FUN SEQUENCE AND SERIES PROBLEMS

Joseph Breen

Solutions will be periodically added at the bottom.

Problems

1. Evaluate
$$\lim_{n\to\infty} \frac{n!}{n^n}.$$
2. Evaluate
$$\lim_{n\to\infty} \frac{n^n}{(2n)!}.$$
3. Evaluate
$$\lim_{n\to\infty} \frac{2^{n^2}}{n!}.$$
4. Evaluate
$$\lim_{n\to\infty} \frac{2^{n^2}}{(2n)!}.$$
5. Evaluate
$$\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n}.$$
(This one is really hard, so don't feel bad if you can't get it. Hint: take the log.)
6. For each statement, answer true or false. If the answer is true, give some justification. If the answer is false, give a counter example.
   (a) If a_n and b_n are sequences such that $0 \leq a_n \leq b_n$ and b_n converges, then a_n converges.
   (b) If a_n and b_n are sequences such that a_n and b_n both diverge, then $a_n + b_n$ diverges.
7. Does the following infinite series converge?
$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{2^n + n^2}.$$
8. Does the following infinite series converge?
$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}.$$
9. Does the following infinite series converge?
$$\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln n}}.$$
10. Does the following infinite series converge?
$$\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln(\ln n)}}.$$

11. For each statement, answer true or false. If the answer is true, give some justification. If the answer is false, give a counter example.

    (a) Suppose a sequence a_n converges. Then the series $\sum a_n$ converges.

    (b) Suppose that $a_n \leq b_n$ and we know the series $\sum b_n$ converges. Then $\sum a_n$ converges.

    (c) Suppose the partial sum sequence of a series is given by $S_N = \frac{1}{\sqrt{N}}$. Then the series converges.

    (d) Suppose that $a_n \geq 0$ and $\sum a_n$ converges. Then $\sum a_n^2$ converges.

    (e) Suppose the sequence b_n converges and the series $\sum a_n$ has partial sum sequence S_N such that $0 \leq S_N \leq b_N$. Then the series $\sum a_n$ converges.

12. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$$

13. Does the following infinite series converge?

$$\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$$

    (*This one is hard. If you try the ratio test, it won't work!*)

14. Suppose that a_n is a sequence such that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{3}$.

    (a) Does the series $\sum n^2 a_n$ converge?

    (b) Does the series $\sum 4^n a_n$ converge?

    (c) Suppose we also know that $a_n > 0$. Does the series $\sum \frac{a_n}{1+a_n}$ converge?

    (d) Suppose we also know that $1 > a_n > 0$. Does the series $\sum \frac{a_n}{1-a_n}$ converge?

15. True or false: Suppose that $\sum a_n$ is a series with partial sum sequence S_N, and suppose that

$$\lim_{N\to\infty} \left| \frac{S_{N+1}}{S_N} \right| = \frac{1}{2}.$$

    Then $\sum a_n$ converges.

16. Does the following infinite series converge?

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{13}{n^2+n+1}\right).$$

17. Does the following series converge absolutely, converge conditionally, or does it diverge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n + (\ln n)^3}.$$

18. Does the following series converge absolutely, converge conditionally, or does it diverge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+\frac{1}{n}}}.$$

Solutions

1. **Evaluate**
$$\lim_{n\to\infty} \frac{n!}{n^n}.$$

Solution. Note that
$$0 \le \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} \le \frac{1\cdot n\cdot n\cdots n}{n\cdot n\cdot n\cdots n} = \frac{1}{n}.$$
Since $\frac{1}{n} \to 0$, it follows that
$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$
□

2. **Evaluate**
$$\lim_{n\to\infty} \frac{n^n}{(2n)!}.$$

Solution. Note that
$$0 \le \frac{n^n}{(2n)!} = \frac{n^n}{(2n)(2n-1)\cdots(n+1)\cdot n!} \le \frac{n^n}{n\cdot n\cdots n\cdot n!} = \frac{n^n}{n^n\cdot n!} = \frac{1}{n!}.$$
Since $\frac{1}{n!} \to 0$, this implies
$$\lim_{n\to\infty} \frac{n^n}{(2n)!} = 0.$$
□

3. **Evaluate**
$$\lim_{n\to\infty} \frac{2^{n^2}}{n!}.$$

Solution. Note that
$$\begin{aligned}\frac{2^{n^2}}{n!} &= \frac{2^{n^2}}{n\cdot(n-1)\cdots 1}\\ &\ge \frac{2^{n^2}}{n\cdot n\cdots n}\\ &= \frac{2^{n^2}}{n^n}\\ &= \left(\frac{2^n}{n}\right)^n.\end{aligned}$$
Note that $\frac{2^n}{n} \ge 2$ for all $n \ge 1$. Thus,
$$\left(\frac{2^n}{n}\right)^n \ge 2^n.$$
In summary,
$$\frac{2^{n^2}}{n!} \ge 2^n.$$
Since the right hand side goes to ∞ as $n \to \infty$, it follows that
$$\lim_{n\to\infty} \frac{2^{n^2}}{n!} = \infty.$$
□

4. **Evaluate**

$$\lim_{n\to\infty} \frac{2^{n^2}}{(2n)!}.$$

Solution. Note that

$$\begin{aligned}\frac{2^{n^2}}{(2n)!} &= \frac{2^{n^2}}{(2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}\\ &\geq \frac{2^{n^2}}{(2n)(2n)(2n)\cdots(2n)\cdot(2n)\cdot(2n)}\\ &= \frac{2^{n^2}}{(2n)^{2n}}\\ &= \frac{(2^n)^n}{(4n^2)^n}\\ &= \left(\frac{2^n}{4n^2}\right)^n.\end{aligned}$$

By L'Hopital's rule,

$$\lim_{n\to\infty} \frac{2^n}{4n^2} = \infty.$$

Thus,

$$\lim_{n\to\infty} \left(\frac{2^n}{4n^2}\right)^n = \infty.$$

Since

$$\frac{2^{n^2}}{(2n)!} \geq \left(\frac{2^n}{4n^2}\right)^n$$

it follows that

$$\lim_{n\to\infty} \frac{2^{n^2}}{(2n)!} = \infty$$

as well.

In fact, this same argument shows that

$$\lim_{n\to\infty} \frac{a^{n^2}}{(kn)!} = \infty$$

for any $a > 1$ and any integer $k \geq 1$. Indeed,

$$\frac{a^{n^2}}{(kn)!} \geq \frac{a^{n^2}}{(kn)^{kn}} = \left(\frac{a^n}{k^k n^k}\right)^n.$$

L'Hopital's rule implies that $\frac{a^n}{k^k n^k} \to \infty$ as $n \to \infty$, and so $\left(\frac{a^n}{k^k n^k}\right)^n \to \infty$ as well. By the squeeze theorem,

$$\lim_{n\to\infty} \frac{a^{n^2}}{(kn)!} = \infty.$$

□

5. **Evaluate**

$$\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n}.$$

Solution. Let $L_n = \frac{(n!)^{\frac{1}{n}}}{n}$. Then

$$\begin{aligned}
\ln L_n &= \ln\left(\frac{(n!)^{\frac{1}{n}}}{n}\right) = \ln\left(\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}\right) = \frac{1}{n}\ln\left(\frac{n!}{n^n}\right) \\
&= \frac{1}{n}\ln\left(\frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n}\right) \\
&= \frac{1}{n}\left(\ln\left(\frac{1}{n}\right) + \ln\left(\frac{2}{n}\right) + \cdots + \ln\left(\frac{n}{n}\right)\right) \\
&= \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right).
\end{aligned}$$

Observe that the above sum is a right Riemann sum for the integral $\int_0^1 \ln(x)\,dx$. Thus, as $n \to \infty$, $\ln L_n \to \int_0^1 \ln(x)\,dx$. So

$$\ln\left(\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\right) = \int_0^1 \ln(x)\,dx.$$

Integrating by parts,

$$\int \ln x\,dx = x\ln x - x.$$

The integral is improper at 0, so taking limits gives

$$\int_0^1 \ln(x)\,dx = (1\cdot\ln 1 - 1) - \lim_{x\to 0^+}(x\ln x - x) = -1 - \lim_{x\to 0^+} x\ln x.$$

The final limit is indeterminate of type $0\cdot\infty$, so we need to use L'Hopital's rule:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

Thus,

$$\int_0^1 \ln x\,dx = -1 - 0 = -1.$$

All of this leads to

$$\ln\left(\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\right) = -1.$$

Exponentiating both sides gives

$$\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} = e^{-1}$$

which is the final answer. □

6. **For each statement, answer true or false. If the statement is true, give an explanation for why. If the statement is false, give an example to prove this.**

   (a) If a_n and b_n are sequences such that $0 \le a_n \le b_n$ and b_n converges, then a_n converges.

   (b) If a_n and b_n are sequences such that a_n and b_n both diverge, then $a_n + b_n$ diverges.

Solution.

(a) False. Consider $b_n = 2$ and $a_n = 1 + (-1)^n$.

(b) False. Consider $a_n = n$ and $b_n = -n$.

□

7. **Does the following infinite series converge?**

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{2^n + n^2}.$$

Solution. Recall that if $x \geq 0$, $0 \leq \arctan x \leq \frac{\pi}{2}$. Thus,

$$0 \leq \frac{\arctan(n)}{2^n + n^2} \leq \frac{\frac{\pi}{2}}{2^n + n^2} \leq \frac{\frac{\pi}{2}}{2^n}.$$

Observe that

$$\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{2^n}$$

is a geometric series with ratio $1/2$, and thus converges. Thus, by direct comparison, the series

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{2^n + n^2}$$

converges.

Note that other comparisons are also possible and fruitful. For example,

$$0 \leq \frac{\arctan(n)}{2^n + n^2} \leq \frac{\frac{\pi}{2}}{n^2}$$

□

8. **Does the following infinite series converge?**

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}.$$

Solution. Note that $\ln n \geq \ln 3 > 1$ for $n \geq 3$ (since $\ln e = 1$ and $\ln(x)$ is increasing). Thus, for $n \geq 3$,

$$0 \leq \frac{1}{(\ln n)^n} \leq \frac{1}{(\ln 3)^n} = \left(\frac{1}{\ln 3}\right)^n.$$

Since

$$\sum_{n=3}^{\infty} \left(\frac{1}{\ln 3}\right)^n$$

is a convergent geometric series, by direct comparison, the series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}.$$

converges. □

9. **Does the following infinite series converge?**

$$\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln n}}.$$

Solution. We use the integral test. Let $f(x) = \frac{1}{(\ln x)^{\ln x}}$. Then f is decreasing, positive, and continuous on the interval $[3, \infty)$. Consider

$$\int_3^\infty \frac{1}{(\ln x)^{\ln x}}\, dx.$$

Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$ and so $dx = e^u\, du$. Since $u \to \infty$ as $x \to \infty$, we have

$$\int_3^\infty \frac{1}{(\ln x)^{\ln x}}\, dx = \int_{\ln 3}^\infty \frac{e^u}{u^u}\, du = \int_{\ln 3}^\infty \left(\frac{e}{u}\right)^u\, du.$$

Note that

$$0 \le \left(\frac{e}{u}\right)^u \le \left(\frac{e}{3}\right)^u$$

for $u \ge 3$. Since $e < 3$, then integral

$$\int_{\ln 3}^\infty \left(\frac{e}{3}\right)^u\, du$$

converges. One can show this carefully by just integrating and evaluating the limit. Thus, by integral comparison, the integral

$$\int_{\ln 3}^\infty \left(\frac{e}{u}\right)^u\, du$$

converges. Thus, by the integral test, the series

$$\sum_{n=3}^\infty \frac{1}{(\ln n)^{\ln n}}$$

converges. $\square$

10. **Does the following infinite series converge?**

$$\sum_{n=3}^\infty \frac{1}{(\ln n)^{\ln(\ln n)}}.$$

Solution. Like in the previous problem, we begin using the integral test. Let $f(x) = \frac{1}{(\ln x)^{\ln(\ln x)}}$. Then f is decreasing, positive, and continuous on the interval $[3, \infty)$. Consider

$$\int_3^\infty \frac{1}{(\ln x)^{\ln(\ln x)}}\, dx.$$

Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$ and so $dx = e^u\, du$. Since $u \to \infty$ as $x \to \infty$, we have

$$\int_3^\infty \frac{1}{(\ln x)^{\ln(\ln x)}}\, dx = \int_{\ln 3}^\infty \frac{e^u}{u^{\ln u}}\, du = \int_{\ln 3}^\infty \left(\frac{e}{u^{\frac{\ln u}{u}}}\right)^u\, du.$$

Consider

$$L = \lim_{u \to \infty} u^{\frac{\ln u}{u}}.$$

We have

$$\ln L = \lim_{u \to \infty} \frac{(\ln u)^2}{u} = 0$$

by L'Hopital's rule. Thus, $L = 1$. Since $u^{\frac{\ln u}{u}} \to 1$, this implies that for u sufficiently large,

$$\frac{e}{u^{\frac{\ln u}{u}}} \ge 1$$

and so

$$\left(\frac{e}{u^{\frac{\ln u}{u}}}\right)^u \to \infty.$$

In particular, the integral

$$\int_{\ln 3}^{\infty} \left(\frac{e}{u^{\frac{\ln u}{u}}}\right)^u \, du$$

diverges. Thus, by the integral test,

$$\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln(\ln n)}}$$

diverges. □

11. **For each statement, answer true or false. If the answer is true, give some justification. If the answer is false, give a counter example.**

   (a) Suppose a sequence a_n converges. Then the series $\sum a_n$ converges.

   (b) Suppose that $a_n \leq b_n$ and we know the series $\sum b_n$ converges. Then $\sum a_n$ converges.

   (c) Suppose the partial sum sequence of a series is given by $S_N = \frac{1}{\sqrt{N}}$. Then the series converges.

   (d) Suppose that $a_n \geq 0$ and $\sum a_n$ converges. Then $\sum a_n^2$ converges.

   (e) Suppose the sequence b_n converges and the series $\sum a_n$ has partial sum sequence S_N such that $0 \leq S_N \leq b_N$. Then the series $\sum a_n$ converges.

Solution.

(a) FALSE. Consider $a_n = 1$. Then the *sequence* a_n converges to 1, but the series $\sum 1$ diverges.

(b) FALSE. In order to use direct comparison, the sequences have to be positive! A counterexample is $b_n = 0$ and $a_n = -1$. $\sum 0$ converges but $\sum -1$ diverges.

(c) TRUE. By definition of an infinite series,

$$\sum a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{\sqrt{N}} = 0$$

so the series converges (to 0).

(d) TRUE. Since $\sum a_n$ converges, the sequence a_n eventually tends to 0. Thus, for n sufficiently large, $a_n \leq 1$. Therefore

$$0 \leq a_n^2 \leq a_n \cdot 1 = a_n$$

Since $\sum a_n$ converges, by direct comparison, $\sum a_n^2$ converges.

(e) FALSE. Let $b_n = 3$ and $a_n = 1 + (-1)^n$. Then the sequence b_n converges to 3. The partial sum sequence of the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1 + (-1)^n$$

is

$$S_N = \{0, 1, 0, 1, 0, 1, \dots\}$$

which satisfies $0 \leq S_N \leq b_N$. But $\lim_{N\to\infty} S_N$ does not exist, so $\sum a_n$ diverges.

□

12. **Does the following series converge?**

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$$

Solution. We apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{((n+1)!)^3}{(3(n+1))!}\cdot\frac{(3n)!}{(n!)^3}\right| \\
&= \lim_{n\to\infty}\left|\frac{((n+1)!)^3}{(n!)^3}\cdot\frac{(3n)!}{(3n+3)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}\right| \\
&= \frac{1}{27}.
\end{aligned}$$

Since $\frac{1}{27} < 1$, by the ratio test, the series converges.

□

13. **Does the following infinite series converge?**

$$\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$$

Solution. The point of this question is that if you try the ratio test (after all, there is a factorial), then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ and so the test is inconclusive.

Getting a feel for this series is quite elusive. Here is one way. Note that

$$\int_1^n \ln x\, dx \le \ln(2) + \ln(3) + \cdots + \ln(n)$$

since the right-hand side is given by an upper Riemann sum for $ln(x)$, which is an increasing function. Drawing a picture with rectangles will help convince you of this. Evaluating the integral on the left,

$$\int_1^n \ln x\, dx = x\ln x - x\Big|_1^n = n\ln n - n + 1.$$

On the right, we can simplify that sum as

$$\ln(2) + \ln(3) + \cdots + \ln(n) = \ln(2\cdot 3\cdots n) = \ln(n!).$$

Thus,

$$n\ln n - n + 1 \le \ln(n!).$$

Exponentiating both sides of this inequality, we have

$$e^{n\ln n - n + 1} \le n!.$$

On the left, we have

$$e^{n\ln n}e^{-n}e^1 = e^{\ln(n^n)}e^{-n}e = n^n e^{-n} e.$$

Thus,

$$n^n e^{-n} e \le n!.$$

Rearranging this inequality gives

$$\frac{e^n\cdot n!}{n^n} \ge e.$$

This implies that $\lim_{n\to\infty}\frac{e^n\cdot n!}{n^n} \ne 0$. By the divergence test, the series diverges. □

14. **Suppose that a_n is a sequence such that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{3}$.**

   (a) Does the series $\sum n^2 a_n$ converge?

   (b) Does the series $\sum 4^n a_n$ converge?

   (c) Suppose we also know that $a_n > 0$. Does the series $\sum \frac{a_n}{1+a_n}$ converge?

   (d) Suppose we also know that $1 > a_n > 0$. Does the series $\sum \frac{a_n}{1-a_n}$ converge?

Solution.

(a) Converges. Note that

$$\lim_{n\to\infty} \left| \frac{(n+1)^2 a_{n+1}}{n^2 a_n} \right| = \frac{1}{3} < 1$$

so that $\sum n^2 a_n$ converges by the ratio test.

(b) Diverges. Note that

$$\lim_{n\to\infty} \left| \frac{4^{n+1} 2 a_{n+1}}{4^n a_n} \right| = 4 \cdot \frac{1}{3} > 1$$

so that $\sum 4^n a_n$ diverges by the ratio test.

(c) Converges. Since $\sum a_n$ converges by assumption by the ratio test, and since $a_n > 0$, we have

$$0 \leq \frac{a_n}{1+a_n} \leq \frac{a_n}{1} = a_n.$$

So by direct comparison, $\sum \frac{a_n}{1+a_n}$ converges.

(d) Converges. We run limit comparison with $\frac{a_n}{1-a_n}$ and a_n. Note that

$$\lim_{n\to\infty} \frac{\frac{a_n}{1-a_n}}{a_n} = \lim_{n\to\infty} \frac{1}{1-a_n} = 1.$$

The last equality follows from the fact that $a_n \to 0$ by the divergence test. Since $0 < 1 < \infty$, by limit comparison, $\sum \frac{a_n}{1-a_n}$ behaves the same as $\sum a_n$. Since the latter converges by assumption, the former converges as well.

□

15. **True or false: Suppose that $\sum a_n$ is a series with partial sum sequence S_N, and suppose that**

$$\lim_{N\to\infty} \left| \frac{S_{N+1}}{S_N} \right| = \frac{1}{2}.$$

Then $\sum a_n$ converges.

Solution. This is true. Since $\lim_{N\to\infty} \left|\frac{S_{N+1}}{S_N}\right| = \frac{1}{2}$, by the ratio test, the *series*

$$\sum_{N=1}^{\infty} S_N$$

converges. In particular, by the divergence test, $\lim_{N\to\infty} S_N = 0$. Thus,

$$\sum a_n = \lim_{N\to\infty} S_N = 0$$

so that $\sum a_n$ converges, and in particular, converges to 0. □

16. **Does the following infinite series converge?**

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{13}{n^2+n+1}\right).$$

Solution. We run limit comparison against the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

By L'Hopital's rule,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln\left(1+\frac{13}{n^2+n+1}\right)}{\frac{1}{n^2}} &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{13}{x^2+x+1}\right)}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{13}{x^2+x+1}} \cdot \frac{-13}{(x^2+x+1)^2} \cdot (2x+1)}{\frac{-2}{x^3}} \\
&= \frac{1}{1+\frac{13}{x^2+x+1}} \cdot \frac{13(2x^4+x^3)}{2(x^2+x+1)^2} \\
&= \frac{1}{1+0} \cdot 13.
\end{aligned}$$

Since $0 < 13 < \infty$, by limit comparison the series

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{13}{n^2+n+1}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

behave the same. Since the latter series is a convergent p-series, it follows that

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{13}{n^2+n+1}\right)$$

converges. □

17. **Does the following series converge absolutely, converge conditionally, or does it diverge?**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(\ln n)^3}.$$

Solution. First, we consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n+(\ln n)^3}.$$

Note that

$$\lim_{n\to\infty} \frac{\frac{1}{n+(\ln n)^3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+(\ln n)^3} = \lim_{n\to\infty} \frac{1}{1+\frac{(\ln n)^3}{n}} = 1.$$

Since $0 < 1 < \infty$, by limit comparison, the series $\sum_{n=1}^{\infty} \frac{1}{n+(\ln n)^3}$ behaves the same as the series $\sum_{n=1}^{\infty} \frac{1}{n}$. Since the latter is a divergent p-series, $\sum_{n=1}^{\infty} \frac{1}{n+(\ln n)^3}$ diverges.

Next, we consider the original series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(\ln n)^3}.$$

Since $\frac{1}{n+(\ln n)^3}$ is a sequence which is positive, decreasing, and tending to 0, by the alternating series test,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(\ln n)^3}$$

converges.

Since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(\ln n)^3}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n+(\ln n)^3}$ diverges, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+(\ln n)^3}$$

converges conditionally. □

18. **Does the following series converge absolutely, converge conditionally, or does it diverge?**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+\frac{1}{n}}}.$$

Solution. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}.$$

We run limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$. Note that

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}}.$$

Since $\ln\left(n^{\frac{1}{n}}\right) = \frac{\ln n}{n} \to 0$, it follows that

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1.$$

Since $0 < 1 < \infty$, by limit comparison, $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ behaves the same as $\sum_{n=1}^{\infty} \frac{1}{n}$. Since the latter is a divergent p-series, the former diverges as well.

Next, consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+\frac{1}{n}}}$. Since the sequence $\frac{1}{n^{1+\frac{1}{n}}}$ is positive, decreasing, and tends to 0, by the alternating series test the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+\frac{1}{n}}}$ converges.

Since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+\frac{1}{n}}}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ diverges, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+\frac{1}{n}}}$$

converges conditionally.

□