FREDHOLM OPERATORS AND THE GENERALIZED INDEX

JOSEPH BREEN

CONTENTS

1. INTRODUCTION

One of the most fundamental problems in mathematics is to solve linear equations of the form $Tf = g$, where T is a linear transformation, g is known, and f is some unknown quantity.

The simplest example of this comes from elementary linear algebra, which deals with solutions to matrix-vector equations of the form $A\mathbf{x} = \mathbf{b}$. More generally, if V, W are vector spaces (or, in particular, Hilbert or Banach spaces), we might be interested in solving equations of the form $Tv = w$ where $v \in V$, $w \in W$, and T is a linear map from $V \to W$.

A more explicit source of examples is that of differential equations. For instance, one can seek solutions to a system of differential equations of the form $A\mathbf{x}(t) = \mathbf{x}'(t)$, where, again, A is a matrix. From physics, we get important partial differential equations such as *Poisson's equation*: $\Delta f = g$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and f and g are functions on $\mathbb{R}^n$. In general, we can take certain partial differential equations and repackage them in the form $Df = g$ for some suitable *differential operator* D between Banach spaces.

In each of these examples, given some sort of operator T and an equation $Tf = g$, we wish to understand the *existence* and *uniqueness* of solutions. Unfortunately, understanding existence and uniqueness in isolation (i.e., as separate entities) often proves to be difficult. The goal of "index theory" is to do the next best thing: assign some kind of an *index* to T which encapsulates information about *both* existence and uniqueness, simultaneously.

Vague Example 1.1. Suppose that T is a linear map from X to Y. We want to understand the space of solutions to the equation $Tf = g$. Suppose that we have a magic number N which measures the uniqueness of solutions, and another magic number M which measures the existence of solutions. Then we can define the **index** of T to be:

$$\text{ind}\,(T) := N - M \tag{1.1}$$

Thus, ind (T) contains information about the existence and uniqueness of solutions, packaged into one number. Computing the index of T might not be as insightful as computing N and M separately, but it might be the case that ind (T) is easier to calculate.

Though this example is devoid of any actual meaning, it is the standard model for the concept of an index.

1.1. **Index Theory in Finite Dimensions.** To warm us up, let's do something a little more concrete. Let V and W be finite dimensional vector spaces, and let $T : V \to W$ be a linear transformation. We want to understand solutions to equations of the form $Tv = w$. We make two observations.

(i) Understanding *uniqueness* of solutions to $Tv = w$ corresponds to understanding *injectivity* of T. Explicitly, if T is injective, then any solution to $Tv = w$ is unique. One way to measure the injectivity of T is to consider the dimension of the kernel of T, defined as $\ker T := \{ v \in V \ : \ Tv = 0 \}$, since T is injective exactly when $\dim \ker T = 0$. The larger the dimension of $\ker T$ is, the less injective T is. Hence, we have the following correspondence:

$$\left\{ \begin{array}{c} \text{uniqueness of solutions} \\ \text{to } Tv = w \end{array} \right\} \cong \left\{ \begin{array}{c} \text{injectivity of} \\ T \end{array} \right\} \longrightarrow \dim \ker T$$

(ii) Similarly, we can understand the *existence* of solutions to $Tv = w$ by understanding the *surjectivity* of T. Explicitly, if T is surjective (i.e. if $\operatorname{im} T = W$) then existence of solutions to $Tv = w$ is guaranteed for any $w \in W$. Define the cokernel of T to be $\operatorname{coker} T := W/\operatorname{im} T$, the quotient of the target space by the image of T. If quotienting makes you nervous, you can think of the cokernel as the orthogonal complement of $\operatorname{im} T$ in W. Hence, we can measure the surjectivity of T by considering the dimension of $\operatorname{coker} T$. If $\dim \operatorname{coker} T = 0$, then $\operatorname{im} T$ is all of W, and so T is surjective. The larger $\dim \operatorname{coker} T$ is, the less surjective T is. Thus, we have the following correspondence:

$$\left\{ \begin{array}{c} \text{existence of solutions} \\ \text{to } Tv = w \end{array} \right\} \cong \left\{ \begin{array}{c} \text{surjectivity of} \\ T \end{array} \right\} \longrightarrow \dim \operatorname{coker} T$$

In the spirit of the vague example from above, $N = \dim \ker T$ is a number which measures the uniqueness of solutions / injectivity of T, and $M = \dim \operatorname{coker} T$ is a number which measures the existence of solutions / surjectivity of T. Modeling equation (1.1), we make the following definition.

Definition 1.2. Let V and W be finite dimensional vector spaces, and let $T : V \to W$ be linear. The **index** of T is

$$\text{ind}\,(T) := \dim \ker T - \dim \operatorname{coker} T$$

Note that if T is invertible, then $\dim\ker T = \dim\operatorname{coker} T = 0$, and so ind $(T) = 0$. This suggests that the index of T measures how far T is from being invertible. Note, however, that ind $(T) = 0$ does *not* imply that T is invertible; all we can say in this case is that $\dim\ker T = \dim\operatorname{coker} T$. Another thing to note is that the Rank-Nullity theorem renders index theory in finite dimensions practically useless, as demonstrated by the next proposition.

Proposition 1.3. *Let V and W be finite dimensional vector spaces, and let $T : V \to W$ be linear. Then*

$$\textit{ind}\,(T) = \dim V - \dim W$$

Proof. Since $\operatorname{coker} T = W/\operatorname{im} T$, we have $\dim\operatorname{coker} T = \dim W/\operatorname{im} T = \dim W - \dim\operatorname{im} T$. Hence,

$$\begin{aligned}\text{ind}\,(T) &= \dim\ker T - \dim\operatorname{coker} T = \dim\ker T - (\dim W - \dim\operatorname{im} T)\\ &= (\dim\ker T + \dim\operatorname{im} T) - \dim W\end{aligned}$$

By the Rank-Nullity theorem, $\dim\ker T + \dim\operatorname{im} T = \dim V$, and we are done. □

In other words, the index of a map between finite dimensional spaces doesn't actually depend on map itself; it only depends on the dimensions of the domain and codomain. Hence, the concept of an index only becomes useful when we move into the world of infinite dimensional vector spaces.

2. The Space of Fredholm Operators

Instead of considering a linear map T between finite dimensional vector spaces V and W, we will study linear operators $T : \mathcal{H} \to \mathcal{H}$ on infinite dimensional complex Hilbert spaces. As in the finite dimensional case, we want to understand the space of solutions to equations of the form $Tf = g$. Also as before, this is equivalent to understanding injectivity and surjectivity of T. We might be tempted to define the index of an operator in the same way as before:

$$\text{ind}\,(T) := \dim\ker T - \dim\operatorname{coker} T$$

But now that we're working in infinite dimensions, we run into the possibility that one or both of $\dim\ker T$ and $\dim\operatorname{coker} T$ are infinite. For this reason, ind (T) is not well-defined for *all* operators (for example, the operator $0 : \mathcal{H} \to \mathcal{H}$ which sends everything to 0 has an infinite-dimensional kernel and an infinite-dimensional cokernel.).

To remedy this, we will define a new class of operators for which the index *is* well-defined.

Definition 2.1. An operator $T : \mathcal{H} \to \mathcal{H}$ is **Fredholm** if $\dim\ker T$ and $\dim\operatorname{coker} T$ are both finite. For a Fredholm operator T, its **Fredholm index** is:

$$\text{ind}\,(T) := \dim\ker T - \dim\operatorname{coker} T$$

We denote the set of Fredholm operators on $\mathcal{H}$ by $\mathfrak{F}(\mathcal{H})$.

We can think about these Fredholm operators as being "almost-invertible" in the sense that the kernel and cokernel are small enough to measure. As in the finite dimensional case, the Fredholm index of an operator gives a measurement for how defective (i.e. not invertible) such an operator is.

Example 2.2. An important class of Fredholm operators is the shift operators. Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for $\mathcal{H}$, and let $U : \mathcal{H} \to \mathcal{H}$ be the operator defined by $Ue_j = e_{j+1}$. The image of U is:

$$\operatorname{im} U = \text{Span}\{e_2, e_3, \dots\}$$

Hence, $\operatorname{coker} U = \mathcal{H}/\operatorname{im} U =$ Span$\{e_1\}$, and so $\dim\operatorname{coker} U = 1$. Since U is injective, $\dim\ker U = 0$, and we have:

$$\text{ind}\,(U) = \dim\ker U - \dim\operatorname{coker} U = 0 - 1 = -1$$

We can generalize this and consider the n-shift operator, which is the nth power of U:

$$U^n e_j = e_{j+n}$$

Then $\operatorname{im} U^n =$ Span$\{e_{n+1}, e_{n+2}, \dots\}$. Repeating the computation above gives ind $(U^n) = -n$.

We can also consider the backwards-shift operator, which is the adjoint[1] of U:

$$U^* e_j = e_{j-1}$$

In this case, U^* is surjective, and $\ker U^* =$ Span$\{e_1\}$. Hence,

$$\text{ind}\,(U^*) = 1 - 0 = 1$$

The backwards n-shift operator $(U^*)^n$ has index n.

This example suggests the following general properties of the index.

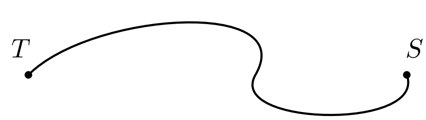
Proposition 2.3.
(i) *The map ind* : $\mathfrak{F}(\mathcal{H}) \to \mathbb{Z}$ *is surjective;*
(ii) *If* $T \in \mathfrak{F}(\mathcal{H})$, *then* $T^* \in \mathfrak{F}(\mathcal{H})$, *and ind* $(T^*) = -$*ind* (T);
(iii) *If* $T, S \in \mathfrak{F}(\mathcal{H})$, *then* $TS \in \mathfrak{F}(\mathcal{H})$, *and ind* $(TS) =$ *ind* $(T) +$ *ind* (S).

Proof. To prove (i), note that $(U^*)^n$ as defined above has index n, and U^n has index n. Thus, there are Fredholm operators of any given index. For (ii) and (iii), we defer to a standard reference (see section 6). $\square$

Other interesting properties of the Fredholm index come from considering the topology/metric structure on $\mathfrak{F}(\mathcal{H})$. In particular, under the operator norm,[2] $\mathfrak{F}(\mathcal{H})$ becomes a topological space, and so we can talk about continuity of functions in and out of the space. From this perspective (viewing $\mathfrak{F}(\mathcal{H})$ as a space itself, with operators as points), we arrive at the most important property of the index map.

Theorem 2.4. *The Fredholm index map ind* : $\mathfrak{F}(\mathcal{H}) \to \mathbb{Z}$ *is continuous, and hence locally constant by the discrete topology on* $\mathbb{Z}$. *Explicitly, given any Fredholm operator* T, *there is an open neighborhood* U *of Fredholm operators containing* T *such that ind* $(S) =$ *ind* (T) *for all* $S \in U$.

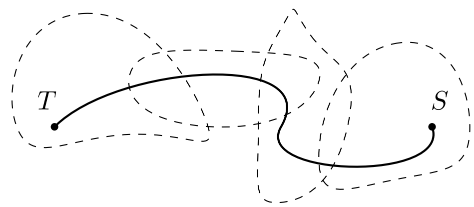
One implication of this theorem is that *the index is constant on connected components of* $\mathfrak{F}(\mathcal{H})$. Indeed, suppose that T and S are two Fredholm operators which are connected by a path in $\mathfrak{F}(\mathcal{H})$.



[1] The adjoint (roughly, conjugate-transpose) of an operator T is the unique operator T^* such that $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in \mathcal{H}$.

[2] The norm of an operator is defined to be $\|T\| := \sup_{\|f\|=1} \|Tf\|$. Vaguely, the norm of T measures how much T stretches the unit sphere in $\mathcal{H}$.

Since the Fredholm index is locally constant, at every point along the path we can find open neighborhoods of constant index. Since a path is a compact space, we can cover the entire path with a finite number of such open sets:



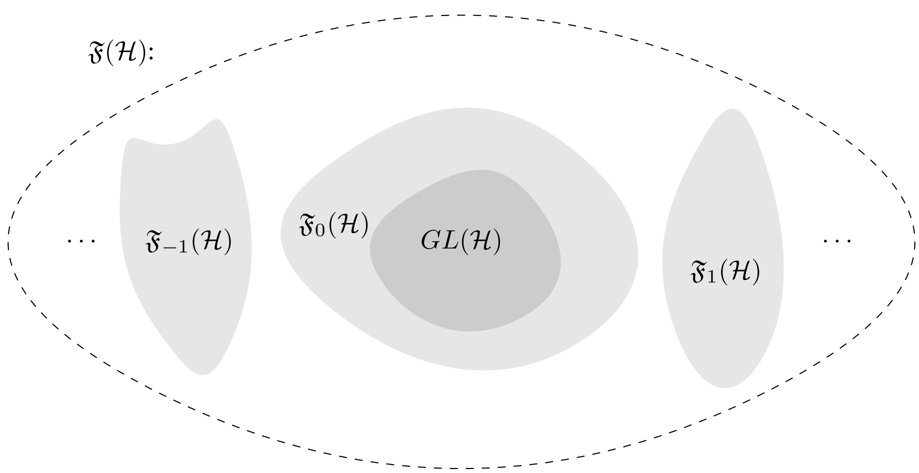
The index is constant on each of these open neighborhoods. Hence, the index must be constant on the union of any two intersecting neighborhoods, which forces the index to be constant along the entire path. Therefore, since any two Fredholm operators which are path-connected have the same index, the index map is constant on connected components. In fact, the converse is also true. If S and T have the same Fredholm index, they can be connected with path!

In other words, the *index partitions the space of Fredholm operators into connected components.* Explicitly,

Theorem 2.5. *The Fredholm index induces a bijection:*

$$\left\{ \begin{array}{c} \textit{connected components} \\ \textit{of } \mathfrak{F}(\mathcal{H}) \end{array} \right\} \longrightarrow \mathbb{Z}$$

We now have a vague idea of what this mysterious infinite-dimensional space of Fredholm operators looks like from a topological perpective:



Here, $\mathfrak{F}_n(\mathcal{H})$ is the set of of Fredholm operators of index n, and $GL(\mathcal{H})$ denotes the invertible operators. Obviously, this is a grotesquely simplified representation of $\mathfrak{F}(\mathcal{H})$; in reality, it is infinite dimensional, and the connected components may twist and tangle in unfathomable ways. Nevertheless, the bijection between connected components and the integers gives us a foothold in understanding the Fredholm operators.

Before moving on, let's recap what we've talked about so far. We are interested in studying linear equations and linear operators in infinite dimensional space:

$$Tf = g$$

In an ideal world, every operator we study would be invertible, but this is not the case. Fortunately, the Fredholm operators are large class of operators which are "almost invertible." The Fredholm index

$$\text{ind}\,(T) = \dim\ker T - \dim\operatorname{coker} T$$

gives us a way to measure how defective T is, and — despite being defined purely algebraically — provides us with insight into the topology of the space of all Fredholm operators; namely, that the connected components of $\mathfrak{F}(\mathcal{H})$ are distinguished by the index map.

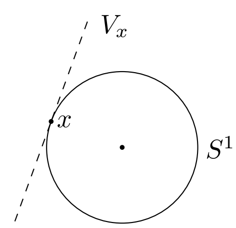
3. Vector Bundles and K-Theory

In Section 4, we will present the Atiyah-Jänich Theorem, a vast generalization of Theorem 2.5, which describes how the space of Fredholm operators can recover an incredible wealth of information about certain topological spaces. In order to do this, we need to switch gears and talk about a seemingly unrelated topic: vector bundles and K-theory.

3.1. **Vector Bundles.**

Linear algebra is the study of finite-dimensional vector spaces. Moving forward, we are interested in studying not just one vector space at a time, but whole families of vector spaces, parametrized by a topological space. A *vector bundle* is a *continuous* family of finite dimensional vector spaces, whatever that means.

In lieu of giving an actual definition, we'll introduce the concept by talking about an important example. Let S^1 denote the unit circle in $\mathbb{R}^2$. At each point $x \in S^1$, let V_x denote the tangent space:

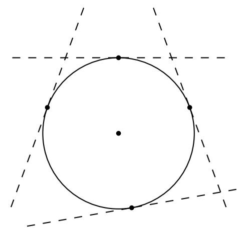


In this case, the tangent space at the specified point is a one-dimensional vector space. At different points on the circle, we get different — albeit isomorphic — vector spaces.

A vector bundle is what we get when we consider *all* of the tangent spaces together in one giant topological space. Explicitly, define the following topological space:

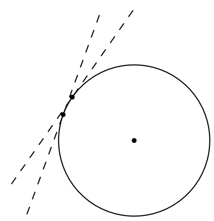
$$E := \bigsqcup_{x \in S^1} V_x$$

This space E, which we call a **vector bundle** over S^1, is the abstract *disjoint* union of the infinitely many dotted lines which are tangent to the circle:



Since every tangent space to the circle is a one-dimensional subspace living inside $\mathbb{R}^2$, another way to view the vector bundle E is as a subspace of $S^1 \times \mathbb{R}^2$. So as a topological space, E has the subspace topology inherited from $S^1 \times \mathbb{R}^2$.

The important thing to note about this example is that the tangent spaces vary "continuously," in that if we pick two points that are close together on the circle, the corresponding tangent spaces are similar:



This sort of continuity is what distinguishes a *vector bundle* from a general *family of vector spaces.* For example, we could define a new topological collection of vector spaces $\tilde{E}$ over the circle S^1 by associating *any* one dimensional vector space to each point:

$$\tilde{E} = \bigsqcup_{x \in S^1} \tilde{V}_x$$

where $\tilde{V}_x$ is some arbitrary one dimensional subspace of $\mathbb{R}^2$. Without any restriction on the $\tilde{V}_x$'s, this collection of vector spaces could be wildly discontinuous, in the sense that if x and y are two points that are close on the circle, $\tilde{V}_x$ and $\tilde{V}_y$ may look completely different.

This gives us a vague idea of what a vector bundle over S^1 looks like. More generally, we can consider a vector bundle E over any topological space X, where each $x \in X$ has an associated vector space E_x which varies in a continuous manner across X. Unsurprisingly, many of the concepts from regular finite dimensional linear algebra generalize nicely to vector bundles. For example, just as we can talk about homomorphisms and isomorphisms of vector spaces, we can talk about homomorphisms and isomorphisms of vector bundles. One way to think about an isomorphism between vector bundles E and F is as a "pointwise" isomorphism of the vector spaces E_x and F_x sitting above each $x \in X$. This allows us to consider isomorphism classes of vector bundles: if E is a vector bundle over X, we will denote by $[E]$ the set of all vector bundles over X which are isomorphic to E. The set of all isomorphism classes of vector bundles over X will be denoted $\mathcal{V}(X)$.

Example 3.1. Suppose that $X = \{x_0\}$ is a single-point space. Then any vector bundle over X is just a single vector space. So if $E = E_{x_0}$ is a vector bundle over X, then $[E]$ is the set of all $\dim E_{x_0}$ dimensional vector spaces. Hence, there is a set bijection between $\mathcal{V}(x_0)$ and $\mathbb{N}$. Every natural number n corresponds to the element of $\mathcal{V}(x_0)$ which consists of n-dimensional vector spaces.

Another vector space operation that generalizes to vector bundles is the *direct sum.* If E and F are vector bundles over a topological space X, we can form their direct sum $E \oplus F$ by taking the pointwise direct sum of vector spaces $E_x \oplus F_x$ at each point in X. This induces an addition operation on $\mathcal{V}(X)$ given by $[E] + [F] := [E \oplus F]$. Said differently, the direct sum operation turns $\mathcal{V}(X)$ into a monoid.[3]

Example 3.2. Consider again the case $X = \{x_0\}$. As we saw above, there is a set bijection between $\mathcal{V}(x_0)$ and $\mathbb{N}$. Since $\mathbb{C}^n \oplus \mathbb{C}^m \cong \mathbb{C}^{n+m}$, it follows that $\mathcal{V}(X)$ and $\mathbb{N}$ are actually isomorphic as monoids, where addition in $\mathbb{N}$ corresponds to the direct sum of the corresponding dimensional isomorphism classes in $\mathcal{V}(X)$.

3.2. K**-Theory.** The monoid structure on $\mathcal{V}(X)$ is nice, but the lack of inverses is annoying (how do we "subtract" vector bundles / vector spaces?). It would be even nicer to have a *group* structure on the isomorphism classes of vector bundles. This is where K-theory comes in.

Before defining K-theory, consider the monoid $\mathbb{N}$. There is a natural way to turn $\mathbb{N}$ into a group, in that we can formally define inverse elements $-1, -2, -3, \dots$. This gives us $\mathbb{Z}$. Hence, we can think about the integers as being a sort of "group-completion" of the natural numbers.

Explicitly, we do this as follows. Define the following set:

$$\mathcal{G}(\mathbb{N}) := \mathbb{N} \times \mathbb{N} / \sim$$

where $(n_1, m_1) \sim (n_2, m_2)$ if and only if $n_1 + m_2 = n_2 + m_1$. In other words, $\mathcal{G}(\mathbb{N})$ is the set of equivalence classes of pairs of natural numbers where we identify (n, m) with $(n + k, m + k)$ for all $k \in \mathbb{N}$. Informally, we can think about the pair (n, m) as representing the integer $n - m$. We can give $\mathcal{G}(\mathbb{N})$ a group structure under the operation

$$[(n_1, m_1)] + [(n_2, m_2)] := [(n_1 + n_2, m_1 + m_2)]$$

The identity element is $[(0, 0)]$, and the inverse of $[(n, m)]$ is $[(m, n)]$. Moreover, there is a group isomorphism of $\mathbb{Z}$ and $\mathcal{G}(\mathbb{N})$ given by $n \mapsto [(n, 0)]$. Hence, $\mathcal{G}(\mathbb{N}) \cong \mathbb{Z}$ is the "group-completion" of $\mathbb{N}$, constructed by formally introducing inverse elements.

This is a special case of the *Grothendieck completion* of an abelian monoid A, which is the most natural way to turn any abelian monoid into a group.

Definition 3.3. Let A be an abelian monoid. The **Grothendieck completion** of A is the group

$$\mathcal{G}(A) := A \times A / \sim \qquad \text{where} \qquad (a_1, b_1) \sim (a_2, b_2) \text{ if } a_1 + b_2 = a_2 + b_1$$

with group operation given by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$.

When we apply the Grothendieck completion to the monoid $\mathcal{V}(X)$ of isomorphism classes of vector bundles over a topological space X, we get K-theory.

Definition 3.4. Let X be a topological space. The K**-theory** (or K**-group**) of X is the abelian group $K(X) := \mathcal{G}(\mathcal{V}(X))$.

So if X is a topological space, the K-group $K(X)$ consists of equivalence classes of pairs $([E], [F])$ of isomorphism classes of vector bundles, which we informally think of as representing $[E] - [F]$.

[3]A *monoid* is basically a group without inverses.

There is a lot that can be said about K-theory; none of it will be covered here. Instead, we close this section with a different perspective on the K-group we just constructed. If you know anything about the fundamental group π_1 of a topological space, then you probably know that it is a *functor* from the category of topological spaces to the category of groups. Vaguely, π_1 takes in a topological space X and spits out a group $\pi_1(X)$ which can be used to understand certain topological aspects of X. More generally, the homotopy groups π_n are all functors which associate groups to topological spaces, each containing their own brand of topological information. Our discussion of vector bundles and K-theory has given us another example of a functor: $K(\cdot)$ takes in a topological space and spits out a group which measures certain kinds of topological information. What's surprising is that this algebraically-topological invariant structure is intrinsically linked with the space of Fredholm operators on a Hilbert space.

4. The Atiyah-Jänich Theorem

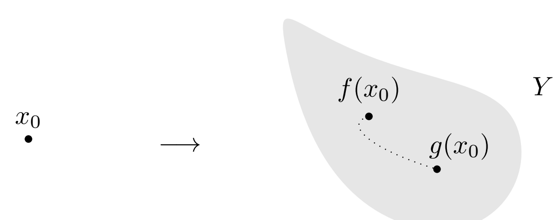
The main result of Section 2 was Theorem 2.5, which gave a bijection between the connected components of the space of Fredholm operators $\mathfrak{F}(\mathcal{H})$ and the integers $\mathbb{Z}$:

$$\left\{ \begin{array}{c} \text{connected components} \\ \text{of } \mathfrak{F}(\mathcal{H}) \end{array} \right\} \xrightarrow{\text{ind}} \mathbb{Z} \tag{4.1}$$

Let's take this equation and rewrite it, beginning with the right-hand side. In the previous section, we saw that the K-group of a single point space $\{x_0\}$ was isomorphic to $\mathbb{Z}$. Hence, the classical Fredholm index induces a bijection:

$$\left\{ \begin{array}{c} \text{connected components} \\ \text{of } \mathfrak{F}(\mathcal{H}) \end{array} \right\} \xrightarrow{\text{ind}} K(x_0) \tag{4.2}$$

Now let's consider the left-hand side. Recall that two continuous functions $f, g : X \to Y$ are *homotopic* if they can be "continuously deformed" into one another. We write $[X, Y]$ to denote the set of homotopy classes (equivalence under homotopy) of functions $X \to Y$, i.e., f and g determine the same element of $[X, Y]$ if and only if they are homotopic. In particular, we can consider two continuous functions $f, g : \{x_0\} \to Y$ out of a single-point space. These functions are homotopic if and only if $f(x_0)$ and $g(x_0)$ can be connected with a path:



The functions f and g are *not* homotopic when $f(x_0)$ and $g(x_0)$ *cannot* be connected with a path. Hence, $[x_0, Y]$ is homeomorphic to the connected components of Y. This means that

$$\left\{ \begin{array}{c} \text{connected components} \\ \text{of } \mathfrak{F}(\mathcal{H}) \end{array} \right\} \cong [x_0, \mathfrak{F}(\mathcal{H})]$$

and so we can rewrite the bijection in (4.2) as:

$$[x_0, \mathfrak{F}(\mathcal{H})] \xrightarrow{\text{ind}} K(x_0) \tag{4.3}$$

In fact, $[x_0, \mathfrak{F}(\mathcal{H})]$ is a group (the group structure comes from the group structure on $\mathfrak{F}(\mathcal{H})$), and so the above bijection is actually a group isomorphism. Look back at Proposition 2.3; this shows that the index map acts as a homomorphism. Shockingly, this isomorphism holds if we replace x_0 with any compact topological space X.

Theorem 4.1 (Atiyah-Jänich)**.** *Let X be a compact topological space. There is a group isomorphism*

$$[X, \mathfrak{F}(\mathcal{H})] \xrightarrow{\sim} K(X)$$

given by a generalized index map.

The proof of this theorem relies on the construction of said generalized index map, which we will now briefly discuss.

4.1. **The Generalized Index Map.** Let $T : X \to \mathfrak{F}(\mathcal{H})$ be a continuous function (so that it represents an element of $[X, \mathfrak{F}(\mathcal{H})]$). Roughly, we can think about T as a family of Fredholm operators, parametrized by X. For instance, if X is the interval $[0, 1]$, then $T : [0, 1] \to \mathfrak{F}(\mathcal{H})$ is a path of Fredholm operators. We wish to define a "generalized index map" which takes in such a function T and returns an element (which we will call ind (T)) of the group $K(X)$.

In the classical case $X = \{x_0\}$ and $K(x_0) \cong \mathbb{Z}$, a continuous function $T : x_0 \to \mathfrak{F}(\mathcal{H})$ was a single Fredholm operator, and the index map was given by:

$$\text{ind}\,(T) = \dim\ker T - \dim\operatorname{coker} T \in \mathbb{Z}$$

Our goal is to mimic this index formula for a *family* of Fredholm operators $T : X \to \mathfrak{F}(\mathcal{H})$. Recall that elements of $K(X)$ look like:

$$[V] - [W]$$

where V and W are vector bundles over X. Thus, our generalized index formula should look like:

$$\text{ind}\,(T) = [\text{some kind of "kernel" vector bundle}] - [\text{some kind of "cokernel" vector bundle}]$$

Indeed, since $T(x)$ is a Fredholm operator for every $x \in X$, $\ker T(x)$ and $\operatorname{coker} T(x)$ are finite dimensional vector spaces for all $x \in X$. One might suspect that the families of kernels and cokernels form vector bundles over X. Explicitly, we can form the topological spaces $\ker T$ and $\operatorname{coker} T$ by attaching $\ker T(x)$ and $\operatorname{coker} T(x)$ to each point $x \in X$:

$$\ker T := \bigsqcup_{x\in X} \ker T(x) \qquad \operatorname{coker} T := \bigsqcup_{x\in X} \operatorname{coker} T(x)$$

If these were both vector bundles, then we could define a generalized index map:

$$\text{ind}\,(T) = [\ker T] - [\operatorname{coker} T] \in K(X)$$

Unfortunately, it is *not* always the case that $\ker T$ and $\operatorname{coker} T$ are vector bundles. There are situations where the dimensions of $\ker T(x)$ and $\operatorname{coker} T(x)$ can jump in a discontinuous manner, which isn't allowed in a vector bundle.

Example 4.2. Suppose that $X = \mathbb{R}$ and $\mathcal{H} = \mathbb{C}$. We can define a family of "multiplication operators" $T : \mathbb{R} \to \mathfrak{F}(\mathbb{C})$ by $x \mapsto T_x$ where $T_x(z) := xz$. Then

$$\dim \ker T(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad \dim \operatorname{coker} T(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and so $\ker T$ and $\operatorname{coker} T$ cannot be vector bundles over $\mathbb{R}$.

One thing to note about the above example is that even though the dimensions of $\ker T(x)$ and $\operatorname{coker} T(x)$ jumped discontinuously, they jumped by the same amount, so that the "overall" dimension remained constant. This suggests that, even though $\ker T$ and $\operatorname{coker} T$ may not be vector bundles, we may be able to alter the function T slightly so that we get a well-defined element of $K(X)$.

The way we can do this is with a generalization of the following procedure. Let $T : \mathcal{H} \to \mathcal{H}$ be a single Fredholm operator. Note that we have an isomorphism of the output Hilbert space:

$$\mathcal{H} \cong \operatorname{im} T \oplus \operatorname{coker} T$$

Let $d = \dim \operatorname{coker} T$. Define

$$\tilde{T} : \mathcal{H} \oplus \mathbb{C}^d \to \mathcal{H} \cong \operatorname{im} T \oplus \operatorname{coker} T$$

by sending $\mathcal{H}$ to $T(\mathcal{H}) = \operatorname{im} T$ and $\mathbb{C}^d$ bijectively onto $\operatorname{coker} T$. By construction, $\tilde{T}$ is surjective, and so $\dim \operatorname{coker} \tilde{T} = 0$. Furthermore, the kernel of T and the kernel $\tilde{T}$ are the same, since the additional mapping of $\mathbb{C}^d$ onto $\operatorname{coker} T$ is injective. Hence, $\dim \ker \tilde{T} = \dim \ker T$, and so:

$$\text{ind}\left(\tilde{T}\right) = \dim \ker \tilde{T} = \dim \ker T$$

Therefore:

$$\text{ind}\,(T) = \text{ind}\left(\tilde{T}\right) - d \tag{4.4}$$

We took a Fredholm operator T, added some extra dimensions to the domain, and created a new *surjective* Fredholm operator $\tilde{T}$. As a result, we get an alternate way to calculate the index of the original operator.

Given a family of Fredholm operators $T : X \to \mathfrak{F}(\mathcal{H})$, we can create a new *surjective* family of Fredholm operators $\tilde{T}$ by adding dimensions at each $x \in X$ in the manner described above *in a continuous manner* across X. This is very hard to do, but it's possible — and by compactness of X, we can do this by only adding a *finite* number of dimensions. Since this new family of Fredholm operators $\tilde{T}$ is surjective, $\operatorname{coker} \tilde{T}(x) = 0$ for all $x \in X$. It turns out that by forcing the cokernel to be trivial, the family of kernels of $\tilde{T}$ is a vector bundle! Analogously to Equation (4.4), if d is the total number of dimensions added to create $\tilde{T}$, we can recover the index of T by subtracting off the trivial vector bundle $X \times \mathbb{C}^d$. Explicitly, we define the **generalized index** of a family of Fredholm operators $T : X \to \mathfrak{F}(\mathcal{H})$ as:

$$\text{ind}\,(T) := [\ker \tilde{T}] - [X \times \mathbb{C}^d] \tag{4.5}$$

which is a well-defined element of $K(X)$. With a lot of other work, it can be shown that the index is homotopy invariant, a group homomorphism, and surjective, and hence a group isomorphism with $K(X)$, completing the proof of Theorem 4.1.

5. Index Theory Examples

The Atiyah-Jänich theorem is a deep and interesting theorem, but it is a statement about K-theory, algebraic topology, cohomology, and the like. Instead of dwelling on complicated things like that, we will finish this exposition with some (simplified) comments on other interesting examples in index theory.

5.1. Toeplitz Operators and the Toeplitz Index Theorem.

The first example we will talk about is that of *Toeplitz operators.* A **Toeplitz matrix** is a matrix with constant diagonals. For example,

$$\begin{bmatrix} 2 & 9 & 0 & 1 & 0 \\ -1 & 2 & 9 & 0 & 1 \\ 5 & -1 & 2 & 9 & 0 \\ 2 & 5 & -1 & 2 & 9 \\ 1 & 2 & 5 & -1 & 2 \end{bmatrix}$$

is a Toeplitz matrix. A **Toeplitz operator** T is essentially an infinite dimensional analogue of Toeplitz matrices. Toeplitz operators look like:

$$T = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ & a_2 & a_1 & a_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

for some sequence $\{a_n\}$ of complex numbers. What kind of space does a Toeplitz operator act on? The real answer is something called the "Hardy space," but we won't discuss that in detail. Instead, recall that a function $f : \mathbb{C} \to \mathbb{C}$ is **analytic** about $z = 0$ if it has a power series expansion:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

Let $\mathbf{H}$ denote the Hilbert space of analytic functions about 0. Then a Toeplitz operator $T : \mathbf{H} \to \mathbf{H}$ acts on $f(z) = \sum_{n=0}^{\infty} c_n z^n$ by a sort of infinite matrix multiplication:

$$T(f)(z) = \sum_{n=0}^{\infty} d_n z^n$$

where

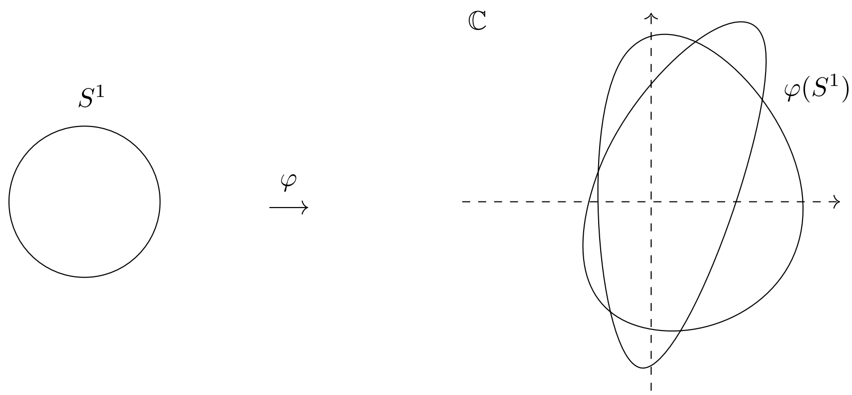
$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ & a_2 & a_1 & a_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}$$

Given what we studied previously, a natural question to ask is the following: when, if ever, is a Toeplitz operator a Fredholm operator?

To answer this question, we make the following observation. Associated to a Toeplitz operator T is an infinite sequence of complex numbers $\{a_n\}_{-\infty}^{\infty}$ given by the diagonals. We can create a bi-infinite power series, giving us a function which we'll call φ:

$$\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

Note that φ isn't necessarily analytic (because of the negative exponent terms) but it's a perfectly fine continuous complex function from $\mathbb{C} \to \mathbb{C}$. We call φ the **symbol** of T, and will actually view it as a function from $S^1 \subseteq \mathbb{C} \to \mathbb{C}$. As a continuous function from $S^1 \to \mathbb{C}$, the image of φ is a loop in the complex plane:



It turns out that we can determine when a Toeplitz operator is Fredholm by asking whether the image of its symbol passes through the origin.

Proposition 5.1. *A Toeplitz operator $T : \mathbf{H} \to \mathbf{H}$ is Fredholm if and only if its symbol $\varphi : S^1 \to \mathbb{C}$ is nonzero everywhere.*

Example 5.2. Consider the Toeplitz operator T whose matrix has 1's on the lower diagonal and 0's everywhere else:

$$\begin{bmatrix} 0 & 0 & 0 & & \cdots \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \ddots \\ & 0 & 1 & 0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Here, $a_1 = 1$ and $a_n = 0$ otherwise, so the symbol of this operator is $\varphi(z) = z$. Since the function z is never 0 on S^1 (the image of S^1 under z is just S^1; the function z doesn't do anything to S^1) it follows that T is a Fredholm operator.

Now that we know when certain Toeplitz operators are Fredholm, the next natural question to ask is: if T is a Fredholm Toeplitz operator, what is its index?

Perhaps not surprisingly, the answer again comes from the symbol φ. For any Fredholm Toeplitz operator, its symbol φ determines a loop in $\mathbb{C}$ that never passes through the origin. Hence, we can define the **winding number** $w(\varphi)$ to be the number of times the curve $\varphi(S^1)$ loops around the origin, going counter-clockwise.

Example 5.3.

(a) The function φ in the picture above (assuming it winds in a counter-clockwise direction) winds around the origin 2 times, so $w(\varphi) = 2$. The function $\bar{\varphi}$ whose image was the same curve but with *clockwise* rotation would have winding number -2.
(b) The winding number of the function z in Example 5.2 is $w(z) = 1$, since the image of S^1 is S^1. One has to be slightly careful, though: the image of S^1 under z^2 is also S^1, but it winds around *twice*. Thus, $w(z^2) = 2$. In general, $w(z^n) = n$.

Another way to define the winding number is in terms of the fundamental group of $\mathbb{C} \setminus 0$. Recall that $\pi_1(\mathbb{C} \setminus 0) \cong \mathbb{Z}$, and that elements of $\pi_1(\mathbb{C} \setminus 0)$ are homotopy classes of continuous functions $S^1 \to \mathbb{C} \setminus 0$. Hence, any continuous nonzero function φ on S^1 determines an integer n which is its image in the fundamental group $\pi_1(\mathbb{C} \setminus 0)$, which coincides exactly with the winding number $w(\varphi)$ as defined above.

From either perspective, it is fairly clear that the winding number of the symbol of a Toeplitz operator is a *topological invariant*; that is, it doesn't change if the operator (and consequently its symbol) is wiggled just a little bit. Miraculously, the winding number of φ also coincides with the Fredholm index of the Toeplitz operator with symbol φ.

Theorem 5.4 (Toeplitz Index Theorem)**.** *Let $T : \mathbf{H} \to \mathbf{H}$ be a Toeplitz operator with nowhere-zero symbol φ. Then*

$$\textit{ind}\,(T) = -w(\varphi)$$

This theorem is exemplifies the surprising connections between analytic/algebraic and topological ideas that index theory uncovers.

5.2. **The Atiyah-Singer Index Theorem.** One of the most important examples of an analytic and topological connection in index theory (and one of the most important mathematical results of the 20th century in general) is the Atiyah-Singer index theorem. As with the Toeplitz index theorem, it says that the analytic (Fredholm) index and a topological index of a certain kind of differential operator are equal. The Atiyah-Singer index theorem is very deep and very complicated, but we will try to to give a general sketch of the idea here.

The Atiyah-Singer index theorem concerns partial differential equations. In the introduction, we alluded to Poisson's equation, given by $\Delta f = g$. A simple case of this is the *Laplace equation in two variables*, given by:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{5.1}$$

We can take an equation like this an write it in "operator form" as follows:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0 \tag{5.2}$$

If we set $D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then (5.2) is of the form $Df = 0$ where D is a linear differential operator. Solutions to the original equation are elements of the kernel of D. Not surprisingly, given various kinds of assumptions, differential operators like D can be Fredholm. How can we calculate the index of a Fredholm differential operator?

To do this, we'll define the **symbol** $P(\xi, \eta)$ of a differential operator like D to be the polynomial obtained by replacing $\partial/\partial x$ with $i\xi$ and $\partial/\partial y$ with $i\eta$, where ξ, η are *real* variables. The symbol of the differential operator in (5.1) is:

$$P(\xi, \eta) = (i\xi)^2 + (i\eta)^2 = -\xi^2 - \eta^2$$

Note that $-\xi^2 - \eta^2 = 0$ implies that $\xi = \eta = 0$. Equations and operators for which this is true (namely, that $P(\xi, \eta) = 0$ implies $\xi = \eta = 0$) are called **elliptic**. Another example of an elliptic partial differential equation / operator is:

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \qquad\qquad D = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

The symbol in this case is $i\xi - \eta$; since ξ and η must be real, this is 0 only when both quantities are 0. A PDE that is *not* elliptic is:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0 \qquad\qquad D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

Here the symbol is $i\xi + i\eta$, which is 0 when $\xi = 1$ and $\eta = -1$, for instance.

We can think of the symbols $P(\xi, \eta)$ as being maps from $\mathbb{R}^2 \cong (\xi, \eta)$-space to $\mathbb{C}$. Actually, we will think of them as being maps from the unit circle S^1 of $\mathbb{R}^2 \cong (\xi, \eta)$-space to $\mathbb{C}$. For an elliptic differential operator, since the symbol is never 0 on the the unit circle S^1, $P(\xi, \eta)$ is in fact a map from S^1 to $\mathbb{C} \setminus 0$. This means that the symbol $P(\xi, \eta)$ has a *winding number*!

Theorem 5.5 (Vague Baby Index Theorem)**.** *Let D be a nice enough elliptic differential operator of two variables, and let $P(\xi, \eta)$ be its symbol. Then ind $(D) = w(P(\xi, \eta))$.*

This theorem shares a lot of similarities with the Toeplitz index theorem: we are equating the Fredholm index of a certain kind of operator to a topological winding number of a its "symbol." That's pretty cool, but we're still a million miles away from the Atiyah-Singer index theorem.

We can generalize our discussion about elliptic differential equations in a couple ways. For one, we could look at equations which depend on more than just two variables. If D is a differential operator with respect to $x_1, \ldots, x_n$, then its symbol will be a polynomial of n variables $\xi_1, \ldots, \xi_n$. Another generalization is to consider vector-valued functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ instead of scalar-valued functions $f : \mathbb{R}^n \to \mathbb{R}$. For example, we could consider general first-order partial differential equations of the form:

$$A_1(\mathbf{x})\frac{\partial \mathbf{f}}{\partial x_1} + \cdots A_n(\mathbf{x})\frac{\partial \mathbf{f}}{\partial x_n} = \mathbf{0} \tag{5.3}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n)$, and each $A_j(\mathbf{x})$ is a $n \times n$ *matrix-valued* function. The symbol of an equation like this would look something like:

$$P(\mathbf{x}, \xi) = P(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = iA_1(\mathbf{x})\xi_1 + \cdots iA_n(\mathbf{x})\xi_n$$

A couple things to note: The domain of the symbol map is now $\mathbb{R}^{2n}$ instead of $\mathbb{R}^2$, so instead of thinking of P as a function from S^1, we will view it as a function from the unit hypersphere $S^{2n-1} \subseteq \mathbb{R}^{2n}$. Moreover, the output of the symbol P is no longer a complex number; it's a complex matrix! Instead of requiring that the symbol be nonzero, we say that an equation like (5.3) is **elliptic** if $P(\mathbf{x}, \xi)$ is an *invertible $n \times n$ matrix* for all nonzero ξ and all $\mathbf{x}$. So instead of assigning a winding number to a map $S^1 \to \mathbb{C} \setminus 0$, we need to assign some other kind of topological invariant to a map $S^{2n-1} \to GL(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ denotes the space of invertible complex matrices of size $n \times n$. Such a quantity exists, and it comes from the fascinating *Bott periodicity theorem*. Let's call this generalization of the winding number the *Bott invariant.* Indeed, when $n = 1$, the Bott invariant coincides with the winding number that we've been using. Moreover, we have the following.

Theorem 5.6 (Another Vague Index Theorem). *Let D be a nice enough elliptic partial differential operator corresponding to a multivariable vector valued equation like* (5.3). *Then ind (D) equals the Bott invariant of its symbol.*

If you're thinking *wow, the Atiyah-Singer index theorem is really cool*, I've got bad news — we're not even close. Fortunately, this theorem presents the basic idea. The real index theorem deals with elliptic partial differential operators on abstract *manifolds*, rather than differential operators of n variables, and equates the Fredholm index to a complicated topological generalization of the Bott invariant. Put succinctly, we have:

Theorem 5.7 (Vague Atiyah-Singer Index Theorem). *Let D be a nice enough elliptic differential operator in an appropriate setting. Then ind (D) equals an important topological index.*

6. Conclusion

The takeaway for what we've talked about is the following: index theory gives us tools to build connections between analysis, topology, geometry, and algebra, and the space of Fredholm operators plays a crucial role in doing so. We've seen the following phenomena:

- (i) The classical Fredholm index $\dim\ker T - \dim\operatorname{coker} T$, an analytic idea which is algebraically defined, gives us insight into the topology of the space of Fredholm operators.
- (ii) The generalized index, defined using Fredholm operators and vector bundles, gives us insight into the topology of any compact space.
- (iii) The Toeplitz index theorem gives an explicit index computation of an important class of operators in terms of the topological winding number of continuous functions.
- (iv) The Atiyah-Singer index theorem relates the analytic and topological index of an important class of partial differential operators.

These ideas are powerful and elegant, and have inspired an incredible amount of mathematics, most of which we did not mention. Here are some resources where you can learn more, if you are so inclined.

- For functional analysis (the theory of Hilbert spaces, operators, infinite-dimensional linear algebra, etc.) the most accessible resource is [1]. Another wonderful text, which also covers the theory of Fredholm and Toeplitz operators, is [2].
- For a detailed development of K-theory, Atiyah's original manuscript [3] is standard; the appendix covers the Atiyah-Jänich theorem. Another accessible resource on topological K-theory is [4].
- A standard reference for the Atiyah-Singer index theorem is [5].
- A fantastic book for all of the topics covered here, and much more, is [6].

References

1. Kreyzig, E., 1978: *Introductory Functional Analysis with Applications*. John Wiley & Sons, 688 pp.
2. Douglas, R., 1998: *Banach Algebra Techniques in Operator Theory*. Springer Science and Business Media, 198 pp.
3. Atiyah, M., 1964: *K-Theory*. W.A. Benjamin, Inc., 166 pp.
4. Park, E., 2008: *Complex Topological K-Theory*. Cambridge University Press, 208 pp.
5. Palais, R., 1965: *Seminar on the Atiyah-Singer Index Theorem*. Princeton University Press, 366 pp.
6. Bleecker, D. and Bernhelm, B., 2013: *Index Theory with Applications to Mathematics and Physics*. International Press of Boston, 766 pp.