that plane as our surface. Explicitly, S is the part of the plane $x + z = 5$ contained in the cylinder $x^2 + y^2 = 81$.

In order to evaluate the surface integral, we need to parametrize S. Since we can solve for z as a function of x and y, let's do that: $z = 5 - x$. So we can make our parameters x and y, in which case $x = x$, $y = y$, and $z = 5 - x$. Explicitly, our parametrization is:

$$\mathbf{S}(x, y) = \langle x,\, y,\, 5 - x\rangle \qquad x^2 + y^2 \leq 81$$

Now that we have our parametrization, we can calculate $d\mathbf{S}$. Since we're doing a vector surface integral,

$$\begin{aligned} d\mathbf{S} &= \mathbf{S}_x \times \mathbf{S}_y \, dx\, dy \\ &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dx\, dy \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} dx\, dy \end{aligned}$$

Next, let's calculate the curl of $\mathbf{F}$:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} xy \\ 3y \\ 5y \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 0 \\ -x \end{pmatrix} \end{aligned}$$

Now we can evaluate the surface integral.

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{x^2+y^2\leq 81} \begin{pmatrix} 2 \\ 0 \\ -x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \, dx\, dy$$
$$= \iint_{x^2+y^2\leq 81} 2 - x \, dx\, dy$$

Now we can convert the integral to polar coordinates, since our region in the xy plane is a disk.

$$= \int_0^{2\pi} \int_0^9 (2 - r\cos\theta) r \, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^9 r^2 - \frac{r^3}{3}\cos\theta \, \Big|_0^9 \, d\theta$$
$$= \int_0^{2\pi} \int_0^9 81 - 243\cos\theta \, d\theta$$
$$= 81\theta - 243\sin\theta \, \Big|_0^{2\pi}$$
$$= 81(2\pi) = 162\pi$$

Hence, by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 162\pi$$

□

7. *Let* $\mathbf{F}(x, y, z) = \langle \sin y,\, x\cos y + \cos z,\, -y\sin z\rangle$. *Let* C *be the curve given by* $\mathbf{r}(t) = \langle \sin t,\, t,\, 2t\rangle$ *for* $0 \leq t \leq 2\pi$.

   (a) Is $\mathbf{F}$ *conservative?*

   (b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution. (a) Since $\mathbf{F}$ is vector field in three dimensions, the best way to test for conservativeness is to take the curl. If the curl of $\mathbf{F}$ is identically the zero vector, then $\mathbf{F}$ is conservative. We have:

$$\nabla \times \mathbf{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \sin y \\ x\cos y + \cos z \\ -y\sin z \end{pmatrix}$$
$$= \begin{pmatrix} -\sin z - (-\sin z) \\ 0 - 0 \\ \cos y - \cos y \end{pmatrix}$$
$$= \mathbf{0}$$

Therefore, $\mathbf{F}$ is conservative.

(b) Notice that we proved in part (a) that $\mathbf{F}$ is conservative. This means that we can use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Explicitly,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$

where f is a potential function for $\mathbf{F}$ and $\mathbf{a}, \mathbf{b}$ are the starting and ending points of the curve C. In order to use the fundamental theorem of line integrals, we need to find a potential function f. By definition, a potential function is a function f such that $\nabla f = \mathbf{F}$. Explicitly, this would mean that

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle \sin y,\, x \cos y + \cos z,\, -y \sin z \rangle$$

Our goal is to figure out what f is. Consider the first component of the vector equation above:

$$\frac{\partial f}{\partial x} = \sin y$$

If we want to solve for f, we can sort of treat this like a differential equation. Separate variables:

$$\partial f = \sin y \, \partial x$$

Now we can integrate both sides:

$$\int \partial f = \int \sin y \, \partial x$$

On the left, since we integrate the differential of f, we get f! On the right, we're doing "partial" integration with respect to x. Thus, since y is a constant, $\sin y$ is a constant, and so the antiderivative with respect to x is $x \sin y$. But we always need to add a constant of integration, which in this case - since y and z are both constants - will be some function of y and z: $h_1(y, z)$. Putting this together means that

$$f(x, y, z) = x \sin y + h_1(y, z)$$

So we almost know what f is, but we have to figure out the "constant" $h_1(y, z)$ is. To do this, we can repeat the same process in the y and z components. In the y component we have

$$\frac{\partial f}{\partial y} = x \cos y + \cos z$$

Integrating both sides with respect to y gives us:

$$f(x, y, z) = x \sin y + y \cos z + h_2(x, z)$$

for some other constant function h_2. Similarly, integrating the z component with respect to z gives us:

$$f(x, y, z) = y \cos z + h_3(x, y)$$

Let's write these all down together in a particular way:

$$\begin{aligned}
f(x,y,z) &= x\sin y + h_1(y,z)\\
f(x,y,z) &= x\sin y + y\cos z + h_2(x,z)\\
f(x,y,z) &= h_3(x,y) + y\cos z
\end{aligned}$$

Remember that *all three equations are the same.* They are all f! We just need to figure out what f is without the constant functions. Observe that I matched up the constant functions of two variables. We see that $h_1(y,z) = y\cos z$, $h_2(x,z) = 0$, and $h_3(x,y) = x\sin y$. Then, using any of those equations, we see that

$$f(x,y,z) = x\sin y + y\cos z$$

is a potential function for $\mathbf{F}$.

The last thing we need to know in order to use the fundamental theorem of line integrals is the starting and endings points of the curve C. Plugging in $t = 0$, we get the starting point:

$$\mathbf{a} = \mathbf{r}(0) = \langle \sin 0,\ 0,\ 2(0)\rangle = \langle 0,\ 0,\ 0\rangle$$

and plugging in $t = 2\pi$ gets us the ending point:

$$\mathbf{b} = \mathbf{r}(2\pi) = \langle \sin 2\pi,\ 2\pi,\ 2(2\pi)\rangle = \langle 0,\ 2\pi,\ 4\pi\rangle$$

Therefore, by the fundamental theorem of line integrals,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} &= f(\mathbf{b}) - f(\mathbf{a})\\
&= f(0,2\pi,4\pi) - f(0,0,0)\\
&= [(0)\sin(2\pi) + (2\pi)\cos(4\pi)] - [(0)\sin(0) + (0)\cos(0)]\\
&= 2\pi
\end{aligned}$$

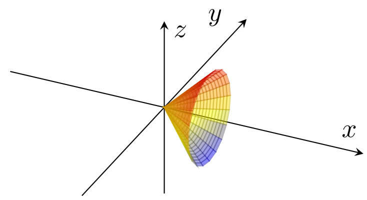
□

8. *Use Stokes' Theorem to evaluate* $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, *where* $\mathbf{F}(x,y,z) = \left\langle \arctan(x^2yz^2),\ x^2y,\ x^2z^2\right\rangle$ *and* S *is the cone* $x = \sqrt{y^2+z^2}$ *from* $0 \le x \le 2$ *oriented in the direction of the positive* x*-axis.*

Solution. We are told to use Stokes' Theorem to evaluate a certain surface integral. Recall that Stokes' Theorem says:

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\cdot d\mathbf{r}$$

where ∂S is the boundary curve for S. What does this boundary curve look like? Well, here is the cone:



The boundary curve ∂S is the circle lying in the plane $x = 2$. So in order to use Stokes' Theorem, we will have to evaluate the line integral of $\mathbf{F}$ around that boundary curve. The first thing we need to do is parametrize that circle, i.e., construct a function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that gives us the coordinates of the circle in terms of t. The first thing to notice is that the x coordinate of any point on the circle is always 2. Therefore, we can set $x(t) = 2$. Since the circle is spinning in the yz-plane, we know that our $y(t)$ and $z(t)$ functions will involve some sort of $\sin(t)/\cos(t)$ combination. Since $x = 2$, the equation for the circle is:

$$2 = \sqrt{y^2 + z^2} \quad \Rightarrow \quad y^2 + z^2 = 2^2$$

Hence, the radius of the circle is 2. So we can set $y(t) = 2\cos(t)$ and $z(t) = 2\sin(t)$. Then:[2]

$$\mathbf{r}(t) = \langle 2,\, 2\cos(t),\, 2\sin(t)\rangle$$

and since we go all the way around the circle, $0 \le t \le 2\pi$. Next, we can compute the tangent vector:

$$\frac{d\mathbf{r}}{dt} = \langle 0,\, -2\sin(t),\, 2\cos(t)\rangle$$

The next step in evaluating a line integral is to convert our vector field $\mathbf{F}$ entirely in terms of t:

$$\begin{aligned}\mathbf{F}(t) &= \left\langle \arctan\left[2^2(2\cos t)(2\sin t)^2\right],\, 2^2(2\cos t),\, 2^2(2\sin t)^2\right\rangle \\ &= \left\langle \arctan\left[32\cos t\sin^2 t\right],\, 8\cos t,\, 16\sin^2 t\right\rangle\end{aligned}$$

Now all we have to do is compute:

$$\begin{aligned}\int_{\partial S} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(t)\cdot d\mathbf{r}\,dt \\ &= \int_0^{2\pi} \left\langle \arctan\left[32\cos t\sin^2 t\right],\, 8\cos t,\, 16\sin^2 t\right\rangle \cdot \langle 0,\, -2\sin t,\, 2\cos t\rangle\; dt \\ &= \int_0^{2\pi} -16\sin t\cos t + 32\sin^2 t\cos t\,dt\end{aligned}$$

To integrate this thing, we can make a u-substitution: $u = \sin t$, in which case $du = \cos t\,dt$, and the bounds go from $u = \sin(0) = 0$ to $u = \sin(2\pi) = 0$. This means that we integrate from 0 to 0, so our final answer is 0! Thus, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\cdot d\mathbf{r} = 0$$

□

[2]Note that, since we go all the way around the circle, it doesn't matter which of y and z are sin and cos. Either way will work!

9. *Fix $a > 0$. Let $\mathbf{F} = \langle xz, x, y \rangle$, and let S be the surface given by:*

$$x^2 + y^2 + z^2 = a^2 \qquad y \geq 0$$

Compute:

$$\iint_S \mathbf{F} \cdot d\mathbf{r}$$

Solution 1. The first step in computing any surface integral is to *parametrize the surface* in terms of two variables, i.e., represent x, y, and z in terms of two variables. We only know how to compute double integrals, so parametrizing the surface S lets us turn the surface integral into a double integral over two variables.

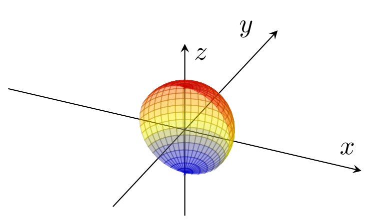
There are a couple ways to parametrize S in this case. First, since S is part of a sphere, I'm going to use spherical coordinates. The spherical coordinate transformations are:

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

Since our surface is a sphere, ρ is fixed! In particular, $\rho = a$. Hence, the spherical coordinate transformation actually gives us a parametrization of S in terms of two variables:

$$x(\varphi, \theta) = a \sin\varphi \cos\theta$$
$$y(\varphi, \theta) = a \sin\varphi \sin\theta$$
$$z(\varphi, \theta) = a \cos\varphi$$

The only thing we have to worry about now is the bounds for the variables φ and θ. Note that we are dealing with half of the sphere, since we are requiring that $y \geq 0$. This gives:



From this picture, we can see that φ (the vertical angle from the z-axis) ranges from 0 to π, and θ (the horizontal rotational angle starting from the positive x axis) ranges from 0 to π. Hence, our complete parametrization is $\mathbf{r}(\varphi, \theta) = \langle x(\varphi, \theta), y(\varphi, \theta), z(\varphi, \theta) \rangle$, where:

$$x(\varphi, \theta) = a \sin\varphi \cos\theta \qquad 0 \leq \varphi \leq \pi$$
$$y(\varphi, \theta) = a \sin\varphi \sin\theta \qquad 0 \leq \theta \leq \pi$$
$$z(\varphi, \theta) = a \cos\varphi$$

Next, the actual calculation of the surface integral will happen in the following way:

$$\iint_S \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \int_0^\pi \mathbf{F}(x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta)) \cdot (\mathbf{r}_\varphi \times \mathbf{r}_\theta)\, d\varphi\, d\theta$$

Our next step is to calculate $\mathbf{r}_\varphi \times \mathbf{r}_\theta$. First, we'll compute the tangent vectors of $\mathbf{r}$:

$$\mathbf{r}_\varphi = \begin{bmatrix} a\cos\varphi\cos\theta \\ a\cos\varphi\sin\theta \\ -a\sin\varphi \end{bmatrix} \qquad \mathbf{r}_\theta = \begin{bmatrix} -a\sin\varphi\sin\theta \\ a\sin\varphi\cos\theta \\ 0 \end{bmatrix}$$

Then,

$$\begin{aligned}
\mathbf{r}_\varphi \times \mathbf{r}_\theta &= \begin{bmatrix} a\cos\varphi\cos\theta \\ a\cos\varphi\sin\theta \\ -a\sin\varphi \end{bmatrix} \times \begin{bmatrix} -a\sin\varphi\sin\theta \\ a\sin\varphi\cos\theta \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} a^2\sin^2\varphi\cos\theta \\ a^2\sin^2\varphi\sin\theta \\ a^2\cos\varphi\sin\varphi(\cos^2\theta + \sin^2\theta) \end{bmatrix} \\
&= \begin{bmatrix} a^2\sin^2\varphi\cos\theta \\ a^2\sin^2\varphi\sin\theta \\ a^2\cos\varphi\sin\varphi \end{bmatrix}
\end{aligned}$$

This actually makes a lot of sense. Here's why: we can factor a $a\sin\varphi$ out of each term to get:

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = a\sin\varphi \begin{bmatrix} a\sin\varphi\cos\theta \\ a\sin\varphi\sin\theta \\ a\cos\varphi \end{bmatrix} = a\sin\varphi \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is telling us that the normal vector to the sphere is a scalar multiple of $\langle x, y, z\rangle$, which geometrically is what we would expect.

Anyways, now that we have the normal vector, let's compute $\mathbf{F}(x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta))$:

$$\mathbf{F} = \begin{bmatrix} xz \\ x \\ y \end{bmatrix} = \begin{bmatrix} (a\sin\varphi\cos\theta)(a\cos\varphi) \\ a\sin\varphi\cos\theta \\ a\sin\varphi\sin\theta \end{bmatrix} = \begin{bmatrix} a^2\sin\varphi\cos\theta\cos\varphi \\ a\sin\varphi\cos\theta \\ a\sin\varphi\sin\theta \end{bmatrix}$$

Now we're ready to calculate the integral:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \int_0^\pi \mathbf{F}(x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta)) \cdot (\mathbf{r}_\varphi \times \mathbf{r}_\theta)\, d\varphi\, d\theta \\
&= \int_0^\pi \int_0^\pi \begin{bmatrix} a^2\sin\varphi\cos\theta\cos\varphi \\ a\sin\varphi\cos\theta \\ a\sin\varphi\sin\theta \end{bmatrix} \cdot \begin{bmatrix} a^2\sin^2\varphi\cos\theta \\ a^2\sin^2\varphi\sin\theta \\ a^2\cos\varphi\sin\varphi \end{bmatrix} d\varphi\, d\theta \\
&= \int_0^\pi \int_0^\pi a^4\sin^3\varphi\cos^2\theta\cos\varphi + a^3\sin^3\varphi\sin\theta\cos\theta + a^3\sin^2\varphi\sin\theta\cos\varphi\, d\varphi\, d\theta
\end{aligned}$$

Here, I'm going to break up the innermost integral into three integrals, just for the sake of making them easier to do. The first term we have to integrate is $a^4 \sin^3\varphi \cos^2\theta \cos\varphi$. Informally, we can use a u-substitution here, with $u = \sin\varphi$ (because $du = \cos\varphi\, du$ is in the integral). Changing the bounds would give us $\sin 0 \le u \le \sin\pi$, which means that $0 \le u \le 0$. So the integral goes from 0 to 0, and so the integral equals 0! Explicitly,

$$\int_0^\pi a^4 \sin^3\varphi \cos^2\theta \cos\varphi\, d\varphi = \int_0^0 a^4 u^3 \cos^2\theta\, du = 0$$

The same thing happens with the third term:

$$\int_0^\pi a^3 \sin^2\varphi \sin\theta \cos\varphi\, d\varphi = \int_0^0 a^3 u^2 \sin\theta\, du = 0$$

To integrate the middle term, we can rewrite $\sin^3\varphi$ as

$$\sin^3\varphi = \sin\varphi \sin^2\varphi = \sin\varphi(1-\cos^2\varphi)$$

then use the substitution $u = \cos\varphi$. Here, we get $du = -\sin\varphi\, d\varphi$, and u goes from 1 to -1.

$$\begin{aligned}\int_0^\pi a^3 \sin^3\varphi \sin\theta\cos\theta\, d\varphi = \int_0^\pi a^3 \sin\varphi(1-\cos^2\varphi)\sin\theta\cos\theta\, d\varphi &= \int_1^{-1} (-1)a^3(1-u^2)\sin\theta\cos\theta\, du \\ &= \int_{-1}^1 a^3(1-u^2)\sin\theta\cos\theta\, du = a^3\sin\theta\cos\theta\left(u - \frac{u^3}{3}\right)\Big|_{-1}^1 \\ &= \frac{4}{3}a^3\sin\theta\cos\theta\end{aligned}$$

Therefore, our big double integral from above become:

$$\iint_S \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \frac{4}{3}a^3 \sin\theta\cos\theta\, d\theta = 0$$

where we get 0 by doing another u-substitution. Therefore, $\iint_S \mathbf{F}\cdot d\mathbf{r} = 0$. □

Solution 2. I mentioned before that there are multiple ways to parametrize the surface S. Here's another way. Since we are considering the part of the sphere where $y \ge 0$, we can solve for y to get:

$$y = \sqrt{a^2 - x^2 - z^2}$$

Now that we have y as a function of x and z, we can get the following parametrization:

$$\begin{aligned} x(u,v) &= u \\ y(u,v) &= \sqrt{a^2-u^2-v^2} \\ z(u,v) &= v \end{aligned}$$

Here, our uv-region would be the disc D of radius a. In this case, the tangent vectors are:

$$\mathbf{r}_u = \begin{bmatrix} 1 \\ \frac{-u}{\sqrt{a^2-u^2-v^2}} \\ 0 \end{bmatrix} \qquad \mathbf{r}_v = \begin{bmatrix} 0 \\ \frac{-v}{\sqrt{a^2-u^2-v^2}} \\ 1 \end{bmatrix}$$

In which case,

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{bmatrix} 1 \\ \frac{-u}{\sqrt{a^2-u^2-v^2}} \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ \frac{-v}{\sqrt{a^2-u^2-v^2}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-u}{\sqrt{a^2-u^2-v^2}} \\ -1 \\ \frac{-v}{\sqrt{a^2-u^2-v^2}} \end{bmatrix}$$

Since we want the normal vector to be oriented *outwards,* we'll multiply the whole vector by -1:

$$\mathbf{n} = \begin{bmatrix} \frac{u}{\sqrt{a^2-u^2-v^2}} \\ 1 \\ \frac{v}{\sqrt{a^2-u^2-v^2}} \end{bmatrix}$$

Substituting our parametrization into $\mathbf{F}$ gives us:

$$\mathbf{F} = \begin{bmatrix} uv \\ u \\ \sqrt{a^2-u^2-v^2} \end{bmatrix}$$

Therefore, the surface integral becomes:

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{r} &= \iint_D \mathbf{F} \cdot \mathbf{n}\, du\, dv \\ &= \iint_D \begin{bmatrix} uv \\ u \\ \sqrt{a^2-u^2-v^2} \end{bmatrix} \cdot \begin{bmatrix} \frac{u}{\sqrt{a^2-u^2-v^2}} \\ 1 \\ \frac{v}{\sqrt{a^2-u^2-v^2}} \end{bmatrix} du\, dv \\ &= \iint_D \frac{u^2 v}{\sqrt{a^2-u^2-v^2}} + u + v\, du\, dv \end{aligned}$$

The $u+v$ part of the integral are easy to integrate, so let's focus on this tough part first:

$$\iint_D \frac{u^2 v}{\sqrt{a^2-u^2-v^2}}\, du\, dv$$

I'm actually going to integrate this with respect to v first. In this case, our bounds would be $-\sqrt{a^2-u^2} \leq v \leq \sqrt{a^2-u^2}$ and $-a \leq u \leq a$:

$$\int_{-a}^{a} \int_{-\sqrt{a^2-u^2}}^{\sqrt{a^2-u^2}} \frac{u^2 v}{\sqrt{a^2-u^2-v^2}}\, dv\, du$$

We could do this integral directly (use a u-substitution by setting the inside of the square root equal to u), but notice that this function is odd with respect to v. Whenever we integrate an odd function from -blah to blah, we get 0. Therefore, the whole thing is 0!

$$\iint_D \frac{u^2 v}{\sqrt{a^2-u^2-v^2}}\, dv\, du = 0$$

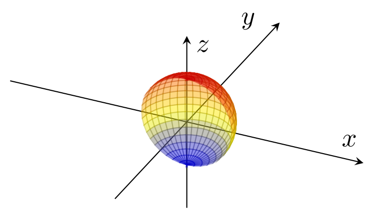
The same thing is going to happen with the term v, since v is an odd function. So if we integrate that with respect to v first, that goes to 0. Similarly, we can integrate u with respect to u first, and since u is odd, that integral goes to 0. So the whole thing is 0:

$$\iint_S \mathbf{F} \cdot d\mathbf{r} = 0$$

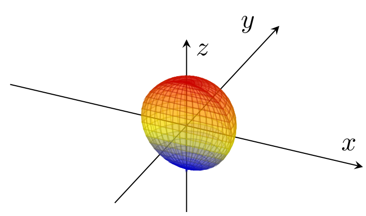
□

Solution 3. We can do this problem in a third way by using the Divergence theorem!

This is one of those times where we have to be careful about how we apply the Divergence theorem, because our surface S is not closed. Remember, we're dealing with half of a sphere:



In order to use the Divergence theorem, we can *close the surface* using a plane that lies in the xz plane:



I'm going to call the plane we just added S_0. Then $S \cup S_0$ is a closed surface, and so we can use the Divergence theorem. Explicitly,

$$\iint_{S\cup S_0} \mathbf{F}\cdot d\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}\, dV$$

where E is the inside of the half-sphere. Let's compute $\nabla\cdot\mathbf{F}$:

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y) = z$$

Now we can compute the triple integral that appears in the divergence theorem, carefully switching to spherical coordinates. Since $y \geq 0$, we'll require that θ goes from 0 to π (draw the projection of the sphere in the xy plane to determine the bounds for θ).

$$\begin{aligned}
\iint_{S\cup S_0} \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \nabla\cdot\mathbf{F}\, dV \\
&= \iiint_E z\, dV \\
&= \int_0^\pi \int_0^\pi \int_0^a (\rho\cos\varphi)\left(\rho^2\sin\varphi\right)\, d\rho\, d\varphi\, d\theta \\
&= \int_0^\pi \int_0^\pi \frac{\rho^4}{4}\cos\varphi\sin\varphi \,\Big|_{\rho=0}^{\rho=a}\, d\varphi\, d\theta \\
&= \int_0^\pi \int_0^\pi \frac{a^4}{4}\cos\varphi\sin\varphi\, d\varphi\, d\theta
\end{aligned}$$

Next, we can do a u-sub with $u = \sin\varphi$, $du = \cos\varphi\, d\varphi$, and u goes from $\sin 0 = 0$ to $\sin\pi = 0$:

$$\begin{aligned} &= \frac{a^4}{4}\int_0^{\pi}\int_0^{0} u\, du\, d\theta \\ &= 0 \end{aligned}$$

(Integrating from 0 to 0 just gives us 0). This tells us that $\iint_{S\cup S_0} \mathbf{F}\cdot d\mathbf{S} = 0$, which is nice, but this isn't what we're after. We want $\iint_S \mathbf{F}\cdot d\mathbf{S}$. But integrals are additive over their domains - so we have:

$$\begin{aligned} \iint_{S\cup S_0} \mathbf{F}\cdot d\mathbf{S} &= 0 \\ \iint_S \mathbf{F}\cdot d\mathbf{S} + \iint_{S_0} \mathbf{F}\cdot d\mathbf{S} &= 0 \\ \iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_{S_0} \mathbf{F}\cdot d\mathbf{S} \end{aligned}$$

So all we have to do now is calculate the surface integral $\iint_{S_0} \mathbf{F}\cdot d\mathbf{S}$ and subtract it off of 0.

The first thing we have to do to calculate the surface integral over S_0 is parametrize S_0, then find the cross product of the tangent vectors. Since S_0 is a plane in the xz-plane, it's equation is $y = 0$. Using our parametrization techniques, we can let $x = u$, $y = 0$, and $z = v$, and use u and v as our parameters. Explicitly:

$$\mathbf{r}(u, v) = \langle u,\, 0,\, v\rangle$$

For this surface, we get:

$$\begin{aligned} d\mathbf{S} &= (\mathbf{r}_u \times \mathbf{r}_v)\; du\, dv \\ &= (\langle 1,\, 0,\, 0\rangle \times \langle 0,\, 0,\, 1\rangle)\; du\, dv \\ &= \langle 0,\, -1,\, 0\rangle\; du\, dv \end{aligned}$$

The only thing we need to check is our orientation. Since we used the Divergence theorem, we were implicitly assuming *outward orientation* on our surface. For the plane S_0 that we added, outward orientation would be pointing in the negative y direction. Since the cross product we computed ($\langle 0,\, -1,\, 0\rangle$) points in the negative y direction, we're good to go. So:

$$\begin{aligned} \iint_{S_0} \mathbf{F}\cdot d\mathbf{S} &= \iint_{(u,v)} \langle xz,\, x,\, y\rangle \cdot \langle 0,\, -1,\, 0\rangle\; du\, dv \\ &= \iint_{(u,v)} \langle uv,\, u,\, 0\rangle \cdot \langle 0,\, -1,\, 0\rangle\; du\, dv \\ &= \iint_{(u,v)} -u\, du\, dv \end{aligned}$$

where our uv-region is the disc of radius a. Hence, we can switch to polar coordinates:

$$\begin{aligned} &= \int_0^{2\pi}\int_0^{a} (r\cos\theta) r\, dr\, d\theta = \int_0^{2\pi} \frac{a^3}{3}\cos\theta\, d\theta \\ &= \frac{a^3}{3}\sin\theta\,\Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

Therefore,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_{S_0} \mathbf{F}\cdot d\mathbf{S} = 0$$

□

10. *Use the Divergence Theorem to evaluate* $\iint_S (2x+2y+z^2)\,dS$ *where* S *is the sphere* $x^2+y^2+z^2=1$.

Solution. Recall that the Divergence says the following: for a vector field $\mathbf{F}$ and simple three dimensional region E, we have

$$\iiint_E \nabla\cdot\mathbf{F}\,dV = \iint_{\partial S} \mathbf{F}\cdot d\mathbf{S}$$

where ∂E is the boundary surface of E. What the problem is asking us to do is evaluate a certain surface integral of a function over a closed sphere S. To use the Divergence Theorem, we want to relate this surface integral to the triple integral of something over the inside of the sphere, but there's an issue: the surface integral that appears in the statement of the Divergence Theorem is $\iint_{\partial S}\mathbf{F}\cdot d\mathbf{S}$, which is the surface integral of a vector field. The integral we want to evaluate is a scalar surface integral: $\iint_S (2x+2y+z^2)\,dS$. That's okay though, because it turns out we can think of any vector surface integral as a scalar surface integral:

$$\iiint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

where $\mathbf{n}$ is a unit normal vector to the surface. So if we can find a vector field $\mathbf{F}$ such that $\mathbf{F}\cdot\mathbf{n} = 2x+2y+z^2$, we could then use this vector field in the Divergence Theorem.

First, what is $\mathbf{n}$? Well, it is a unit normal vector to a sphere. There's a couple ways to do this. I did it once in the problem right before this one by parametrizing the sphere and taking the cross product of the tangent vectors. Here's a slightly easier way: think about the sphere as a level surface of the following function:

$$g(x,y,z) = x^2+y^2+z^2$$

Explicitly, the unit sphere S is the level surface $g(x,y,z) = 1$. What do we know about gradient vectors? They're always normal to level surfaces! So let's calculate the gradient of $g(x,y,z)$:

$$\nabla g(x,y,z) = \langle 2x,\, 2y,\, 2z\rangle = 2\,\langle x,\, y,\, z\rangle$$

Hence, the normal vector to a sphere is a multiple of the vector $\langle x,\, y,\, z\rangle$. But we want a unit normal vector, so let's normalize $\langle x,\, y,\, z\rangle$:

$$\mathbf{n} = \frac{\langle x,\, y,\, z\rangle}{\|\langle x,\, y,\, z\rangle\|} = \frac{\langle x,\, y,\, z\rangle}{\sqrt{x^2+y^2+z^2}}$$

But on the sphere, $\sqrt{x^2+y^2+z^2} = 1$. Hence, the unit normal vector is $\mathbf{n} = \langle x,\, y,\, z\rangle$. Thus, we want a vector field $\mathbf{F}$ such that

$$\mathbf{F}(x,y,z)\cdot\langle x,\, y,\, z\rangle = 2x+2y+z^2$$

In other words, if $\mathbf{F}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle$, then we need

$$P(x,y,z)x + Q(x,y,z)y + R(x,y,z)z = 2x + 2y + z^2$$

Let's make the obvious choices: $P(x,y,z) = 2$, $Q(x,y,z) = 2$, $R(x,y,z) = z$. So our vector field is $\mathbf{F} = \langle 2,\, 2,\, z\rangle$. Now we can use the Divergence Theorem!

$$\begin{aligned}
\iint_S (2x + 2y + z^2)\, dS &= \iint_S \mathbf{F}\cdot\mathbf{n}\, dS \\
&= \iint_S \mathbf{F}\cdot d\mathbf{S} \\
&= \iiint_E \nabla\cdot\mathbf{F}\, dV \\
&= \iiint_E \left(\frac{\partial}{\partial x}2 + \frac{\partial}{\partial y}2 + \frac{\partial}{\partial z}z\right)\, dV \\
&= \iiint_E 1\, dV
\end{aligned}$$

But since E is the inside of a sphere and, in general, $\iiint_E dV$ gives the volume of E, it follows that $\iiint_E 1\, dV = \frac{4}{3}\pi$. Therefore,

$$\iint_S (2x + 2y + z^2)\, dS = \frac{4}{3}\pi$$

□