FUN INTEGRALS

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Problems

1. Evaluate
$$\int \frac{\arcsin(e^x)}{e^{4x}}\,dx.$$
2. Evaluate
$$\int_1^\infty \frac{1}{(x+1)\sqrt{x-1}}\,dx.$$
3. Evaluate
$$\int_0^\infty \ln\left(1+\frac{1}{x^2}\right)\,dx.$$
4. Evaluate
$$\int \frac{\ln(1+x)}{x^{\frac{3}{2}}}\,dx.$$
5. Evaluate
$$\int \frac{1}{x^4+4}\,dx.$$
6. Evaluate
$$\int \sqrt{\tan\theta}\,d\theta.$$
7. Evaluate
$$\int_0^\infty \frac{1}{1+e^x}\,dx.$$
8. Evaluate
$$\int \frac{1}{x-\sqrt{1-x^2}}\,dx.$$

Hints

1. Let $t = e^x$ and then integrate by parts.
2. Let $u^2 = x - 1$.
3. Integrate by parts with $u = \ln\left(1 + \frac{1}{x^2}\right)$.
4. Integrate by parts with $u = \ln(1 + x)$.
5. Turn the bottom into a difference of squares and factor. For a further hint, note that $(x^2 + 2)^2 = x^4 + 4x^2 + 4$.
6. Let $u = \sqrt{\tan\theta}$, then see the previous problem.
7. Let $e^{\frac{x}{2}} = \tan\theta$.
8. Let $x = \sin\theta$.

Solutions

1. **Evaluate**

$$\int \frac{\arcsin(e^x)}{e^{4x}}\,dx.$$

Solution. Let $t = e^x$. Then $dt = e^x\,dx$, so that $dx = \frac{dt}{t}$. So

$$\int \frac{\arcsin(e^x)}{e^{4x}}\,dx = \int \frac{\arcsin(t)}{t^4}\,\frac{dt}{t} = \int \frac{\arcsin(t)}{t^5}\,dt.$$

Next, integrate by parts with $u = \arcsin(t)$ and $dv = t^{-5}\,dt$:

$$\int \frac{\arcsin(t)}{t^5}\,dt = -\frac{\arcsin(t)}{4t^4} + \frac{1}{4}\int \frac{1}{t^4\sqrt{1-t^2}}\,dt.$$

Next we make a trig substitution: let $t = \cos\theta$. Then $dt = -\sin\theta\,d\theta$. Isolating the latter integral, this gives

$$\int \frac{1}{t^4\sqrt{1-t^2}}\,dt = -\int \frac{1}{\cos^4\theta\sqrt{1-\cos^2\theta}}\sin\theta\,d\theta = -\int \sec^4\theta\,d\theta.$$

We can use trig identities to evaluate this integral:

$$\begin{aligned} -\int \sec^4\theta\,d\theta &= -\int \sec^2\theta\cdot\sec^2\theta\,d\theta \\ &= -\int (\tan^2\theta + 1)\sec^2\theta\,d\theta. \end{aligned}$$

Letting $u = \tan\theta$ (and so $du = \sec^2\theta\,d\theta$) gives

$$-\int (\tan^2\theta+1)\sec^2\theta\,d\theta = -\int (u^2+1)\,du = -\frac{u^3}{3} - u + C.$$

Thus,

$$\int \frac{\arcsin(e^x)}{e^{4x}}\,dx = -\frac{\arcsin(t)}{4t^4} + \frac{1}{4}\left(-\frac{\tan^3\theta}{3} - \tan\theta\right) + C.$$

Since $t = \cos\theta$, $\tan\theta = \frac{\sqrt{1-t^2}}{t}$, thus

$$\int \frac{\arcsin(e^x)}{e^{4x}}\,dx = -\frac{\arcsin(t)}{4t^4} - \frac{1}{4}\frac{\sqrt{1-t^2}^{\frac{3}{2}}}{3t^3} - \frac{1}{4}\frac{\sqrt{1-t^2}}{t} + C.$$

Finally, $t = e^x$. Therefore,

$$\int \frac{\arcsin(e^x)}{e^{4x}}\,dx = -\frac{\arcsin(e^x)}{4e^{4x}} - \frac{1}{4}\frac{\sqrt{1-e^{2x}}^{\frac{3}{2}}}{3e^{3x}} - \frac{1}{4}\frac{\sqrt{1-e^{2x}}}{e^x} + C.$$

□

2. **Evaluate**

$$\int_1^\infty \frac{1}{(x+1)\sqrt{x-1}}\,dx.$$

Solution. Let $u^2 = x - 1$. Then $2u\,du = dx$. The antiderivative is then

$$\begin{aligned} \int \frac{1}{(x+1)\sqrt{x-1}}\,dx &= \int \frac{1}{(2+u^2)\sqrt{u^2}}\,2u\,du = 2\int \frac{1}{2+u^2}\,du = \sqrt{2}\arctan(u/\sqrt{2}) + C \\ &= \sqrt{2}\arctan(\sqrt{x-1}/\sqrt{2}) + C. \end{aligned}$$

Next, note the integral is improper at both ∞ and at 1. Even though I constantly talk about how we need to split the integral up into two parts in class, we can handle it in one go by using two different limiting parameters:

$$\begin{aligned}
\int_1^\infty \frac{1}{(x+1)\sqrt{x-1}}\,dx &= \lim_{S\to 1^+}\lim_{R\to\infty}\int_S^R \frac{1}{(x+1)\sqrt{x-1}}\,dx \\
&= \lim_{S\to 1^+}\lim_{R\to\infty}\left[\sqrt{2}\arctan(\sqrt{x-1}/\sqrt{2})\right]\Big|_S^R \\
&= \lim_{S\to 1^+}\lim_{R\to\infty}\left[\sqrt{2}\arctan(\sqrt{x-1}/\sqrt{2}) - \sqrt{2}\arctan(\sqrt{S-1}/\sqrt{2})\right] \\
&= \sqrt{2}\left(\frac{\pi}{2} - \arctan(0)\right) \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

□

3. **Evaluate**

$$\int_0^\infty \ln\left(1+\frac{1}{x^2}\right)\,dx.$$

Solution. First, we compute the antiderivative. Integrate by parts with $u = \ln\left(1+\frac{1}{x^2}\right)$ and $dv = dx$:

$$\begin{aligned}
\int \ln\left(1+\frac{1}{x^2}\right)\,dx &= x\cdot\ln\left(1+\frac{1}{x^2}\right) - \int \frac{-2x^{-3}}{1+\frac{1}{x^2}}\cdot x\,dx \\
&= x\ln\left(1+\frac{1}{x^2}\right) + 2\int \frac{1}{x^2+1}\,dx \\
&= x\ln\left(1+\frac{1}{x^2}\right) + 2\arctan(x) + C.
\end{aligned}$$

Like the previous problem, the integral is improper at both ∞ and at 0. I'll do the same thing I did before and instead of splitting it up, I'll use two limiting parameters:

$$\begin{aligned}
\int_0^\infty \ln\left(1+\frac{1}{x^2}\right)\,dx &= \lim_{S\to 0^+}\lim_{R\to\infty}\int_S^R \ln\left(1+\frac{1}{x^2}\right)\,dx \\
&= \lim_{S\to 0^+}\lim_{R\to\infty}\left[x\ln\left(1+\frac{1}{x^2}\right) + 2\arctan(x)\right]\Bigg|_S^R.
\end{aligned}$$

Note that, by L'Hopital's rule,

$$\begin{aligned}
\lim_{S\to 0^+} S\ln\left(1+\frac{1}{S^2}\right) + 2\arctan(S) &= \lim_{S\to 0^+}\frac{\ln\left(1+\frac{1}{S^2}\right)}{S^{-1}} \\
&= \lim_{S\to 0^+}\frac{\frac{1}{1+\frac{1}{S^2}}\cdot -2S^{-3}}{-S^{-2}} \\
&= \lim_{S\to 0^+}\frac{2S^2}{S^3+S} \\
&= \lim_{S\to 0^+}\frac{2S}{S^2+1} \\
&= 0.
\end{aligned}$$

Next, again by L'Hopital,

$$\begin{aligned}\lim_{R\to\infty} R\ln\left(1+\frac{1}{R^2}\right)+2\arctan(R) &= \lim_{S\to 0^+}\frac{\ln\left(1+\frac{1}{R^2}\right)}{R^{-1}}+\pi \\ &= \lim_{R\to\infty}\frac{2R}{R^2+1}+\pi \\ &= \pi.\end{aligned}$$

Thus,

$$\int_0^\infty \ln\left(1+\frac{1}{x^2}\right)\,dx = \pi - 0 = \pi.$$

□

4. **Evaluate**

$$\int \frac{\ln(1+x)}{x^{\frac{3}{2}}}\,dx.$$

Solution. Integrate by parts with $u=\ln(1+x)$ and $dv = x^{-\frac{3}{2}}\,dx$:

$$\int \frac{\ln(1+x)}{x^{\frac{3}{2}}}\,dx = -2x^{-\frac{1}{2}}\ln(1+x)+2\int\frac{x^{-\frac{1}{2}}}{1+x}\,dx.$$

Isolate the latter integral, and make the substitution $t=\sqrt{x}$. Then $2t\,dt = dx$, so

$$\int\frac{x^{-\frac{1}{2}}}{1+x}\,dx = \int\frac{t^{-1}}{1+t^2}\,2t\,dt = 2\int\frac{1}{1+t^2}\,dt = 2\arctan(t)+C = 2\arctan(\sqrt{x})+C.$$

Thus,

$$\int \frac{\ln(1+x)}{x^{\frac{3}{2}}}\,dx = -2x^{-\frac{1}{2}}\ln(1+x)+4\arctan(\sqrt{x})+C.$$

□

5. **Evaluate**

$$\int\frac{1}{x^4+4}\,dx.$$

Solution. Note that

$$x^4+4 = (x^2+2)^2-4x^2 = (x^2-2x+2)(x^2+2x+2).$$

The last equality follows from difference of squares. Thus,

$$\int\frac{1}{x^4+4}\,dx = \int\frac{1}{(x^2-2x+2)(x^2+2x+2)}\,dx.$$

Note that both quadratic factors in the denominator are irreducible, since $2^2-4\cdot 2<0$. Thus, we compute a partial fractions decomposition with the following set up:

$$\frac{1}{(x^2-2x+2)(x^2+2x+2)} = \frac{Ax+B}{x^2-2x+2}+\frac{Cx+D}{x^2+2x+2}.$$

Clearing denominators gives

$$1 = (Ax+B)(x^2+2x+2)+(Cx+D)(x^2-2x+2).$$

Comparing x^3 coefficients gives $A + C = 0$ and comparing the constant terms gives $2B + 2C = 1$. Comparing the x^2 coefficients gives $2A + B - 2C + D = 0$. Since $B + D = \frac{1}{2}$, this equation implies that $A - C = -\frac{1}{4}$. Adding this to the first equation we generated gives $2A = -\frac{1}{4}$, hence

$$A = -\frac{1}{8} \quad \text{and} \quad C = \frac{1}{8}.$$

Comparing x coefficients gives $2A + 2B + 2C - 2D = 0$. Since $A + C = 0$ this implies $B - D = 0$. Since $B + D = \frac{1}{2}$, we have

$$B = \frac{1}{4} \quad \text{and} \quad D = \frac{1}{4}.$$

Therefore,

$$\frac{1}{(x^2 - 2x + 2)(x^2 + 2x + 2)} = \frac{-\frac{1}{8}x + \frac{1}{4}}{x^2 - 2x + 2} + \frac{\frac{1}{8}x + \frac{1}{4}}{x^2 + 2x + 2}.$$

Completing the square in each denominator then gives

$$\int \frac{1}{x^4 + 4}\,dx = \int \frac{-\frac{1}{8}x + \frac{1}{4}}{(x-1)^2 + 1}\,dx + \int \frac{\frac{1}{8}x + \frac{1}{4}}{(x+1)^2 + 1}\,dx =: I_1 + I_2.$$

We compute each integral separately. For I_1, let $u = x - 1$. Then

$$I_1 = \int \frac{-\frac{1}{8}(u+1) + \frac{1}{4}}{u^2 + 1}\,du = \int \frac{-\frac{1}{8}u + \frac{1}{8}}{u^2 + 1}\,du = -\frac{1}{16}\ln((x-1)^2 + 1) + \frac{1}{8}\arctan(x-1) + C.$$

Similarly, letting $u = x + 1$ in I_2 gives

$$I_2 = \int \frac{\frac{1}{8}u + \frac{1}{8}}{u^2 + 1}\,dx = \frac{1}{16}\ln((x+1)^2 + 1) + \frac{1}{8}\arctan(x+1) + C.$$

Putting all of this together,

$$\int \frac{1}{x^4 + 4}\,dx = -\frac{1}{16}\ln((x-1)^2 + 1) + \frac{1}{8}\arctan(x-1) + \frac{1}{16}\ln((x+1)^2 + 1) + \frac{1}{8}\arctan(x+1) + C.$$

□

6. **Evaluate**

$$\int \sqrt{\tan\theta}\,d\theta.$$

Solution. Let $u = \sqrt{\tan\theta}$. Then $u^2 = \tan\theta$, and so

$$2u\,du = \sec^2\theta\,d\theta \qquad \Rightarrow \qquad d\theta = \frac{2u}{\sec^2\theta}\,du = \frac{2u}{\tan^2\theta + 1}\,du = \frac{2u}{u^4 + 1}\,du.$$

So

$$\int \sqrt{\tan\theta}\,d\theta = \int \frac{2u^2}{u^4 + 1}\,du.$$

Motivated by the previous problem, we factor the denominator of the integrand as:

$$\frac{2u^2}{u^4 + 1} = \frac{2u^2}{(u^2 + 1)^2 - 2u^2} = \frac{2u^2}{(u^2 - \sqrt{2}u + 1)(u^2 + \sqrt{2}u + 1)}.$$

Doing a partial fractions decomposition similar to the previous problem gives

$$\frac{2u^2}{(u^2 - \sqrt{2}u + 1)(u^2 + \sqrt{2}u + 1)} = \frac{\frac{1}{\sqrt{2}}u}{u^2 - \sqrt{2}u + 1} - \frac{\frac{1}{\sqrt{2}}u}{u^2 + \sqrt{2}u + 1}.$$

FINISH □

7. **Evaluate**

$$\int_0^\infty \frac{1}{1+e^x}\,dx.$$

Solution. Let $e^{\frac{x}{2}} = \tan\theta$. Then

$$\frac{1}{2}e^{\frac{x}{2}}\,dx = \sec^2\theta\,d\theta \qquad \Rightarrow \qquad dx = \frac{2}{\tan\theta}\sec^2\theta\,d\theta$$

and so the antiderivative transforms into

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{1+\tan^2\theta}\,\frac{2}{\tan\theta}\sec^2\theta\,d\theta = 2\int\frac{1}{\tan\theta}\,d\theta = 2\int\frac{\cos\theta}{\sin\theta}\,d\theta = 2\ln|\sin\theta| + C.$$

Since $e^{\frac{x}{2}} = \tan\theta$, $\sin\theta = \frac{e^{\frac{x}{2}}}{\sqrt{1+e^x}}$. (Draw the usual triangle to figure that part out.) So

$$\int \frac{1}{1+e^x}\,dx = \ln\left(\frac{e^{\frac{x}{2}}}{\sqrt{1+e^x}}\right) + C.$$

Thus,

$$\begin{aligned}\int_0^\infty \frac{1}{1+e^x}\,dx &= \lim_{R\to\infty} \ln\left(\frac{e^{\frac{x}{2}}}{\sqrt{1+e^x}}\right)\Bigg|_0^R \\ &= \lim_{R\to\infty} \ln\left(\frac{e^{\frac{R}{2}}}{\sqrt{1+e^R}}\right) - \ln\left(\frac{1}{\sqrt{2}}\right).\end{aligned}$$

Note that

$$\lim_{R\to\infty}\frac{e^{\frac{R}{2}}}{\sqrt{1+e^R}} = 1$$

and so

$$\lim_{R\to\infty} \ln\left(\frac{e^{\frac{R}{2}}}{\sqrt{1+e^R}}\right) = \ln(1) = 0.$$

Therefore,

$$\int_0^\infty \frac{1}{1+e^x}\,dx = -\ln\left(\frac{1}{\sqrt{2}}\right) = \frac{\ln 2}{2}.$$

□