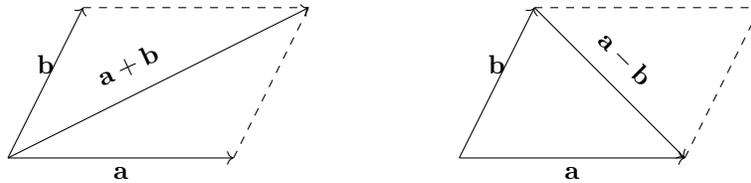


CONDENSED FINAL REVIEW

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Vectors

Given two vectors \mathbf{a} and \mathbf{b} , adding and subtracting (geometrically) happens in the following way:



Arithmetically adding and subtracting vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ happens component wise:

$$\mathbf{a} \pm \mathbf{b} = \langle a_1 \pm b_1, a_2 \pm b_2 \rangle$$

The **length** / **magnitude** / **norm** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is:

$$\|\mathbf{a}\| = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The distance between two points / vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is just:

$$\|\mathbf{a} - \mathbf{b}\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

A **unit vector** in the direction of \mathbf{a} is: $\frac{\mathbf{a}}{\|\mathbf{a}\|}$.

Dot Product

The **dot product** returns a *scalar*:

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

Alternatively:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

where θ is the angle between \mathbf{a} and \mathbf{b} . Using this, we can calculate the angle between vectors \mathbf{a} and \mathbf{b} by solving for θ :

$$\theta = \arccos \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right)$$

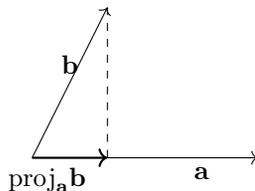
Two vectors are **orthogonal** / **perpendicular** if their dot product is 0:

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \iff \quad \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal}$$

Note that $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

Projections

When we **project \mathbf{b} onto \mathbf{a}** , we smash \mathbf{b} orthogonally onto \mathbf{a} :



The **projection of \mathbf{b} onto \mathbf{a}** is:

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a}$$

The **component of \mathbf{b} in the \mathbf{a} direction** is the signed length of the projection vector:

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$$

Cross Product

The **cross product** is only defined for vectors in three dimensions, and returns a *vector*:

$$\mathbf{a} \times \mathbf{b} = \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

One way to remember this formula is by calculating the determinant of a special matrix, using cofactor expansion across the top row:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

The cross product $\mathbf{a} \times \mathbf{b}$ is a vector *orthogonal to both \mathbf{a} and \mathbf{b}* . Its direction is determined by the **right-hand rule**: in curling the fingers of your right hand from \mathbf{a} to \mathbf{b} , the cross product $\mathbf{a} \times \mathbf{b}$ sticks in the direction of your thumb.

The length of the cross product of \mathbf{a} and \mathbf{b} is:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$

where θ is the angle between \mathbf{a} and \mathbf{b} . The length of the cross product represents *the area of the parallelogram formed by \mathbf{a} and \mathbf{b}* .

Note that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. Also, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

The volume of the parallelepiped determined by vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Lines

There are three ways to represent lines in three dimensions:

- **Vector Form:** Looks like:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

where \mathbf{r}_0 is any point on the line, and \mathbf{v} is a direction vector.

- **Parametric Form:** Looks like:

$$\begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases}$$

We can translate parametric form into vector form (and vice versa) by $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$.

- **Cartesian Form:** Looks like:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

To find parametric equations of a line, it is best to find a *point on the line* \mathbf{r}_0 and a *direction vector* \mathbf{v} .

The vector equation of a line *segment* from points \mathbf{a} to \mathbf{b} is given by:

$$\mathbf{r}(t) = (1 - t)\mathbf{a} + t\mathbf{b} \quad 0 \leq t \leq 1$$

Planes

There are two ways to represent planes in three dimensions:

- **Vector Form:** Looks like:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

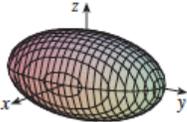
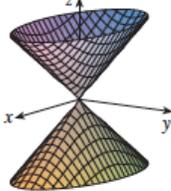
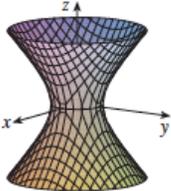
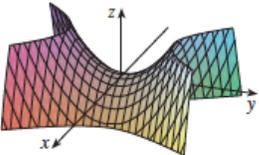
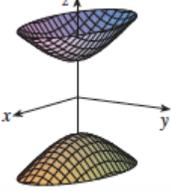
where \mathbf{r}_0 is any point on the plane, and $\mathbf{r} = \langle x, y, z \rangle$ is the vector of variables, and \mathbf{n} is a *normal* vector to the plane, i.e., a vector perpendicular to the plane.

- **Cartesian Form:** Looks like:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d$$

In either case, a normal vector to the plane is given by $\mathbf{n} = \langle a, b, c \rangle$.

To find an equation of a plane, it is best to find a *point on the plane* \mathbf{r}_0 and a *normal vector* \mathbf{n} . One tactic to finding a normal vector is to find two vectors that lie in the plane (i.e. are parallel to the plane) and take their cross product.

Surface	Equation	Surface	Equation
<p>Ellipsoid</p> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.</p>	<p>Cone</p> 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.</p>
<p>Elliptic Paraboloid</p> 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.</p>	<p>Hyperboloid of One Sheet</p> 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p>Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.</p>
<p>Hyperbolic Paraboloid</p> 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p>Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.</p>	<p>Hyperboloid of Two Sheets</p> 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.</p>

Other Coordinate Systems

In two dimensions:

- **Polar Coordinates:** (r, θ)

Polar to Rectangular	Rectangular to Polar
$x = r \cos \theta$	$r^2 = x^2 + y^2$
$y = r \sin \theta$	$\tan \theta = \frac{y}{x}$

Here, r describes the distance from the point (x, y) to the origin, and θ is its angle off the x -axis.

In three dimensions:

- **Cylindrical Coordinates:** (r, θ, z)

Cylindrical to Rectangular	Rectangular to Cylindrical
$x = r \cos \theta$	$r^2 = x^2 + y^2$
$y = r \sin \theta$	$\tan \theta = \frac{y}{x}$
$z = z$	$z = z$

Cylindrical coordinates are just like “vertically shifted” polar coordinates. The only new information is the rectangular z coordinate.

- **Spherical Coordinates:** (ρ, θ, φ)

Spherical to Rectangular	Rectangular to Spherical
$x = \rho \sin \varphi \cos \theta$	$\rho^2 = x^2 + y^2 + z^2$
$y = \rho \sin \varphi \sin \theta$	$\tan \theta = \frac{y}{x}$
$z = \rho \cos \varphi$	$\cos \varphi = \frac{z}{\rho}$

Here, ρ is the distance from the (x, y, z) point to the origin, θ is the rotational angle in the xy plane (just like in polar/cylindrical coordinates), and φ is the vertical angle from the z axis.

Parametric Curves

A **parametric curve** is a function that takes in a real number t and spits out a point in space:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

For example, the standard way to **parametrize** the unit circle is:

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \leq t \leq 2\pi$$

Calculus with Curves

The **slope** of a parametric curve $\langle x(t), y(t) \rangle$ in two dimensions is given by:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

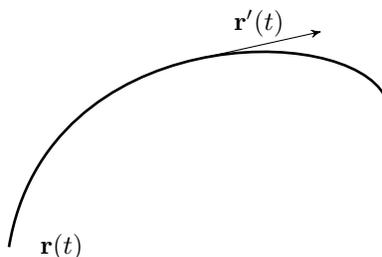
The **arc-length** of a parametric curve $\langle x(t), y(t) \rangle$ from $t = a$ to $t = b$ is given by:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Given a parametric curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, the **tangent vector** to $\mathbf{r}(t)$ is calculated by taking the derivatives of the components:

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

Intuitively, the tangent vector is the direction the curve is travelling in at a given time t :



Particle Motion

Position: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$

Speed: $\|\mathbf{v}(t)\|$.

Unit Tangent Vector: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

Unit Normal Vector: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$

Unit Binormal Vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

Tangential Component of Acceleration	Normal Component of Acceleration
$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\ \mathbf{r}'(t)\ }$	$a_N = \frac{\ \mathbf{r}'(t) \times \mathbf{r}''(t)\ }{\ \mathbf{r}'(t)\ ^3}$

Curvature for $y = f(x)$	Curvature for $\mathbf{r}(t)$
$k(x) = \frac{ f''(x) }{[1+(f'(x))^2]^{\frac{3}{2}}}$	$k(t) = \frac{\ \mathbf{r}'(t) \times \mathbf{r}''(t)\ }{\ \mathbf{r}'(t)\ ^3}$

Projectile Motion

Special Case: two dimensions, only gravity

Initial position (x_0, y_0) , initial speed v_0 , initial angle θ . Then

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle x_0 + v_0 \cos(\theta)t, y_0 + v_0 \sin(\theta)t - \frac{1}{2}gt^2 \right\rangle$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle = \langle v_0 \cos(\theta), v_0 \sin(\theta) - gt \rangle$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t) \rangle = \langle 0, -g \rangle$$

General Case: three dimensions, constant acceleration

If $\mathbf{a}(t) = \langle a_1, a_2, a_3 \rangle$, $\mathbf{v}(0) = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{r}(0) = \langle r_1, r_2, r_3 \rangle$, then

$$\mathbf{v}(t) = \langle a_1t + v_1, a_2t + v_2, a_3t + v_3 \rangle$$

and

$$\mathbf{r}(t) = \left\langle \frac{1}{2}a_1t^2 + v_1t + r_1, \frac{1}{2}a_2t^2 + v_2t + r_2, \frac{1}{2}a_3t^2 + v_3t + r_3 \right\rangle$$

Multivariable Functions

Domains of Functions

Questions to ask:

- Do I ever divide by zero?
- Do I ever take the square root of a negative number?
- Do I ever take the natural log of a non-positive number?

Level Curves, Level Surfaces, Contour plots

Let $f(x, y)$ be a function. A **level curve** is:

$$f(x, y) = k$$

for some number k . A **contour plot** is a collection of level curves:

$$f(x, y) = k_1 \quad f(x, y) = k_2 \quad f(x, y) = k_3 \quad \dots$$

Let $f(x, y, z)$ be a function. A **level surface** is:

$$f(x, y, z) = k$$

for some number k .

Multivariable Limits and Continuity

Let $f(x, y)$ be a function. The **limit of f as (x, y) approaches (a, b)** :

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y)$$

exists if we get the same value for *every possible path* of approach in the domain. *If two different approaches two different values, then the limit does not exist.*

Same thing for functions of three variables:

$$\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z)$$

The same principles hold.

Limit Techniques / Limit Flowchart

A general approach to evaluating the following limit:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y)$$

- Continuity / Plugging In

If f is continuous at the point (a, b) , then you can just plug the point in and calculate $f(a, b)$. If nothing bad happens (like dividing by 0), then that's the answer!

- Different Path Approaches

If you can find two different approaches that give different results, then the limit does not exist. If the target point (a, b) is the origin $(0, 0)$, some common paths are $x = 0$, $y = 0$, $y = x$, etc. For these paths, the limits become, respectively:

$$\lim_{y \rightarrow 0} f(0, y) \quad \lim_{x \rightarrow 0} f(x, 0) \quad \lim_{x \rightarrow 0} f(x, x)$$

Note: you can use *any* path of approach; these are just some easy ones to check.

- **Factoring**

If $f(x, y)$ is a fraction and yields an indeterminate form like $\frac{0}{0}$, sometimes you can factor the top or bottom to conveniently cancel terms. Note: you can't use L'Hopitals Rule with a multivariable limit: if you convert the limit to one variable, then you can use it, but not in general.

- **Conjugate** If $f(x, y)$ contains a term that looks like $\sqrt{blah1} - \sqrt{blah2}$, sometimes it helps to multiply the top and bottom by the conjugate, $\sqrt{blah1} + \sqrt{blah2}$.

- **Polar Coordinates**

If $f(x, y)$ contains terms that look like $x^2 + y^2$ or $\sqrt{x^2 + y^2}$ or something that reminds you of circles, and $(a, b) = (0, 0)$, sometimes it is useful to convert $f(x, y)$ into polar coordinates. Then the limit becomes:

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{r \rightarrow 0} f(r \cos \theta, r \sin \theta)$$

- **Squeeze Theorem / Comparison**

If you can find upper and lower bounds for $f(x, y)$, evaluating the limits of those bounds can "squeeze" in the limit of $f(x, y)$. This is sometimes useful if you have a trig function inside $f(x, y)$.

Continuity of Multivariable Functions

A function $f(x, y)$ is **continuous** at (a, b) if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

Hence,

$$f(x, y) = \begin{cases} g(x, y) & (x, y) \neq (a, b) \\ c & (x, y) = (a, b) \end{cases}$$

is continuous at (a, b) if and only if $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = c$.

Partial Derivatives

Let $f(x, y)$ be a function. The **partial derivative of f with respect to x** , denoted

$$\frac{\partial f}{\partial x}(x, y) \quad \text{or} \quad f_x(x, y)$$

The **partial derivative of f with respect to y** is defined and denoted similarly.

Calculating the partial derivative of f with respect to f can be expressed as:

$$\frac{\partial}{\partial x} f(x, y) = \frac{d}{dx} f(x, C) \quad \text{where } C \text{ is a constant}$$

We can compute **second partial derivatives** by just taking the derivative again. Note: taking the partial with respect to x then y would be denoted:

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}$$

If the second partials of f exist and are continuous, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{yx} = f_{xy}$$

Interpreting Partial Derivatives with Contour Plots

Some things to note:

- For the first partials, just look at if the numbers on the level curves are increasing or decreasing.
- Always move in the positive direction when looking at how the function changes.
- When level curves get closer together, that means the change is becoming more *extreme*, either positively or negatively. So if the level curves are increasing in value and getting closer together, the second partial derivative in that direction will be positive.
- Level curves represent *paths of no change*. If the direction you are moving in is tangent to a level curve, then the partial derivative with respect to that direction is 0.

Tangent Planes and Linear Approximations

The **tangent plane** of a function $f(x, y)$ at (a, b) is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

This is also called the **linear approximation** to f at (a, b) .

For three variables, the formula for the **linear approximation** to $f(x, y, z)$ at (a, b, c) is:

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

Actual change:

$$\Delta z = f(a + h_1, b + h_2) - f(a, b)$$

Approximate change:

$$dz = T(a + h_1, b + h_2) - T(a, b) = T(a + h_1, b + h_2) - f(a, b)$$

Quadratic Approximations

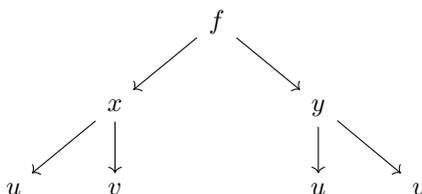
The **quadratic approximation** to f at (a, b) is given by adding some new terms to the linear approximation:

$$Q(x, y) = T(x, y) + \frac{1}{2} [f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2]$$

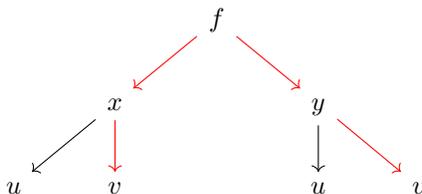
This is also called the **second-order Taylor polynomial** to f at (a, b) .

The Chain Rule

Draw a tree. To find the total change of f with respect to a variable, add each leg of the change that involves that variable. For example, in the diagram below,



$f(x, y)$ depends on x and y , and x and y both depend on u and v . If we want to find $\frac{\partial f}{\partial v}$, there are two “legs” of change to find. These are highlighted in red:



Hence,

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Implicit Differentiation

Given $F(x, y, z) = c$, we can compute:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

and similarly for other variables. (Or do what I do and just take the derivatives of the equation implicitly and then solve.)

Gradients and Directional Derivatives

The **gradient** of a function $f(x, y)$ (respectively $f(x, y, z)$) is:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The gradient represents the **direction of fastest increasing rate of change**. The actual rate of change in that direction is $\|\nabla f\|$. The direction of fastest *decreasing* rate of change is $-\nabla f$.

The gradient ∇f is perpendicular / normal to level curves $f(x, y) = k$ and level surfaces $f(x, y, z) = k$.

Given a *unit* vector \mathbf{u} , the **directional derivative** of f in the direction \mathbf{u} is:

$$D_{\mathbf{u}}f = \mathbf{u} \cdot \nabla f$$

This represents the rate of change in the \mathbf{u} direction. Note:

$$D_{\mathbf{i}}f = \frac{\partial f}{\partial x} \quad D_{\mathbf{j}}f = \frac{\partial f}{\partial y}$$

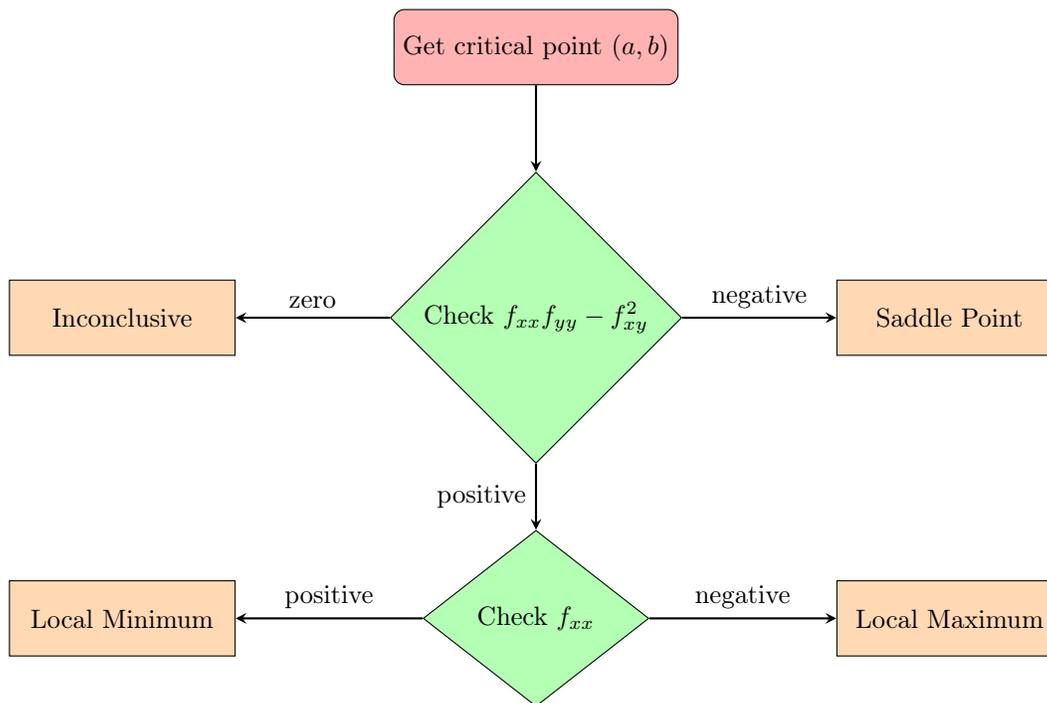
Maximizing and Minimizing Functions

Local Extrema

The **critical points** of a function $f(x, y)$ occur when

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle = \langle 0, 0 \rangle$$

To classify critical points, use the **second derivative test**:



The **local minimum/maximum value** is $f(a, b)$.

Absolute Extrema

The **absolute extrema** of a function $f(x, y)$ are the biggest and smallest values of f . Given a function $f(x, y)$ and a closed region E , to find the absolute extrema,

1. Find values at all critical points;
2. Parametrize the boundary to get a function of one variable, and find the absolute extrema of that function using single variable calculus techniques.
3. Compare all the values you have and pick the biggest and smallest numbers.

Lagrange Multipliers

Given a function $f(x, y, z)$ and a constraint equation $g(x, y, z) = c$, critical points occur when

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

for some number λ .