Green's Theorem

Joseph Breen

Introduction

One of the most important theorems in vector calculus is *Green's Theorem.* Here are some notes that discuss the intuition behind the statement, subtleties about how and when to apply it, and some applications.

First, let's give a formal statement of the theorem:

> **Green's Theorem**: *Let C be a simple*[1]*, closed, positively-oriented differentiable curve in $\mathbb{R}^2$, and let D be the region inside C. If P and Q are continuously differentiable real-valued functions defined on D, then:*
>
> $$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Before getting into the nitty-gritty details of the theorem, it helps to have some intuition as to what it even says.

Intuition

To better understand the physical meaning behind Green's Theorem, it helps to consider it's alternate (but equivalent) form:

> **Green's Theorem**: *Let C be a simple, closed, positively-oriented differentiable curve in $\mathbb{R}^2$, and let D be the region inside C. If $\mathbf{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$ is a continuously differentiable vector field defined on D, then:*
>
> $$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$$

While this "vector" version of Green's Theorem is perhaps more difficult to use computationally, it is easier to understand conceptually. Here's why:

On the left-hand side, we have the line integral of a vector field around a closed curve. Recall that a line integral of a vector field measures how well the vector field "travels" with the curve. So a line integral of a

[1] This means that C doesn't cross over itself.

vector field around a closed curve measures how much the vector field *circulates* around the curve. Therefore, the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is sometimes referred to as the **circulation** of $\mathbf{F}$ around C.

Now let's think about the right-hand side. Recall that the **curl** of a vector field is $\nabla \times \mathbf{F}$, and in a vague way, it measures the infinitely small "rotation" of $\mathbf{F}$. So one way to think about the curl is that it is the *infinitely small circulation of* $\mathbf{F}$. In particular, the quantity $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ [2] is the circulation of $\mathbf{F}$ around an infinitely small circle. Hence, the double integral

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

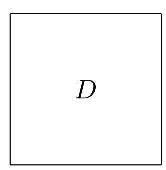
is the sum of all the infinitely small circulations in the region D.

Let's think about this in the context of Green's Theorem. The left-hand side is the *macroscopic circulation* of $\mathbf{F}$ around the boundary of D, and the right-hand side is the sum of *macroscopic circulation* inside D. Succinctly, Green's Theorem says that

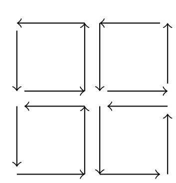
$$\textit{macroscopic circulation} = \textit{sum of microscopic circulation}$$

Cool!

Another way to think about this is that if we take a region D and add up the microscopic circulations inside D, they all cancel each other out on the inside and all we're left with is the circulations on the border. The following picture demonstrates this concept. Consider the following region D:

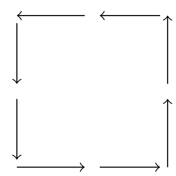


We can fill up D with microscopic loops (here we will only do four square loops, one in each corner), all traveling counter-clockwise:



For each arrow *inside* the square, there is a second arrow right next to it, pointing the opposite direction. These cancel each other out, leaving us with just the boundary:

[2] Remember, this is just the length of the vector $\nabla \times \mathbf{F}$.

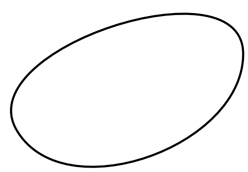


This is exactly the macroscopic circulation around the region D!

Orientation

Now it's time to discuss one of the trickiest parts about correctly using Green's Theorem: *orientation.* Look back at the statement of the theorem — one of the assumptions about the boundary curve is that it is *positively oriented.* What does this mean?

Let's consider a closed curve in the plane:



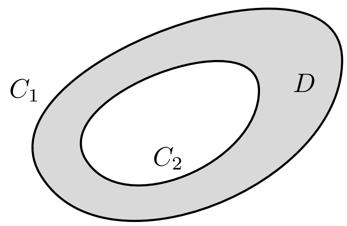
Because Green's Theorem deals with a line integral around the boundary, which implies that we travel along the curve in some direction, we need to pay attention to how the curve is parametrized. We can either travel clockwise along the curve, or counter-clockwise:

These two possible ways of traversing the curve give two possible *orientations* of the curve. The left curve (the clockwise direction) has a *negative* orientation, and the right curve (the counter-clockwise direction) has a *positive* orientation. Another way to think about positive orientation is that in travelling along the curve, the interior of the region is to the *left.* This interpretation will come in handy soon.

Green's Theorem only works when the curve is oriented positively — if we use Green's Theorem to evaluate a line integral oriented negatively, our answer will be off by a minus sign!

Multiple Boundary Curves

For simple curves (curves with no holes), orientation and how it applies to Green's Theorem is pretty easy. The issue gets a little trickier if we have a region with multiple boundary curves. For example, suppose we have the following donut-like region D, bounded by curves C_1 and C_2:

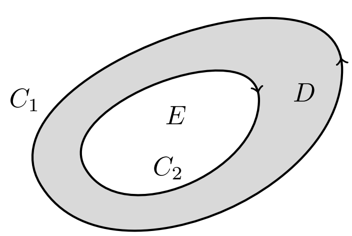


Say we want to calculate the some sort of line integral around the boundary of D. This would be the line integral around $\partial D = C_1 \cup C_2$, the union of the two curves:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

We might be tempted to use Green's Theorem to turn this line integral into a double integral over D, but we have to be careful. $\partial D = C_1 \cup C_2$ isn't a "simple, closed" curve, so we can't use the theorem as stated.

The first thing we need to do to tackle this problem is to be explicit about the orientations of C_1 and C_2. We want them to be positively oriented, which means the interior of the region needs to be to our *left* when we travel along the curves. This means that we will move along C_1 in a counter-clockwise fashion, and along C_2 in a clockwise fashion (take note that in this case, positive orientation does *not* mean counter-clockwise!). Also, let's label the inside of the donut E:



Here's something that we *can* do: if we think of the interior of C_1 as one egg-like region — namely, the region $D \cup E$ — we can use Green's Theorem:

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_{D \cup E} (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA \tag{1}$$

Next, notice that we can split the double integral on the right side of this equation into two separate double integrals: one over D, and one over E:

$$\iint_{D\cup E} (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA + \iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \tag{2}$$

Consider the double integral over E. E is a region bounded by a simple, closed curve C_2, so we can use Green's Theorem again! But we have to be careful — since C_2 is going clockwise, it is oriented negatively *relative to* E. This will give us a negative sign in the equation:

$$\iint_E (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = -\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} \tag{3}$$

Putting equations (1), (2), and (3) together gives us:

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA - \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} \qquad \Rightarrow \qquad \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \tag{4}$$

But notice that:

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1\cup C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

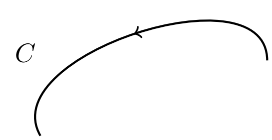
So rewriting (4) gives us:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \tag{5}$$

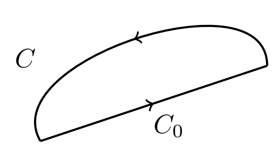
This is exactly the statement of Green's Theorem! So we *can* use Green's Theorem for a region with holes / multiple boundary curves, *as long as C_1 and C_2 are both oriented positively*, which means that the outer curve travels counter-clockwise, and the inner curve travels clockwise (so that D is to the left of both curves).

Nonclosed Curves

Another context in which Green's Theorem is useful is in calculating line integrals over curves which are not necessarily closed. Suppose I give you the following curve C:



We want to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. We can't use Green's Theorem as is, because C is not a closed curve. But, we could add in a line segment C_0 to close it off:



The curve $C \cup C_0$ is closed, so we can apply Green's Theorem:

$$\oint_{C\cup C_0} \mathbf{F}\cdot d\mathbf{r} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$$

Then we can split up the line integral on the left hand side:

$$\int_C \mathbf{F}\cdot d\mathbf{r} + \int_{C_0} \mathbf{F}\cdot d\mathbf{r} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$$

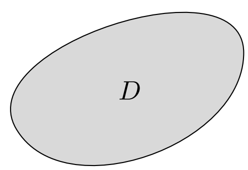
Remember that the original line integral we wanted to solve was $\int_C \mathbf{F}\cdot d\mathbf{r}$. We can subtract the integral over C_0 from both sides to get:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA - \int_{C_0} \mathbf{F}\cdot d\mathbf{r}$$

In summary, we can use Green's Theorem to calculate line integrals of an arbitrary curve by closing it off with a curve C_0 and subtracting off the line integral over this added segment.

Area

Another application of Green's Theorem is that is gives us one way to calculate areas of regions. Let's start with a region D:



We can calculate the area of D by integrating 1 over D as a double integral:

$$A = \iint_D 1\,dx\,dy \tag{6}$$

Look back to the double integral in the first statement of Green's Theorem:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

If we can find functions P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then by (6), we can use Green's Theorem to calculate the area of D using a line integral around the boundary! Here a couple possibilities for P and Q:

1. Suppose that $Q(x, y) = x$ and $P(x, y) = 0$. Then $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = 0$. Thus,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$$

Then using (6), we have:

$$\begin{aligned} A &= \iint_D 1\,dx\,dy \\ &= \iint_D (1-0)\,dx\,dy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \\ &= \oint_{\partial D} P\,dx + Q\,dy \\ &= \oint_{\partial D} 0\,dx + x\,dy \\ &= \oint_{\partial D} x\,dy \end{aligned}$$

2. Another possibility is if $Q(x, y) = 0$ and $P(x, y) = -y$. Then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-1) = 1$$

And a similar computation shows us that:

$$\begin{aligned} A &= \iint_D 1\,dx\,dy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \\ &= \oint_{\partial D} P\,dx + Q\,dy \\ &= -\oint_{\partial D} y\,dx \end{aligned}$$

So Green's Theorem lets us calculate areas using line integrals:

$$A = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx \tag{7}$$