STRATEGIES FOR DETERMINING CONVERGENCE

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Suppose I'm faced with an infinite series $\sum a_n$ and some higher power wants me to determine whether or not the series converges or diverges. If I don't know what to do, here are the questions I ask myself:

1. **Is there any obvious algebraic simplification I can do?**

   If so, do it! For example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{e^{2\ln n}}.$$

2. **Do the terms look like they don't go to zero?**

   If so, try the *divergence test*. Oftentimes if they sequence looks odd or unusual, this may be the case. For example:

$$\sum_{n=1}^{\infty} \frac{n^2 + 7^n}{1 + \sin(n)} \qquad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n.$$

3. **Is the series of a special form?**

   By that I mean:

   (a) **Does it have only terms with numbers raised to nth powers (or rather, powers that are linear in n)?**

   If so, it might be *geometric*. For example:

$$\sum_{n=1}^{\infty} \frac{3^n + (-1)^n}{2^{2n}} \qquad \sum_{n=37}^{\infty} e^{-3n+1}2^{2n+84}.$$

   (b) **Is it a p-series?**

   If so...it's a p-series. For example:

$$\sum_{n=1}^{\infty} \frac{1}{n^{25}} \qquad \sum_{n=3} n^{-0.73}.$$

   (c) **Do I see a $(-1)^n$ or some other alternating behavior?**

   If so, try the *alternating series test*. For example:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(\ln(n^2 + \ln n))} \qquad \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2 + 1}.$$

   If the alternating series test fails, it probably failed because the terms don't go to zero, so try the *divergence test* again. For example:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n + 1}.$$

   (d) **Is it a rational function, or am I asked to evaluate it?**

   If so, it might be *telescoping*. Do partial fractions if necessary and compute partial sums.

4. **Do I see a factorial, or a combination of exponentials and polynomials?**

   If so, try the *ratio test*. For example,

$$\sum_{n=1}^{\infty} \frac{n^4}{(2n)!} \qquad \sum_{n=1}^{\infty} \frac{n^2+1}{4^n}.$$

5. **Do I see a complicated exponential expression?**

   If so, try the root test. For example,

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

6. **Are there obvious dominant terms that I can isolate, or is there fluff I can throw away?**

   If so, try one of the comparison tests. Generate a term to compare to by isolating the dominant behavior. For example:

$$\sum_{n=3}^{\infty} \frac{n^2+2n+1}{\sqrt{n^5-\ln n+2}} \qquad \sum_{n=1}^{\infty} \frac{n+\sin n}{n^{\frac{3}{2}}+\sqrt{n}}.$$

7. **Are there bounded trig functions or logs?**

   If so, maybe try direct comparison by bounding the trig functions or using a comparison like $\ln n \leq n^a$. For example:

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^5} \qquad \sum_{n=1}^{\infty} \frac{\ln(n+56)}{n^{1.001}}.$$

8. **Does the sequence look like a function I can integrate, or are there logs?**

   If so, try the integral test. The integral test together with a substitution can be helpful for many series involving logs, if nothing else works.

9. **All of that failed, now what.**

   A last ditch resort is to go back and try the limit comparison test again, but with a clever or non-obvious choice. If you know anything about Taylor polynomials, this can sometimes be enlightening. For example,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right).$$