LIOUVILLE AND WEINSTEIN DOMAINS

JOSEPH BREEN

1. Introduction

In this set of notes, we discuss two important kinds of structures on symplectic manifolds: Liouville structures and Weinstein structures. These structures arise when considering the relationship between contact manifolds as boundaries of symplectic manifolds, and as level sets of symplectic manifolds. An exhaustive reference for much of this material is [1].

2. Definitions and Examples

Definition 2.1. A **Liouville form** on a symplectic manifold (W, ω) is a 1-form λ such that $\omega = d\lambda$. The vector field X such that $i_X\omega = \lambda$ is the **Liouville vector field** of λ.

Observe that if X is a Liouville vector field for a Liouville form λ, Cartan's formula gives

$$\mathcal{L}_X\omega = i_X d\omega + d i_X\omega = d\lambda = \omega.$$

The equation $\mathcal{L}_X\omega = \omega$ implies that the flow ϕ_t of X satisfies $\phi_t^*\omega = e^t\omega$. In words, the symplectic form expands as one flows along X. A simple Stokes' theorem argument then shows that there are no closed exact symplectic manifolds:

$$0 < \int_W \omega^n = \int_W d(\lambda \wedge \omega^{n-1}) = \int_{\partial W} \lambda \wedge \omega^{n-1}$$

hence $\partial W \neq \emptyset$. In particular, this shows that no closed symplectic manifold admits a *globally* defined Liouville vector field, though such vector fields always exist locally.

Definition 2.2. A **Liouville domain** is a compact[1] symplectic manifold (W, ω, X) with boundary, together with a globally defined Liouville vector field X which points transversally out of the boundary.

Positive transversality at the boundary is a natural condition to consider, as this implies that ∂W is a contact manifold with contact form $\alpha := \lambda\,|_{\partial W}$. Indeed, note that

$$\lambda \wedge (d\lambda)^{n-1} = i_X d\lambda \wedge (d\lambda)^{n-1} = \frac{1}{n}\, i_X (d\lambda)^n.$$

Since $(d\lambda)^n$ is a volume form on W and $X \pitchfork \partial W$, $i_X(d\lambda)^n$ restricts to a volume form on ∂W, hence α is contact. This means that a Liouville domain is an exact symplectic filling of its (contact) boundary.

Next, we introduce even more structure.

Definition 2.3. A **Weinstein domain** (W, ω, X, f) is a Liouville domain (W, ω, X) equipped with a Morse function $f : W \to \mathbb{R}$ such that f is locally constant on ∂W and such that X is *gradient-like* for f, i.e.,

$$X(f) \geq \delta(|X|^2 + |df|^2)$$

for some choice of Riemannian metric on W and some $\delta > 0$.

[1] One can also define (open) Liouville manifolds and also Liouville cobordisms, but for now we will focus on Liouville domains. Read Section 4 for some comments on Liouville cobordisms.

Observe that if X is gradient-like for f, then Cauchy-Schawrz gives.

$$\delta |X|^2 \leq \delta(|X|^2 + |df|^2) \leq |X(f)| \leq |df||X| \quad \Rightarrow \quad \delta|X| \leq |df|.$$

On the other hand,

$$\delta |df|^2 \leq \delta(|X|^2 + |df|^2) \leq |X(f)| \leq |df||X| \quad \Rightarrow \quad |df| \leq \frac{1}{\delta}|X|.$$

Thus,

$$\delta|X| \leq |df| \leq \frac{1}{\delta}|X|.$$

In particular, the zeroes of X occur exactly at critical points of f, and $X(f) > 0$ away from critical points. This justifies the terminology. Furthermore, note that if ∇f is the gradient vector field of $f : M \to \mathbb{R}$ for some choice of Riemannian metric, then

$$(\nabla f)(f) = df(\nabla f) = |\nabla f|^2 = \frac{1}{2}(|\nabla f|^2 + |df|^2)$$

so that gradient vector fields are indeed gradient-like.

Another way to think about the gradient-like condition in Definition 2.3 is as follows: the fact that X is gradient-like for f implies that the Liouville form λ restricts to a contact form on each regular level-set of f. Thus, a Weinstein domain is a symplectic manifold which more or less decomposes into layers of contact manifolds.

Next, we consider some examples.

Example 2.4. Here is the simplest example of a Weinstein domain. Consider the closed unit disc $W = \{x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$ together with the standard symplectic form $\omega = dx \wedge dy$. Let $\lambda = \frac{1}{2}(x\,dy - y\,dx)$. Then $d\lambda = \omega$, hence λ is a Liouville form. An easy computation verifies that the radial vector field $X = \frac{1}{2}(x\,\partial_x + y\,\partial_y)$ satisfies $i_X\omega = \lambda$:

$$i_{\frac{1}{2}(x\,\partial_x + y\,\partial_y)}(dx \wedge dy) = \frac{1}{2}x\,dy - \frac{1}{2}y\,dx = \lambda.$$

The Liouville vector field X is outwardly transverse to $\partial W = S^1$, so W is a Liouville domain.

Let $f : W \to \mathbb{R}$ be given by $f(x, y) = \frac{1}{4}(x^2 + y^2)$. Then f is Morse and constant on ∂W. Choose the standard Riemannian metric on $\mathbb{R}^2$. Then $X = \nabla f$, hence X is gradient-like for f and thus W is a Weinstein domain.

This example extends in the obvious way to the closed unit ball in $\left(\mathbb{R}^{2n}, \sum_{j=1}^n dx_j \wedge dy_j\right)$.

Example 2.5. One interesting way of finding Liouville domains is to consider positive regions of convex hypersurfaces.[2]

Indeed, let $\Sigma^{2n} \subseteq (M^{2n+1}, \xi)$ be a closed, oriented convex hypersurface in a contact manifold. Then Σ has a neighborhood contactomorphic to $\left(\mathbb{R}_{(t)} \times \Sigma, \ker(u\,dt + \beta)\right)$, where $u : \Sigma \to \mathbb{R}$ and $\beta \in \Omega^1(\Sigma)$ do not depend on t. The contact condition $\alpha \wedge (d\alpha)^n > 0$ on Σ implies that the form

$$\theta = (d\beta)^{n-1} \wedge (u\,d\beta + n\,\beta \wedge du)$$

is a volume form on Σ. Recall (see Chapter 2 of [2]) that the characteristic foliation of Σ is directed by the vector field Y satisfying

$$i_Y\theta = \beta \wedge (d\beta)^{n-1}.$$

Next, let $R_+ = \{u > 0\}$ be the positive region of the convex hypersurface. On R_+, we can write the contact form as $dt + \beta'$ with $\beta' = \frac{\beta}{u}$. The contact condition implies that $(d\beta')^n > 0$ on Σ, hence β' is a Liouville form on R_+. Let X be the Liouville vector field with respect to β'. One can show that $X = nuY$.

Finally, recall that the characteristic foliation on a convex hypersurface is transverse to the dividing set. Since the dividing set is $\{u = 0\}$ and since $X = nuY$, if we consider $R_+^\varepsilon = \{u \geq \varepsilon > 0\}$ for some small ε, the Liouville vector field X will be outwardly transverse to ∂R_+^ε.

[2]For details on this example and convex hypersurfaces in general, see [11].

It is natural to wonder if there are examples of Liouville domains that are not Weinstein domains. Such things exist, but they are relatively nontrivial to construct. In order to do so, we'll use the fact that Weinstein domains have a significant restriction on their topology, in terms of handle decompositions.

Proposition 2.6. *Let (W^{2n}, ω, X, f) be a Weinstein domain. The index of each critical point of f does not exceed n.*

Proof. Let ϕ_t be the flow of X. As previously remarked, since $\mathcal{L}_X\omega = \omega$, the flow satisfies $\phi_t^*\omega = e^t\omega$.
Let p be a critical point of f. Let

$$\Lambda_p = \left\{ q \in W \ : \ \lim_{t\to\infty} \phi_t(q) = p \right\}$$

be the stable manifold of p. For any $q \in \Lambda_p$,

$$\omega_q = e^{-t}\phi_t^*\omega_q.$$

Since $\phi_t(q) \to p$ as $t \to \infty$, sending $t \to \infty$ in the above equation gives $\omega_q = 0 \cdot \omega_p = 0$. So ω vanishes on Λ_p, hence Λ_p is an isotropic submanifold of W. It is a standard fact from symplectic linear algebra that an isotropic subspace of a symplectic vector space has dimension at most half the dimension of the full vector space; thus, $\dim \Lambda_p \leq n$. Since X is (upward) gradient-like for f, this implies that the index of f at p is $\leq n$. □

Corollary 2.7. *For $n \geq 2$, the boundary of a $2n$-dimensional Weinstein domain is connected.*

Proof. By the previous proposition, every $2n$-dimensional Weinstein domain admits a handle decomposition involving k-handles for $k \leq n$. Observe that a $2n$-manifold has disconnected boundary only if it contains a $(2n-1)$-handle, since the belt sphere $\partial\mathbb{D}^\ell$ of a handle $\mathbb{D}^k \times \mathbb{D}^\ell$ is disconnected if and only if $\ell = 1$. If $n \geq 2$, a Weinstein domain has no $(2n-1)$-handles, and hence has connected boundary. □

With this fact, one can find Liouville domains which do not admit Weinstein structures by constructing a Liouville domain with disconnected boundary. The first example was given by McDuff [3], and was further generalized in [5], [6], and [8]. Such examples are still relatively rare. Note the importance of *outward* tranversality of the Liouville vector field at the boundary: it is much easier to find compact manifolds with disconnected boundary where the Liouville vector field flows in from one boundary component and out the other. For example, take the standard unit ball from Example 2.4 and remove a smaller open ball centered at the origin.

Example 2.8 ([5], [6], [7])**.** Here is an example of a Liouville domain with disconnected boundary, hence an example of a Liouville domain which does not admit a Weinstein structure. The geometric picture behind this example rests on the existence of a so called *Anosov flow* on a 3-manifold, which lets one construct and combine two oppositely oriented contact structures to get the desired Liouville structure, see [5]. What follows is a particular instance of this, with full computations included for completeness.

We will build a symplectic manifold of the form $[0,1] \times M^3$ in a few steps.

Step 1: Constructing M^3.

First, let $A \in SL_2(\mathbb{Z}) = \{A \in M_2(\mathbb{Z}) \ : \ \det A = 1\}$ with $trA > 2$. Concretely, one may consider $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Because $\det A = 1$, A^{-1} is an integer matrix, thus A descends to a diffeomorphism of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Consequently, the mapping torus

$$M := T^2 \times [0,1] \big/ \sim \quad \text{where} \quad \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), 1\right) \sim \left(A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), 0\right)$$

is a smooth, closed 3-manifold.

Step 2: Constructing two 1-forms on M^3.

The matrix A has two positive real eigenvalues, since $(trA)^2 - 4 > 0$. Since $\det A = 1$, the eigenvalues are $e^{\nu}, e^{-\nu}$ for some $\nu > 0$. Let v and w be corresponding eigenvectors, normalized so that $dx \wedge dy(v, w) = 1$, where (x, y) are standard coordinates on $\mathbb{R}^2$.

Define two 1-forms on M as follows:

$$\alpha = e^{-\nu z}\, i_v(dx \wedge dy) - e^{\nu z}\, i_w(dx \wedge dy) \quad \text{and} \quad \beta = e^{-\nu z}\, i_v(dx \wedge dy) + e^{\nu z}\, i_w(dx \wedge dy).$$

where z denotes the coordinate in the $[0, 1]$ direction of M. We first need to verify that α and β are indeed 1-forms on M; that is, that they are well-defined with respect to $\sim$. Consider a vector of the form $c_1 v + c_2 w$ at the point $(x, y, 1)$. Then

$$e^{-\nu}\, dx \wedge dy(v, c_1 v + c_2 w) = c_2 e^{-\nu}.$$

On the other hand, since $(x, y, 1) \sim (A(x, y), 0)$, the fact that

$$e^{-\nu \cdot 0}\, dx \wedge dy(v, A(c_1 v + c_2 w)) = dx \wedge dy(v, c_1 e^{\nu} v + c_2 e^{-\nu} w) = c_2 e^{-\nu}$$

shows that the form $e^{-\nu z}\, i_v(dx \wedge dy)$ is well-defined on M. A similar computation shows that $e^{\nu z}\, i_w(dx \wedge dy)$ is well-defined on M, hence α and β are both well-defined on M.

Observe that α and β satisfy the following properties.

Lemma 2.9. *With α and β as above,*

(i) $\alpha \wedge \beta = 2\, dx \wedge dy$
(ii) $d\alpha = -\nu\, dz \wedge \beta$
(iii) $d\beta = -\nu\, dz \wedge \alpha$

Proof.

(i) This follows from

$$\begin{aligned} i_v(dx \wedge dy) \wedge i_w(dx \wedge dy) &= ((i_v dx)\, dy - (i_v dy)\, dx) \wedge ((i_w dx)\, dy - (i_w dy)\, dx) \\ &= (i_v dx \cdot i_w dy - i_w dx \cdot i_v dy)\, dx \wedge dy \\ &= dx \wedge dy \end{aligned}$$

since $dx \wedge dy(v, w) = 1$.

(ii) Compute:

$$d\alpha = -\nu e^{-\nu z}\, dz \wedge i_v(dx \wedge dy) - \nu e^{\nu z}\, dz \wedge i_w(dx \wedge dy) = -\nu\, dz \wedge \beta.$$

(iii) Similar to (ii).

□

Step 3: Using the two 1-forms to define a Liouville form on $W = [0, 1] \times M^3$.

Next, consider the 4-manifold $W = [0, 1] \times M$. Let $\omega = d(s\alpha + (1 - s)\beta)$, where s denotes the $[0, 1]$ coordinate. We claim that ω is an exact symplectic form on W. Indeed, using the lemma from the previous step, we compute

$$\begin{aligned} \omega^2 &= [d(s\alpha + (1 - s)\beta)]^2 \\ &= [ds \wedge (\alpha - \beta) - \nu\, dz \wedge (s\beta + (1 - s)\alpha)]^2 \\ &= -2\nu\, ds \wedge (\alpha - \beta) \wedge dz \wedge (s\beta + (1 - s)\alpha) \\ &= 2\nu\, ds \wedge dz \wedge (2s\, dx \wedge dy + 2(1 - s)\, dx \wedge dy) \\ &= 2\nu\, ds \wedge dz \wedge dx \wedge dy \\ &> 0. \end{aligned}$$

Step 4: Showing the Liouville vector field is transverse to ∂W.

Let $X = (2s-1)\,\partial_s - \frac{1}{\nu}\,\partial_z$. We claim that X is the Liouville vector field for ω. Indeed,

$$\begin{aligned} i_X\omega &= i_X\left[ds \wedge (\alpha - \beta) - \nu\, dz \wedge (s\beta + (1-s)\alpha)\right] \\ &= (2s-1)(\alpha-\beta) + (s\beta + (1-s)\alpha) \\ &= s\alpha + (1-s)\beta. \end{aligned}$$

Finally, X is clearly outwardly transverse to ∂W because of the $(2s-1)\,\partial_s$ component. Thus, W is a Lioville domain with disconnected boundary, hence a Liouville domain which is not Weinstein.

3. Weinstein Handles

In this section, we discuss one way of building new Weinstein domains from old ones by attaching handles of a certain type. This construction is originally from [4].

Consider $\mathbb{R}^{2n}$ with the standard symplectic form $\omega = \sum_{j=1}^n dx_i \wedge dy_i$. For each $k \in \{0, 1, \ldots, n\}$, define

$$X_k = \sum_{j=1}^{n-k} \left(\frac{1}{2}x_j\,\partial_{x_j} + \frac{1}{2}y_j\,\partial_{y_j}\right) + \sum_{j=n-k+1}^{n} \left(2x_j\,\partial_{x_j} - y_j\,\partial_{y_j}\right).$$

The vector field X_k has k "hyperbolic"-type components in the (x_j, y_j)-directions for $j \in \{n-k+1, n\}$. One can check that X_k is a Liouville vector field:

$$i_{X_k}\omega = \sum_{j=1}^{n-k} \left(\frac{1}{2}x_j\,dy_j - \frac{1}{2}y_j\,dx_j\right) + \sum_{j=n-k+1}^{n} (2x_j\,dy_j + y_j\,dx_j)$$

and so

$$\begin{aligned} di_{X_k}\omega &= d\left(\sum_{j=1}^{n-k} \left(\frac{1}{2}x_j\,dy_j - \frac{1}{2}y_j\,dx_j\right) + \sum_{j=n-k+1}^{n} (2x_j\,dy_j + y_j\,dx_j)\right) \\ &= \sum_{j=1}^{n} dx_j \wedge dy_j = \omega. \end{aligned}$$

Also observe that X_k is the gradient of the (Morse) function

$$f_k = \sum_{j=1}^{n-k} \left(\frac{1}{4}x_j^2 + \frac{1}{4}y_j^2\,dx_j\right) + \sum_{j=n-k+1}^{n} \left(x_j^2 - \frac{1}{2}y_j^2\right)$$

hence X_k is gradient-like for f_k.

Consider the closed unit k-disc in the $(y_{n-k+1}, \ldots, y_n)$-region. This disc is isotropic with respect to ω. For $\varepsilon > 0$, we consider the following tubular neighborhood of this disc:

$$H_k^\varepsilon := \left\{\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n-k} y_j^2 \le \varepsilon\right\} \cap \left\{\sum_{j=n-k+1}^{n} y_j^2 \le 1\right\}.$$

Then H_k^ε is diffeomorphic to $\mathbb{D}^k \times \mathbb{D}^{2n-k}$, hence we view H_k^ε as a k-handle. Finally, observe that the Liouville vector field X_k is transverse to ∂H_k^ε:

$$2\left(\sum_{j=1}^{n} x_j\,dx_j + \sum_{j=1}^{n-k} y_j\,dy_j\right)(X_k) = \sum_{j=1}^{n-k}(x_j^2 + y_j^2) + 4\sum_{j=n-k+1}^{n} x_j^2 = \varepsilon + 3\sum_{j=n-k+1}^{n} x_j^2 \ge \varepsilon > 0.$$

and

$$\left(\sum_{j=n-k+1}^{n} y_j\,dy_j\right)(X_k) = -\sum_{j=n-k+1}^{n} y_j^2 = -1 < 0.$$

In particular, X_k flows in one boundary component and out the other, see Figure 1.

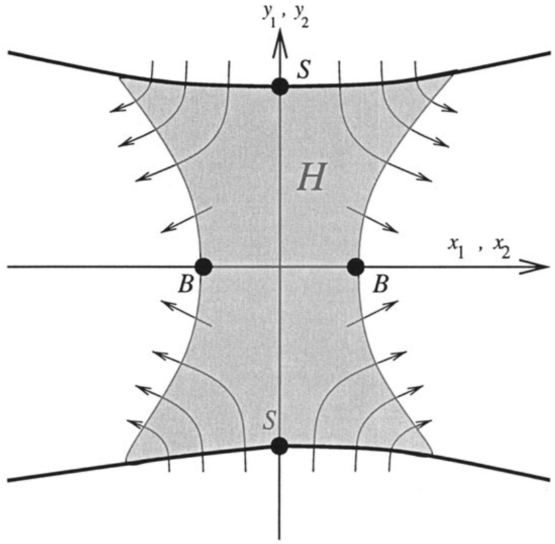

FIGURE 1. The standard (critical) 4-dimensional 2-handle, i.e., the case $n = 2$ and $k = 2$, courtesy of [9]. The same schematic picture applies to handles of all index.

Definition 3.1. We call H_k^ε the **standard Weinstein k-handle**, where there is an implicit dependence on n. A Weinstein handle is **critical** if $k = n$, and **subcritical** otherwise.

We would like to attach a Weinstein handle to a Liouville domain in a way such that the Liouville vector fields glue together nicely to give a new Liouville domain. The key to making this work is to use standard neighborhood theorems of isotropic submanifolds. Recall that an isotropic submanifold of a contact manifold is one in which the tangent spaces lie in the contact structure at all points.

Lemma 3.2. *The attaching sphere $S = S^{k-1} = \partial\mathbb{D}^k \times \{0\}$ of $H_k^\varepsilon = \mathbb{D}^k \times \mathbb{D}^{2n-k}$ is an isotropic sphere in the contact manifold $\partial_- H_k^\varepsilon$, where by $\partial_- H_k^\varepsilon$ we mean the part of the boundary of H_k^ε for which the Liouville vector field flows inward.*

Proof. The contact form on this part of the boundary is

$$\alpha = i_{X_k}\omega\Big|_{\partial_- H_k^\varepsilon} = \left(\sum_{j=1}^{n-k}\left(\frac{1}{2}x_j\,dy_j - \frac{1}{2}y_j\,dx_j\right) + \sum_{j=n-k+1}^{n}(2x_j\,dy_j + y_j\,dx_j)\right)\Bigg|_{\partial_- H_k^\varepsilon}.$$

The attaching sphere S is given by the equations $\sum_{j=n-k+1}^{n} y_j^2 = 1$ and $x_1 = \cdots x_n = y_1 = \cdots y_{n-k} = 0$. So

$$\alpha\mid_S = \sum_{j=n-k+1}^{n} y_j\,dx_j.$$

Since $dx_j = 0$ along S, it follows that $\alpha\mid_S = 0$ and hence S is isotropic. □

In order to carefully describe how to attach Weinstein handles along isotropic spheres, one needs to further discuss *conformal symplectic normal bundles* of isotropic manifolds in contact manifolds. The key here is that the attaching sphere S in H_k^ε is an isotropic sphere with trivial conformal symplectic normal bundle, and hence must be attached a long an isotropic sphere in the boundary of the Liouville domain which also has trivial conformal symplectic normal bundle. Moreover, a choice of trivialization of this conformal symplectic normal bundle gives rise to a choice of framing. With this data, we get the following.

Theorem 3.3 ([4], [10])**.** *Let W be a Liouville domain of dimension $2n$, and let $S^{k-1} \subseteq \partial W$ be an embedded isotropic sphere with trivial conformal symplectic normal bundle. The manifold W' obtained by attaching an index k Weinstein handle along S^{k-1} admits the structure of a Liouville domain. Moreover, if W is a Weinstein domain, then W' also admits the structure of a Weinstein domain.*

For a discussion on conformal symplectic normal bundles and the the proof of the above theorem, see Chapter 6 of [2], or [4].

4. Legendrian Surgery

One can also interpret the attachment of Weinstein handles in the context of performing surgery on a contact manifold. First, we need to introduce a slight generalization of Liouville and Weinstein domains.

Definition 4.1. A **Liouville cobordism** is a compact symplectic manifold $(W, \omega = d\lambda, X)$ with boundary $\partial W = \partial_- W \sqcup \partial_+ W$, together with a globally defined Liouville vector field X which is outwardly transverse to $\partial_+ W$ and inwardly transverse to $\partial_- W$. One can define **Weinstein cobordisms** in the same way.

Example 4.2. A Liouville domain is a Liouville cobordism with $\partial_- W = \emptyset$.

Example 4.3. Let $(M, \xi = \ker\alpha)$ be a contact manifold and let $(W = [-1,1]_t \times M, d(e^t\alpha))$ be its symplectization. Note that

$$i_{\partial_t} d(e^t\alpha) = i_{\partial_t}(e^t\, dt \wedge \alpha + e^t\, d\alpha) = e^t\alpha$$

and so the Liouville vector field is ∂_t, which is inwardly transverse to $\{-1\} \times M$ and outwardly transverse to $\{1\} \times M$.

Next, we focus our attention on critical Weinstein handles, i.e., Weinstein handles with index $k = n$. For a critical handle, the attaching sphere S is an isotropic submanifold of the boundary of maximal dimension, hence Legendrian. It turns out that the conformal symplectic normal bundle of a Legendrian submanifold has rank 0, hence is canonically trivialized, and thus S comes with a canonical framing.

Suppose (M^{2n+1}, ξ) is a contact manifold and $S^n \subset M$ is an embedded Legendrian sphere. By the above, there is a canonical framing and hence one can perform *Legendrian surgery*, that is, elementary surgery along this sphere. This surgery corresponds to attaching an n-handle to the cylindrical cobordism $W := [-1,1] \times M$ along $\partial_+ W := \{1\} \times M$, producing a new cobordism W' such that $\partial_- W' = M$ and $\partial_+ W'$ is the surgered manifold. Since $[-1,1]_t \times M$ carries an exact symplectic structure given by $\omega = d(e^t\alpha)$ and Liouville vector field ∂_t, W is a Liouville (and in fact, Weinstein) cobordism. The following theorem then essentially follows from Theorem 3.3.

Theorem 4.4 ([4])**.** *Let $S^n \subset (M, \xi)$ be an embedded Legendrian sphere in a contact manifold M. Let M' be the manifold obtained by elementary surgery along S^n. Then the cobordism W from M to M' obtained by attaching a critical Weinstein handle to $[-1,1] \times M$ along $S^n \subset \{1\} \times M$ is a Weinstein cobordism. In particular, M' is a contact manifold and the contact structure on M' coincides with ξ outside the neighborhood of the surgery.*

References

1. Cieliebak, K., Eliashberg, Y. *From Stein to Weinstein and Back.* 2010
2. Geiges, H.. *An Introduction to Contact Topology.* 2008.
3. McDuff, D.. *Symplectic manifolds with contact type boundaries.* 1991.
4. Weinstein, A.. *Contact surgery and symplectic handlebodies.* 1990
5. Mitsumatsu, Y.. *Anosov flows and non-Stein symplectic manifolds.* 1995
6. Geiges, H.. *Examples of Symplectic* 4*-manifolds with Disconnected Boundary of Contact Type.* 1995
7. Mori, A.. *Reeb foliations on* S^5 *and contact* 5*-manifolds violatin the Thurston-Bennequin inequality.* 2009
8. Massot, P., Niederkruger, K., Wendl, C.. *Weak and strong fillability of higher dimensional contact manifolds.* 2011

9. Ozbagci, B., Stipsicz, A.. *Surgery on Contact* 3*-Manifolds and Stein Surfaces*. 2004.
10. Eliashberg, Y.. *Topological characterization of Stein manifolds of dimension* > 2. 1990.
11. Giroux, E.. *Convexite en topologie de contact*. 1991.