Computing Taylor Polynomials and Taylor Series

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In this note, we'll get some practice computing Taylor polynomials and Taylor series of functions. Just to review, the **nth-degree Taylor polynomial of f at a** is the polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{1}$$

Similarly, the **Taylor series of f at a** is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots . \tag{2}$$

So a Taylor series is essentially a Taylor polynomial of infinite degree. This means that computing a Taylor series also gives you all of the Taylor polynomials. These formulas can be intimidating and confusing, but they just describe a recipe for how to compute coefficients of polynomials: the coefficient in front of the $(x-a)^n$ term is the number $\frac{f^{(n)}(a)}{n!}$.

In general, if you're asked to compute a Taylor series (or a Taylor polynomial), you have two main techniques:

1. Compute the coefficients $\frac{f^{(n)}(a)}{n!}$ by hand, doing your best to pattern-match and come up with a general expression.

2. Use a simpler Taylor series that you've already computed, if possible.

Here are some examples which demonstrate these techniques to varying degrees. We'll start off with something relatively simple.

Example 1: *Compute the Taylor series of $f(x) = e^{2x}$ centered at 0.*

Solution 1 (Direct Computation): If we want the Taylor series of f centered at 0, then by equation (2), we need to write down the following power series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

Note that all we need to write this down are the numbers $f(0), f'(0), f''(0), \ldots$, etc.. To get a formula for the general term, we'll need to find the pattern amongst all these numbers to get $f^{(n)}(0)$.

The best way to do this is to start computing $f'(0), f''(0), \ldots$ and keep going until we spot the pattern. Let's calculate a few of the numbers we need:

$$\begin{aligned}
f(x) &= e^{2x} & &\Rightarrow & f(0) &= 1 \\
f'(x) &= 2e^{2x} & &\Rightarrow & f'(0) &= 2 \\
f''(x) &= 2^2e^{2x} & &\Rightarrow & f''(0) &= 2^2 \\
f^{(3)}(x) &= 2^3e^{2x} & &\Rightarrow & f^{(3)}(0) &= 2^3 \\
f^{(4)}(x) &= 2^4e^{2x} & &\Rightarrow & f^{(4)}(0) &= 2^4
\end{aligned}$$

Note that I'm *not* simplifying the numbers 2^2, 2^3, etc. I think it's a good idea to leave things un-simplified to better understand the pattern!

Speaking of, do you see the pattern? The last two numbers we computed were $f^{(3)}(0) = 2^3$ and $f^{(4)}(0) = 2^4$. If you want to compute some more, you can, but it's not to hard to see that the number of derivatives of f that we took corresponded exactly to the number of 2's. So from this we can conclude that $f^{(n)}(0) = 2^n$. This means that the general term in the Taylor series looks like

$$\frac{2^n}{n!}x^n$$

and so the Taylor series is $\sum_{n=0}^{\infty} \frac{2^n}{n!}x^n$. Done! □

Solution 2 (Using a Known Taylor Series): Let's say you remembered that the Taylor series for e^x centered at 0 is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, which is a good one to have memorized. Then to get the Taylor series centered at 0 for e^{2x}, we can just stick in a $2x$ everywhere we see an x in the original Taylor series to get:

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n.$$

Unsurprisingly, this is the same answer we got in the first solution. □

Let's try an example that involves some more sophisticated pattern matching.

Example 2: *Compute the Taylor series of $f(x) = \ln(1+x)$ centered at 0.*

Solution 1 (Direct Computation): Like before, to write down the Taylor series of f at 0 we just need to write down the power series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

and just as before, this amounts to computing the numbers $f^n(0)$. Once we do that, we're done!

Also as before, to accomplish this I'm going to start computing derivatives of our function, grouping things in suggestive ways and not simplifying. Hopefully if we compute enough we can spot the pattern.

First, $f(x) = \ln(1+x)$. So

$$f'(x) = \frac{1}{1+x}.$$

By the power rule,

$$f''(x) = \frac{(-1)}{(1+x)^2}.$$

By the power rule again,

$$f^{(3)}(x) = \frac{(-2)(-1)}{(1+x)^3} = \frac{(-1)^2 \cdot 2 \cdot 1}{(1+x)^3}.$$

Computing a few more derivatives gives

$$f^{(4)}(x) = \frac{(-1)^3 \cdot 3 \cdot 2 \cdot 1}{(1+x)^4}$$

and

$$f^{(5)}(x) = \frac{(-1)^4 \cdot 4 \cdot \cdot 3 \cdot 2 \cdot 1}{(1+x)^5}.$$

At this point, the general pattern is clear. First, each derivative contains a power of $1+x$ in the denominator, and it looks like that power matches the number of derivatives we take. Next, there is a power of -1 in the numerator, and it looks like that power is one smaller than the number of derivatives we take. Finally, there is a factorial looking thing in the numerator which starts at one number smaller than the number of derivatives we take. Putting all this together, we can conclude

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Nice. But remember that we actually only care about $f^{(n)}(0)$, not the function $f^{(n)}(x)$. So plugging 0 in for x in the above gives us $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. One comment: this doesn't quite work for $n = 0$, which makes sense because $f^{(0)}(x) = \ln(1+x)$ and that doesn't match the pattern. So we can treat that term separately. Since $f(0) = \ln(1) = 0$, we can actually just ignore the term corresponding to $n = 1$. So plugging all of this into the formula for the Taylor series (and then simplifying a bit) gives us the final answer:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

□

Solution 2 (Using a Known Taylor Series): Here's an example of an alternate solution that uses the second technique, although there is some extra integration trickery involved.

Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$. This is simply the formula for a convergent geometric series. This shows us that $\sum_{n=0}^{\infty} x^n$ is the Taylor series for the function $\frac{1}{1-x}$ centered at 0. (If you don't believe me, then you can compute the Taylor series by hand, like we just did. It's a pretty similar computation!) This is what we'll use.

Here's the trick: note that

$$f'(x) = \frac{d}{dx} \ln(1+x) = \frac{1}{1+x}.$$

So the derivative of our function f kind of looks like the function $\frac{1}{1-x}$ that we know the Taylor series of. It's not exactly the same, but we can still use this to our advantage as follows:

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

for $|x| < 1$.

So we've expanded $f'(x)$ into a power series. That's nice, but we're trying to find the Taylor series for $f(x)$, not $f'(x)$. But how can we recover $f(x)$ from $f'(x)$? We can integrate! Integrating both sides of the equation

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n x^n$$

gives

$$\int f'(x)\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx.$$

On the left side we get $f(x)$, plus possibly a constant. On the right side, to integrate a power series we can just integrate each term individually. By the power rule,

$$\int (-1)^n x^n \, dx = (-1)^n \frac{x^{n+1}}{n+1} + C.$$

(Note that $(-1)^n$ is a constant with respect to x!). So putting this all together, we get

$$f(x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

for $|x| < 1$. It's a little annoying to have the constant of integration C, but since we know that $f(x) = \ln(1+x)$, we can actually determine C by plugging in $x = 0$ to the above equation. Every term in the series vanishes (since $0^{n+1} = 0$ for all $n \geq 0$) and since $\ln(1) = 0$, we get $C = 0$.

So all of this together shows us that

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

for $|x| < 1$. In other words, the Taylor series that we were after is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

If any of this is to make sense, then this better be the same as the answer as we got in the first solution:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

They don't look exactly the same, but I claim they are the same, after reindexing the sums. The easiest way to see this is to just start writing out the terms of each:

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .\end{aligned}$$

They are indeed the same! □