INFINITE SERIES PROBLEMS

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Problems

1. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{2^n}.$$

2. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{2n+\sin(2n)}{\sqrt{n^8+1}}.$$

3. Does the following series converge or diverge? If it converges, evaluate it.
$$\sum_{n=2}^{\infty} \frac{2^{3-n}+(-1)^{3n}}{3^{2n+1}}.$$

4. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+(\ln n)^2}.$$

5. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{n^2\,3^n}{(2n)!}.$$

6. Does the following series converge or diverge?
$$\sum_{n=2}^{\infty} \frac{(\ln n)^2}{\sqrt{n^3-1}}.$$

7. Does the following series converge or diverge?
$$\sum_{n=2}^{\infty} \left(1+\frac{1}{\ln n}\right)^{\cos(1/n)}.$$

8. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{\sin^4(n)}{n^4}.$$

9. Does the following series converge absolutely, converge conditionally, or diverge?
$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}+\sqrt{n+1}}.$$

10. Does the following series converge absolutely, converge conditionally, or diverge?
$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n!}}.$$

11. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}.$$

12. Does the following series converge absolutely, converge conditionally, or diverge?
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}+n^{\frac{1}{3}}+1}.$$

13. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{-\frac{1}{n^2}}}.$$

14. Does the following series converge or diverge?
$$\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}.$$

Solutions

1. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{2^n}.$$

Solution. Note that

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\ln(n+2)}{2^{n+1}} \cdot \frac{2^n}{\ln(n+1)} \\ &= \lim_{n\to\infty} \frac{1}{2} \frac{\ln(n+2)}{\ln(n+1)} \\ &= \lim_{x\to\infty} \frac{1}{2} \frac{\ln(x+2)}{\ln(x+1)} \\ (L'H) &= \lim_{x\to\infty} \frac{1}{2} \frac{x+1}{x+2} \\ &= \frac{1}{2}. \end{aligned}$$

Since $\frac{1}{2} < 1$, by the ratio test, this series converges. □

2. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{2n + \sin(2n)}{\sqrt{n^8+1}}.$$

Solution. We run limit comparison with

$$\frac{n}{\sqrt{n^8}} = \frac{n}{n^4} = \frac{1}{n^3}.$$

Note that

$$\lim_{n\to\infty} \frac{\frac{2n+\sin(2n)}{\sqrt{n^8+1}}}{\frac{n}{\sqrt{n^8}}} = \lim_{n\to\infty} \frac{2 + \frac{\sin(2n)}{n}}{\sqrt{1+\frac{1}{n^8}}} = 2.$$

The above limit computation follows from the following squeeze theorem argument:

$$\left| \frac{\sin(2n)}{n} \right| \leq \frac{1}{n} \to 0.$$

Since $0 < 2 < \infty$, by the limit comparison test, the series $\sum_{n=1}^{\infty} \frac{2n+\sin(2n)}{\sqrt{n^8+1}}$ behaves the same as the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Since the latter is a convergent p-series, the former converges as well. □

3. **Does the following series converge or diverge? If it converges, evaluate it.**

$$\sum_{n=2}^{\infty} \frac{2^{3-n} + (-1)^{3n}}{3^{2n+1}}.$$

Solution. Note that

$$\frac{2^{3-n}+(-1)^{3n}}{3^{2n+1}} = \frac{2^3 2^{-n}}{3^{2n+1}} + \frac{((-1)^3)^n}{3^{2n+1}} = 8\frac{1}{2^n \cdot 3 \cdot (3^2)^n} + \frac{(-1)^n}{3 \cdot (3^2)^n}$$
$$= \frac{8}{3}\left(\frac{1}{18}\right)^n + \frac{1}{3}\left(\frac{-1}{9}\right)^n.$$

Thus,

$$\sum_{n=2}^{\infty} \frac{2^{3-n}+(-1)^{3n}}{3^{2n+1}} = \sum_{n=2}^{\infty} \frac{8}{3}\left(\frac{1}{18}\right)^n + \sum_{n=2}^{\infty} \frac{1}{3}\left(\frac{-1}{9}\right)^n.$$

Both of these series are convergent geometric series because each ratio is between -1 and 1, and thus the original series converges. Furthermore, the series converges to

$$\sum_{n=2}^{\infty} \frac{2^{3-n}+(-1)^{3n}}{3^{2n+1}} = \frac{\frac{8}{3}\left(\frac{1}{18}\right)^2}{1-\frac{1}{18}} + \frac{\frac{1}{3}\left(\frac{-1}{9}\right)^2}{1+\frac{1}{9}}.$$

□

4. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+(\ln n)^2}.$$

Solution. Note that $1+(\ln n)^2$ is an increasing sequence, since $\ln n$ is increasing and compositions of increasing functions are increasing. Furthermore, note that

$$\lim_{n\to\infty} \frac{1}{1+(\ln n)^2} = 0.$$

Thus, by the alternating series test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+(\ln n)^2}$$

converges. □

5. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{n^2\, 3^n}{(2n)!}.$$

Solution. Because of the factorial, we use the ratio test. Observe that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2\, 3^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{n^2\, 3^n} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} \cdot \frac{3(n+1)^2}{n^2}$$
$$= \lim_{n\to\infty} \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \cdot 3\left(1+\frac{1}{n}\right)^2$$
$$= \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} \cdot 3\left(1+\frac{1}{n}\right)^2.$$

Since $3\left(1+\frac{1}{n}\right)^2 \to 3$, it follows that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot 3 = 0.$$

Since $0<1$, by the ratio test, this series converges. □

6. **Does the following series converge or diverge?**

$$\sum_{n=2}^{\infty} \frac{(\ln n)^2}{\sqrt{n^3-1}}.$$

Solution. Because logarithms grow very slowly (in particular, $\ln n << n^a$ for any $a > 0$) we expect the dominant behavior of $\sqrt{n^3} = n^{\frac{3}{2}}$ in the denominator to force this series to converge. However, because the logarithms on top *do* grow to infinity, we can't expect that running limit comparison with $\frac{1}{n^{\frac{3}{2}}}$ is going to be conclusive. Thus, we will run limit comparison with $\frac{1}{n^{\frac{5}{4}}}$, since $1 < \frac{5}{4} < \frac{3}{2}$. This slight difference in exponent accounts for the minor growth contribution of the logarithms in the numerator.

Note that

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{\sqrt{n^3-1}}}{\frac{1}{n^{\frac{5}{4}}}} = \lim_{n\to\infty} \frac{(\ln n)^2}{\frac{\sqrt{n^3-1}}{\sqrt{n^{\frac{5}{2}}}}} = \lim_{n\to\infty} \frac{(\ln n)^2}{\sqrt{n^{\frac{1}{2}} - n^{-\frac{5}{2}}}}.$$

We could use L'Hopital on this limit as written, but it would probably be difficult with the expression in the denominator. Instead, we'll do some clever algebra to break this into simpler pieces:

$$\lim_{n\to\infty} \frac{(\ln n)^2}{\sqrt{n^{\frac{1}{2}} - n^{-\frac{5}{2}}}} = \lim_{n\to\infty} \frac{1}{\frac{\sqrt{n^{\frac{1}{2}} - n^{-\frac{5}{2}}}}{\sqrt{(\ln n)^4}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^{\frac{1}{2}}}{(\ln n)^4} - \frac{1}{n^{\frac{5}{2}}(\ln n)^4}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\left(\frac{n^{\frac{1}{8}}}{\ln n}\right)^4 - \frac{1}{n^{\frac{5}{2}}(\ln n)^4}}}.$$

Note that $\frac{1}{n^{\frac{5}{2}}(\ln n)^4} \to 0$ as $n \to \infty$. Next, by L'Hopital's rule,

$$\lim_{n\to\infty} \frac{n^{\frac{1}{8}}}{\ln n} = \lim_{x\to\infty} \frac{x^{\frac{1}{8}}}{\ln x} = \lim_{x\to\infty} \frac{1}{8} x^{\frac{1}{8}} = \infty.$$

Putting all of this together implies

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{\sqrt{n^3-1}}}{\frac{1}{n^{\frac{5}{4}}}} = 0.$$

Note that this is an endpoint case of limit comparison. Because we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{4}}}$ is a convergent p-series and because the above limit is 0, we can conclude by limit comparison that the series

$$\sum_{n=2}^{\infty} \frac{(\ln n)^2}{\sqrt{n^3-1}}$$

converges as well. □

7. **Does the following series converge or diverge?**

$$\sum_{n=2}^{\infty} \left(1 + \frac{1}{\ln n}\right)^{\cos(1/n)}.$$

Solution. Note that

$$\lim_{n\to\infty} \left(1 + \frac{1}{\ln n}\right)^{\cos(1/n)} = \lim_{n\to\infty} (1+0)^{\cos(0)} = 1^1 = 1.$$

Since $1 \neq 0$, by the divergence test, the series diverges. □

8. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{\sin^4(n)}{n^4}.$$

Solution. Note that

$$0 \leq \frac{\sin^4(n)}{n^4} \leq \frac{1}{n^4}.$$

Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

is a convergent p-series, by the direct comparison test, the series

$$\sum_{n=1}^{\infty} \frac{\sin^4(n)}{n^4}$$

converges as well. □

9. **Does the following series converge absolutely, converge conditionally, or diverge?**

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}+\sqrt{n+1}}.$$

Solution. First, note that $(-1)^2n = ((-1)^2)^n = 1$ so that

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}+\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}.$$

Next, note that

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} &= \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} \\ &= 1. \end{aligned}$$

Since $0 < 1 < \infty$, by the limit comparison test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}+\sqrt{n+1}}$ behaves the same as the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Since the latter is a divergent p-series, the former diverges as well. □

10. **Does the following series converge absolutely, converge conditionally, or diverge?**

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n!}}.$$

Solution. Because of the factorial, we begin with the ratio test. Recall that the ratio test gives *absolute* convergence if successful. Note that

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^2}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{(-1)^n n^2} \right| \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \sqrt{\frac{n!}{(n+1)!}} \\ &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \cdot \sqrt{\frac{1}{n+1}} \\ &= 1 \cdot 0 \\ &= 0. \end{aligned}$$

Since $0 < 1$, this series converges absolutely. □

11. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}.$$

Solution. Note that $\lim_{n\to\infty} e^{\frac{1}{n}} = e^0 = 1$. Motivated by this, we run limit comparison with $\frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{\frac{e^{\frac{1}{n}}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} e^{\frac{1}{n}} = 1$$

the limit comparison test implies that the series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ behaves the same as the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the latter is a convergent p-series, the former converges as well. □

12. **Does the following series converge absolutely, converge conditionally, or diverge?**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n + \sqrt{n} + n^{\frac{1}{3}} + 1}.$$

Solution. First, we consider the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n + \sqrt{n} + n^{\frac{1}{3}} + 1} \right| = \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n} + n^{\frac{1}{3}} + 1}.$$

Note that

$$\lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n}+n^{\frac{1}{3}}+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{\sqrt{n}} + \frac{1}{n^{\frac{2}{3}}} + \frac{1}{n}} = 1.$$

Since $0 < 1 < \infty$, by the limit comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}+n^{\frac{1}{3}}+1}$ behaves the same as $\sum_{n=1}^{\infty} \frac{1}{n}$. Since the latter is a divergent p-series, the former diverges as well.

Next, we consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n + \sqrt{n} + n^{\frac{1}{3}} + 1}.$$

Note that

$$\lim_{n\to\infty} \frac{1}{n + \sqrt{n} + n^{\frac{1}{3}} + 1} = 0$$

and that $\frac{1}{n+\sqrt{n}+n^{\frac{1}{3}}+1}$ is a decreasing sequence, since $n + \sqrt{n} + n^{\frac{1}{3}} + 1$ is an increasing sequence. Thus, by the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}+n^{\frac{1}{3}}+1}$ converges.

Thus, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}+n^{\frac{1}{3}}+1}$ converges conditionally. □

13. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{-\frac{1}{n^2}}}.$$

Solution. Note that

$$\lim_{n\to\infty} \frac{1}{3^{-\frac{1}{n^2}}} = \lim_{n\to\infty} 3^{\frac{1}{n^2}} = 3^0 = 1.$$

Since $\frac{1}{3^{-\frac{1}{n^2}}} \to 1$, the limit of the sequence

$$\frac{(-1)^n}{3^{-\frac{1}{n^2}}}$$

does not exist. Thus, by the divergence test, the series diverges. □

14. **Does the following series converge or diverge?**

$$\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}.$$

Solution. Because of the factorial, we use the ratio test. Note that

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{n!}\right| &= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{2^{n^2}}{2^{(n+1)^2}} \\
&= \lim_{n\to\infty} \frac{n+1}{2^{(n+1)^2-n^2}} \\
&= \lim_{n\to\infty} \frac{n+1}{2^{2n+1}} \\
&= \lim_{x\to\infty} \frac{x+1}{2^{2x+1}} \\
(L'H) &= \lim_{x\to\infty} \frac{1}{2^{2x+1} \cdot \ln 2 \cdot 2} \\
&= 0.
\end{aligned}$$

Since $0 < 1$, by the ratio test, this series converges. □