Intuition for Convergence and Divergence

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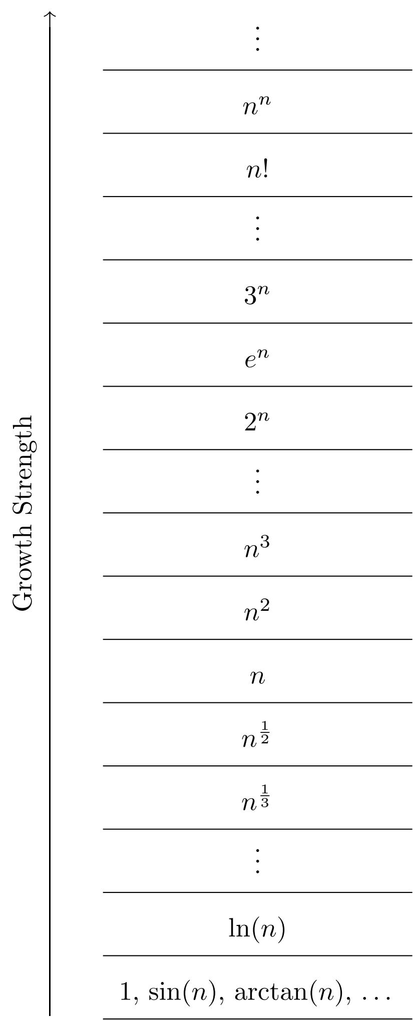
Here I'm going to try to describe how to intuitively think about whether or not a series converges. This is not necessary in order to do well on an exam, but (in my opinion) it certainly helps. So here's the deal:

In the world of functions, there is sort of a hierarchy of how fast functions grow. At the very bottom are constant functions (e.g. $f(x) = 5$) and *almost* constant functions[1] (e.g. $f(x) = \sin x$). These functions don't grow as $x \to \infty$. Above these growth-less functions are functions like $f(x) = x$, $f(x) = x^2$, etc. We can consider this growth hierarchy in terms of functions of n; i.e., sequences. Here it is, in all its glory:



At the bottom of the growth hierarchy are constant functions and other bounded functions, and anything that doesn't grow infinitely big as $n \to \infty$. Next up is $\ln(n)$. It turns out that $\ln(n)$, while it does grow to infinity, grows *so slowly* that it is slower than any positive power of n. So for example, $\ln(n)$ grows slower

[1] Bounded functions.

than $n^{0.0000001}$. Above $\ln(n)$ we see that every successive positive power of n grows faster than the one before it. Furthermore, every exponential function (like e^n) grows faster than *any polynomial in n*. Likewise, $n!$ grows faster than any exponential function, and n^n grows even faster[2].

So why is all of this so useful? Here's why: the gowth hierarchy determines the convergence or divergence of basically any sequence or series you can cook up. This is because *every* series out there arises as a ratio of two functions. Explicitly, every series is of the form $\sum \frac{u_n}{v_n}$, and will converge only when v_n grows sufficiently faster than u_n. So for example, the series $\sum \frac{n}{e^n}$ converges because e^n grows way faster than n. This is where you have to be careful, though. It would be really nice if we could tell if a series converges or not by just checking if the bottom part of the ratio is higher in the growth hierarchy than the top, but **this isn't good enough.** The classic example of this is the harmonic series, $\sum \frac{1}{n}$: the bottom (n) certainly grows faster than the top (1), but *it doesn't grow fast enough.* This is where the "sufficiently faster" idea comes in to play. So there are some very delicate ratios that act as sort of a tipping point for convergence and divergence. The harmonic series is one of these tipping points: $\sum \frac{1}{n}$ diverges, but $\sum \frac{1}{n^{1.0000000001}}$ converges. It takes some practice to get used to what sorts of ratios converge and diverge, but that's where the intuition lies.

The reason why this is so powerful is that we can take a super complicated series and view it from a high-level perspective in terms of the growth-hierarchy. Say I give you the following series:

$$\sum \frac{n^2 + 2n + \sin n + \log n}{n^4 + 300n^3 - 25n + 4000}$$

If we look at the top, the growth hierarchy tells us that the only term that really matters (in the infinite!) is n^2. Similarly, on the bottom, the only term that matters for infinite growth is n^4. We can intuitively conclude that our series will behave the same as:

$$\sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$$

which converges. Therefore, our original series converges! Formally, we could use the limit comparison test to show that the behavior of these two series is the same. In fact, this is a great way to cook up potential comparison series for the limit comparison test. Now, let's get crazy:

$$\sum \frac{1 + \sin(\ln(4n^{200} - \pi n^2 + 34))^{\arctan(\frac{1}{n})} + n^{10^{10^{10}}} - \cos(n!) + 4^n - \sqrt{n+3}}{\ln(\ln(\ln(n^3 + \frac{1}{n}))) + \sin^2(e^n)\cos^2(e^{2n})n^2 + 3^{2n} - n^{\frac{1}{10}}}$$

As ugly as this is, some careful inspection will reveal that the most dominant term on the top is 4^n, and the most dominant term on the bottom is 3^{2n}. Therefore, this insane series behaves the same as:

$$\sum \frac{4^n}{3^{2n}} = \sum \frac{4^n}{9^n} = \sum \left(\frac{4}{9}\right)^n$$

which is a convergent geometric series. So it converges! Showing this formally would take a lot of tedious applications of limit comparison, direct comparison, L'Hopital's Rule, and probably some other stuff, but the intuitive behavior is staring you right in the face.

[2]Whether you realize it or not, you already know all of this growth-hierarchy stuff — it's basically L'Hopital's Rule!