STRATEGIES FOR DETERMINING CONVERGENCE

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Suppose I’m faced with an infinite series \( \sum a_n \) and some higher power wants me to determine whether or not the series converges or diverges. If I don’t know what to do, here are the questions I ask myself:

1. **Is there any obvious algebraic simplification I can do?**
   
   If so, do it! For example:
   
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{e^{2 \ln n}}.
   \]

2. **Do the terms look like they don’t go to zero?**

   If so, try the divergence test. Oftentimes if they sequence looks odd or unusual, this may be the case. For example:
   
   \[
   \sum_{n=1}^{\infty} \frac{n^2 + 7^n}{1 + \sin(n)} \quad \text{and} \quad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n.
   \]

3. **Is the series of a special form?**

   By that I mean:

   (a) **Does it have only terms with numbers raised to \( n \)th powers (or rather, powers that are linear in \( n \))?**

   If so, it might be geometric. For example:
   
   \[
   \sum_{n=1}^{\infty} \frac{3^n + (-1)^n}{2^{2n}} \quad \text{and} \quad \sum_{n=37}^{\infty} e^{-3n+1} 2^{2n+84}.
   \]

   (b) **Is it a \( p \)-series?**

   If so...it’s a \( p \)-series. For example:
   
   \[
   \sum_{n=1}^{\infty} \frac{1}{n^{25}} \quad \text{and} \quad \sum_{n=3}^{\infty} n^{-0.73}.
   \]

   (c) **Do I see a \( (-1)^n \) or some other alternating behavior?**

   If so, try the alternating series test. For example:
   
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(\ln(n^2 + \ln n))} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2 + 1}.
   \]

   If the alternating series test fails, it probably failed because the terms don’t go to zero, so try the divergence test again. For example:
   
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n + 1}.
   \]

   (d) **Is it a rational function, or am I asked to evaluate it?**

   If so, it might be telescoping. Do partial fractions if necessary and compute partial sums.
4. Do I see a factorial, or a combination of exponentials and polynomials?

If so, try the ratio test. For example,

\[ \sum_{n=1}^{\infty} \frac{n^4}{(2n)!} \quad \sum_{n=1}^{\infty} \frac{n^2 + 1}{4^n}. \]

5. Do I see a complicated exponential expression?

If so, try the root test. For example,

\[ \sum_{n=1}^{\infty} \left( \frac{1 + \frac{1}{n}}{n} \right)^{-n^2} \quad \sum_{n=1}^{\infty} \frac{1}{n^n}. \]

6. Are there obvious dominant terms that I can isolate, or is there fluff I can throw away?

If so, try one of the comparison tests. Generate a term to compare to by isolating the dominant behavior. For example:

\[ \sum_{n=3}^{\infty} \frac{n^2 + 2n + 1}{\sqrt{n^5 - \ln n + 2}} \quad \sum_{n=1}^{\infty} \frac{n + \sin n}{n^2 + \sqrt{n}}. \]

7. Are there bounded trig functions or logs?

If so, maybe try direct comparison by bounding the trig functions or using a comparison like \( \ln n \leq n^a \). For example:

\[ \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^5} \quad \sum_{n=1}^{\infty} \frac{\ln(n + 56)}{n^{1.001}}. \]

8. Does the sequence look like a function I can integrate, or are there logs?

If so, try the integral test. The integral test together with a substitution can be helpful for many series involving logs, if nothing else works.

9. All of that failed, now what.

A last ditch resort is to go back and try the limit comparison test again, but with a clever or non-obvious choice. If you know anything about Taylor polynomials, this can sometimes be enlightening. For example,

\[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n^2} \right). \]