How to Evaluate a Surface Integral

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There are two kinds of surface integrals. Surface integrals of *scalar* valued functions $f(x, y, z)$ and surface integrals of *vector* valued functions $\mathbf{F}(x, y, z)$. Here is a very brief summary of how to directly evaluate such integrals.

Scalar Surface Integrals

Evaluate $\iint_S f(x, y, z)\, dS$.

1. **Parametrize the surface S.**

   Define a parametrization $\mathbf{r}(u, v)$ of the surface S (see above section about surface parametrizations). Note that this includes a description of the bounds for u and v.

2. **Calculate dS.**

   In the scalar case, $dS = \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv$. You need to calculate the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, take their cross product, and compute magnitude of that vector.

3. **Convert the function $f(x, y, z)$ into a uv-expression.**

   Using the parametrization $\mathbf{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$, plug the expressions into $f(x, y, z)$ to get the function f in terms of u and v: $f(u, v)$.

4. **Combine everything together and compute the double integral.**

   The $u - v$ bounds from your parametrization become the bounds for the double integral, $f(x, y, z)$ becomes $f(u, v)$, and dS becomes what you calculated in step 2.

$$\iint_S f(x, y, z)\, dS = \iint_{(u,v)\text{ region}} f(u, v)\ \|\mathbf{r}_u \times \mathbf{r}_v\|\ du\, dv$$

Vector Surface Integrals

Evaluate $\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}$.

1. **Parametrize the surface S.**

   Define a parametrization $\mathbf{r}(u, v)$ of the surface S (see above section about surface parametrizations). Note that this includes a description of the bounds for u and v.

2. **Calculate $d\mathbf{S}$.**

   In the vector case, $d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$. You need to calculate the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ and take their cross product. Note that you do not need to take the length of the cross product, unlike the scalar case.

(Equivalently,

$$d\mathbf{S} = \mathbf{n}\,\|\mathbf{r}_u \times \mathbf{r}_v\|\; du\,dv$$

where **n** is a *unit* normal vector to the surface; I don't like writing it this way.)

3. **Convert the function $\mathbf{F}(x, y, z)$ into a uv-expression.**

   Using the parametrization $\mathbf{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$, plug the expressions into $f(x, y, z)$ to get the function f in terms of u and v: $\mathbf{F}(u, v)$.

4. **Combine everything together and compute the double integral.**

   The $u - v$ bounds from your parametrization become the bounds for the double integral, $\mathbf{F}(x, y, z)$ becomes $\mathbf{F}(u, v)$, and $d\mathbf{S}$ becomes what you calculated in step 2. Compute the dot product of $\mathbf{F}$ and $d\mathbf{S}$ and do the double integral.

$$\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S} = \iint_{(u,v)\text{ region}} \mathbf{F}(u, v) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$