FUN POWER SERIES PROBLEMS

Joseph Breen

Solutions will be periodically added at the bottom.

Problems

1. Evaluate each of the following infinite series. Don't just show they converge, actually evaluate them!

   (a)
$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

   (b)
$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

   (c)
$$\sum_{n=0}^{\infty} \frac{n+1}{n!}$$

   (d)
$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}$$

2. Let $f(x) = x\sin(x^2)$. What is $f^{(147)}(0)$? What about $f^{(148)}(0)$?

   Hint: you probably don't want to take 147 *derivatives of* f.

3. Evaluate the following integral as an infinite series:
$$\int_0^1 \frac{1}{x^x}\, dx.$$

4. Determine whether the following series converge or diverge. (The point of these exercises is that thinking about Maclaurin series can help you!)

   (a)
$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right).$$

   (b)
$$\sum_{n=1}^{\infty} \left(1 - 3^{-\frac{1}{n}}\right).$$

   (c)
$$\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{\sqrt{n}}\right).$$

   (d)
$$\sum_{n=1}^{\infty} \left(\int_0^{\frac{1}{n}} \frac{\sin(x^2)}{x}\, dx\right).$$

(e)

$$\sum_{n=1}^{\infty} \ln\left(\frac{1}{1-\sin^2\left(\frac{1}{n}\right)}\right).$$

(f)

$$\sum_{n=1}^{\infty} \frac{\arcsin\left(\frac{1}{n}\right)}{(\ln n)^2}.$$

(g)

$$\sum_{n=1}^{\infty} \tan\left(n^{-\frac{1}{2}}\right)\sin\left(n^{-1}\right).$$

(h)

$$\sum_{n=1}^{\infty} \frac{\left[\ln\left(1+\frac{1}{n}\right)\right]\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}{n}.$$

5. Determine the interval of convergence for each of the following power series.

(a)

$$\sum_{n=1}^{\infty} \frac{n!(x-2)^{2n}}{n^n}.$$

(b)

$$\sum_{n=1}^{\infty} \frac{x^{3n}}{(2n)!}.$$

(c)

$$\sum_{n=1}^{\infty} \frac{3^n\, n!\,(x-1)^n}{n^n}.$$

6. In this long, multi-part exercise, I'm going to walk us through the derivations of some really cool formulas that compute π. In particular, we will show:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$$

and

$$\pi = \sum_{n=0}^{\infty} \frac{3(2n)!}{16^n (n!)^2 (2n+1)}.$$

Wow!

Before we prove those formulas, some comments. The first formula is beautiful in its simplicity, but it's actually a pretty bad way to estimate the value of π. For example, I had WolframAlpha evaluate the 10th partial sum of the first series:

$$\sum_{n=0}^{10} (-1)^n \frac{4}{2n+1} = 3.232...$$

Not bad; could be much better. Let's try the 20th partial sum:

$$\sum_{n=0}^{20} (-1)^n \frac{4}{2n+1} = 3.189...$$

A little better, but still not great. Let's try the 100th partial sum:

$$\sum_{n=0}^{100}(-1)^n\frac{4}{2n+1} = 3.151...$$

Getting closer, but we added 101 terms of that series together and we only have one correct digit of of π past the decimal! On the other hand, the second formula is much better at computing π. For example, I had WolframAlpha compute the 5th partial sum:

$$\sum_{n=0}^{5}\frac{3(2n)!}{16^n(n!)^2(2n+1)} = 3.141576...$$

It only takes 6 terms of that series to compute 4 correct digits of π past the decimal! Let's do the 15th partial sum:

$$\sum_{n=0}^{5}\frac{3(2n)!}{16^n(n!)^2(2n+1)} = 3.1415926535859...$$

The true decimal expansion of π is

$$\pi = 3.1415926535897...$$

Only 16 terms of that series gives us a very accurate estimate to π. Sweet!

Okay, now the exercises:

(a) The first formula is not too hard to prove. Show that

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{4}{2n+1}.$$

Hint: what is $\int_0^1 \frac{1}{1+x^2}\,dx$*? Expand the integrand into a power series.*

(b) The second formula is a bit harder. I'll break it down into multiple steps. First, let $f(x) = (1-x)^{-\frac{1}{2}}$. Show that

$$f^{(n)}(x) = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n}(1-x)^{-\frac{2n+1}{2}}.$$

Hint: start taking derivatives and try to write down the pattern carefully.

(c) Show that

$$\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n} = \frac{(2n)!}{4^n\,n!}$$

and use this to show that the MacLaurin series of $f(x)$ is

$$\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2}x^n.$$

Hint: $5\cdot 3\cdot 1 = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 4\cdot 2} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2^3\cdot 3\cdot 2\cdot 1}$.

(d) Using the previous part, show that we have the following power series expansion (on its interval of convergence):

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2}x^{2n}.$$

(e) Integrate the above power series term by term to show that, on the interval of convergence,

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} \frac{x^{2n+1}}{2n+1}.$$

(f) Show that

$$\pi = \sum_{n=0}^{\infty} \frac{3(2n)!}{16^n (n!)^2 (2n+1)}.$$

Hint: what is $\arcsin(1/2)$*?*

One last comment. Here is another crazy formula involving π and an infinite series:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{12\,(-1)^n\,(6n)!\,(545140134n + 13591409)}{(3n)!\,(n!)^3\,640320^{3n+\frac{3}{2}}}.$$

I don't know how to prove this, so don't ask me. You can read about it here:

`https://en.wikipedia.org/wiki/Chudnovsky_algorithm`.

Solutions

1. **Evaluate each of the following infinite series. Don't just show they converge, actually evaluate them!**

   (a)
$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

   (b)
$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

   (c)
$$\sum_{n=0}^{\infty} \frac{n+1}{n!}$$

   (d)
$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}$$

   Solution.

   (a) Recall that
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
   for $|x| < 1$. Differentiating both sides gives
$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$
   and then mutliplying through by x gives
$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$
   Plugging in $x = \frac{1}{2}$ yields
$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2.$$

   (b) From the previous part, we have
$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$
   Differentiating this equality gives
$$\frac{(1-x)^2 + 2x(1-x)}{(1-x)^2} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$
   and multiplying through by x, we have
$$x\frac{(1-x)^2 + 2x(1-x)}{(1-x)^2} = \sum_{n=1}^{\infty} n^2 x^n.$$

Plugging in $x = \frac{1}{2}$ gives

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} \cdot \frac{\frac{1}{4} + \frac{1}{2}}{1/4} = \frac{3}{2}.$$

(c) Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Multiplying through by x gives

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}.$$

Differentiating both sides yields

$$e^x + xe^x = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

Plugging in $x = 1$ gives

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} = e^1 + 1 \cdot e^1 = 2e.$$

(d) . This series reminds of the power series for cosine:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Evaluating at $x = 1$ gives

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} = \cos 1$$

but this is not exactly the series we want to evaluate. It's close though!

I would like to find a MacLaurin series where the coefficients are just $\frac{1}{(2n)!}$ instead of $\frac{(-1)^n}{(2n)!}$. It turns out that $f(x) = \cosh x$ does the trick. Note that

$$f^{(2n)}(0) = \cosh(0) = 1 \quad \text{and} \quad f^{(2n+1)}(0) = \sinh(0) = 0.$$

This implies that the Maclaurin series for $\cosh x$ is

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Evaluating this at 1 yields

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \cosh(1) = \frac{e + e^{-1}}{2}.$$

□

2. Let $f(x) = x \sin(x^2)$. What is $f^{(147)}(0)$? What about $f^{(148)}(0)$?

Solution. We begin by finding the MacLaurin series for f. Since

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

it follows that

$$x\sin(x^2) = x\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!}.$$

We know that the coefficient of the x^{147} term in the above series is $\frac{f^{(147)}(0)}{147!}$. Because $4n+3=147$ if $n=36$, it follows that

$$\frac{f^{(147)}(0)}{147!} = \frac{(-1)^{36}}{(2(36)+1)!} = \frac{1}{73!}.$$

Therefore,

$$f^{(147)}(0) = \frac{147!}{73!}.$$

Finally, the above power series also implies that the coefficient in front of x^{148} is 0, since $4n+3$ never equals 148. Therefore, $f^{(148)}(0)=0$. □

3. **Evaluate the following integral as an infinite series:**

$$\int_0^1 \frac{1}{x^x}\,dx.$$

Solution. We begin by expanding the integrand $\frac{1}{x^x}$ as an infinite series in the following way: write

$$\frac{1}{x^x} = \frac{1}{e^{x\ln x}} = e^{-x\ln x}.$$

Next, recall that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. This implies that

$$\frac{1}{x^x} = \sum_{n=0}^{\infty}\frac{(-x\ln x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n(\ln x)^n.$$

Now we integrate term by term:

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 x^n(\ln x)^n\,dx.$$

Note that the above integral is improper at 0. To evaluate the antiderivative $\int x^n(\ln x)^n\,dx$, we integrate by parts. Let $u=(\ln x)^n$ and $dv = x^n\,dx$. Then $du = n(\ln x)^{n-1}\frac{1}{x}\,dx$ and $v=\frac{x^{n+1}}{n+1}$, and the integration by parts formula gives

$$\int x^n(\ln x)^n\,dx = (\ln x)^n\frac{x^{n+1}}{n+1} - \frac{n}{n+1}\int x^n(\ln x)^{n-1}\,dx.$$

Thus,

$$\begin{aligned}
\int_0^1 x^n(\ln x)^n\,dx &= \lim_{R\to 0^+}(\ln x)^n\frac{x^{n+1}}{n+1}\bigg|_R^1 - \frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}\,dx \\
&= \lim_{R\to 0^+} -\frac{(\ln R)}{(n+1)R^{-(n+1)}} - \frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}\,dx \\
(L'H) &= \lim_{R\to 0^+} -\frac{\frac{1}{R}}{-(n+1)^2R^{-(n+2)}} - \frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}\,dx \\
&= -\frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}\,dx.
\end{aligned}$$

To summarize the above computation,

$$\int_0^1 x^n (\ln x)^n \, dx = -\frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} \, dx.$$

Repeating the same procedure and integrating by parts n times gives

$$\begin{aligned}
\int_0^1 x^n (\ln x)^n \, dx &= -\frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} \, dx \\
&= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+1}\right) \int_0^1 x^n (\ln x)^{n-2} \, dx \\
&\vdots \\
&= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+1}\right) \cdots \left(-\frac{1}{n+1}\right) \int_0^1 x^n \, dx \\
&= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+1}\right) \cdots \left(-\frac{1}{n+1}\right) \left(\left.\frac{x^{n+1}}{n+1}\right|_0^1 \right) \\
&= (-1)^n \frac{n!}{(n+1)^{n+1}}.
\end{aligned}$$

Thus,

$$\int_0^1 \frac{1}{x^x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n (\ln x)^n \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot (-1)^n \frac{n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}}.$$

Reindexing a gives a lovely final answer:

$$\int_0^1 \frac{1}{x^x} \, dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

□

4. **Determine whether the following series converge or diverge. (The point of these exercises is that thinking about Maclaurin series can help you!)**

   (a)
   $$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right).$$

   (b)
   $$\sum_{n=1}^{\infty} \left(1 - 3^{-\frac{1}{n}}\right).$$

   (c)
   $$\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{\sqrt{n}}\right).$$

   (d)
   $$\sum_{n=1}^{\infty} \left(\int_0^{\frac{1}{n}} \frac{\sin(x^2)}{x} \, dx\right).$$

   (e)
   $$\sum_{n=1}^{\infty} \ln\left(\frac{1}{1 - \sin^2\left(\frac{1}{n}\right)}\right).$$

(f)

$$\sum_{n=1}^{\infty} \frac{\arcsin\left(\frac{1}{n}\right)}{(\ln n)^2}.$$

(g)

$$\sum_{n=1}^{\infty} \tan\left(n^{-\frac{1}{2}}\right)\sin\left(n^{-1}\right).$$

(h)

$$\sum_{n=1}^{\infty} \frac{\left[\ln\left(1+\frac{1}{n}\right)\right]\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}{n}.$$

Solution.

(a) Recall that the MacLaurin series for $\arctan x$ is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots.$$

Thus, for large values of n, $\arctan(1/n^2) \approx 1/n^2$. Thus, we run limit comparison with $\frac{1}{n^2}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\arctan(1/n^2)}{1/n^2} &= \lim_{n\to\infty} \frac{\arctan(1/x^2)}{1/x^2} \\
(L'H) &= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x^4}} \cdot -\frac{2}{x^3}}{-\frac{2}{x^3}} \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^4}} = 1.
\end{aligned}$$

Since $0 < 1 < \infty$, by limit comparison, $\sum \arctan(1/n^2)$ behaves the same as $\sum 1/n^2$. Since the latter is a convergent p-series, the former converges as well.

(b) Recall that

$$3^x = e^{(\ln 3)\cdot x} = 1 + ((\ln 3)x) + \frac{((\ln 3)x)^2}{2!} + \cdots.$$

So for large values of n,

$$1 - 3^{-\frac{1}{n}} \approx \frac{\ln 3}{n}.$$

Thus, we run limit comparison with $\frac{1}{n}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1-3^{-\frac{1}{n}}}{\frac{1}{n}} &= \lim_{x\to\infty} \frac{1-3^{-\frac{1}{x}}}{\frac{1}{x}} \\
(L'H) &= \lim_{x\to\infty} \frac{-3^{-\frac{1}{x}} \cdot \ln 3 \cdot \frac{1}{x^2}}{\frac{-1}{x^2}} \\
&= \ln 3.
\end{aligned}$$

Since $0 < \ln 3 < \infty$, by limit comparison, $\sum \left(1 - 3^{-\frac{1}{n}}\right)$ behaves the same as $\sum 1/n$. Since the latter is a divergent p-series, the former diverges as well.

(c) Because the MacLaurin series for $\sin x$ is

$$\sin x = x - \frac{1}{3!}x^3 + \cdots$$

we expect $\sin^3(x) \approx x^3$ for small x. Thus, $\sin^3(n^{-1/2}) \approx \frac{1}{n^{3/2}}$. Thus, we run limit comparison with $\frac{1}{n^{3/2}}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sin^3(n^{-1/2})}{n^{-3/2}} &= \lim_{x\to\infty} \frac{\sin^3(x^{-1/2})}{x^{-3/2}} \\
(L'H) &= \lim_{x\to\infty} \frac{-\frac{3}{2}\sin^2(x^{-1/2})\cos(x^{-1/2})x^{-3/2}}{-\frac{3}{2}x^{-5/2}} \\
(\cos 0 = 1) &= \lim_{x\to\infty} \frac{\sin^2(x^{-1/2})}{\frac{1}{x}} \\
(L'H) &= \lim_{x\to\infty} \frac{-\sin(x^{-1/2})\cos(x^{-1/2})x^{-3/2}}{-x^{-2}} \\
(\cos 0 = 1) &= \lim_{x\to\infty} \frac{\sin(x^{-1/2})}{x^{-1/2}} \\
&= 1
\end{aligned}$$

as the the final limit is well known. (One could also use L'Hopital one more time).

Since $0 < 1 < \infty$, by limit comparison, $\sum \sin^3(n^{-1/2})$ behaves the same as $\sum \frac{1}{n^{3/2}}$. Since the latter is a convergent p-series, the former converges as well.

$\square$