**How to Evaluate a Surface Integral**

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There are two kinds of surface integrals. Surface integrals of *scalar* valued functions $f(x, y, z)$ and surface integrals of *vector* valued functions $\mathbf{F}(x, y, z)$. Here is a very brief summary of how to directly evaluate such integrals.

**Scalar Surface Integrals**

Evaluate $\int_S f(x, y, z) \, dS$.

1. **Parametrize the surface $S$.**

   Define a parametrization $\mathbf{r}(u, v)$ of the surface $S$ (see above section about surface parametrizations). Note that this includes a description of the bounds for $u$ and $v$.

2. **Calculate $dS$.**

   In the scalar case, $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv$. You need to calculate the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, take their cross product, and compute magnitude of that vector.

3. **Convert the function $f(x, y, z)$ into a $uv$-expression.**

   Using the parametrization $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, plug the expressions into $f(x, y, z)$ to get the function $f$ in terms of $u$ and $v$: $f(u, v)$.

4. **Combine everything together and compute the double integral.**

   The $u - v$ bounds from your parametrization become the bounds for the double integral, $f(x, y, z)$ becomes $f(u, v)$, and $dS$ becomes what you calculated in step 2.

$$\int_S f(x, y, z) \, dS = \int_{(u, v) \text{ region}} f(u, v) \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv$$

**Vector Surface Integrals**

Evaluate $\int_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}$.

1. **Parametrize the surface $S$.**

   Define a parametrization $\mathbf{r}(u, v)$ of the surface $S$ (see above section about surface parametrizations). Note that this includes a description of the bounds for $u$ and $v$.

2. **Calculate $d\mathbf{S}$.**

   In the vector case, $d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$. You need to calculate the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ and take their cross product. Note that you do not need to take the length of the cross product, unlike the scalar case.
(Equivalently,
\[ dS = n \| r_u \times r_v \| \, du \, dv \]
where \( n \) is a \textit{unit} normal vector to the surface; I don’t like writing it this way.)

3. \textbf{Convert the function} \( F(x, y, z) \) \textbf{into a} \textit{uv-expression}.
Using the parametrization \( r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \), plug the expressions into \( f(x, y, z) \) to get the function \( f \) in terms of \( u \) and \( v \): \( F(u, v) \).

4. \textbf{Combine everything together and compute the double integral}.
The \( u - v \) bounds from your parametrization become the bounds for the double integral, \( F(x, y, z) \) becomes \( F(u, v) \), and \( dS \) becomes what you calculated in step 2. Compute the dot product of \( F \) and \( dS \) and do the double integral.
\[
\iint_S F(x, y, z) \cdot dS = \iint_{(u, v) \ \text{region}} F(u, v) \cdot (r_u \times r_v) \, du \, dv
\]