A Short Note on the Gradient Vector

OR, THE BENEFIT OF MULTIPLE PERSPECTIVES IN MULTIVARIABLE CALCULUS

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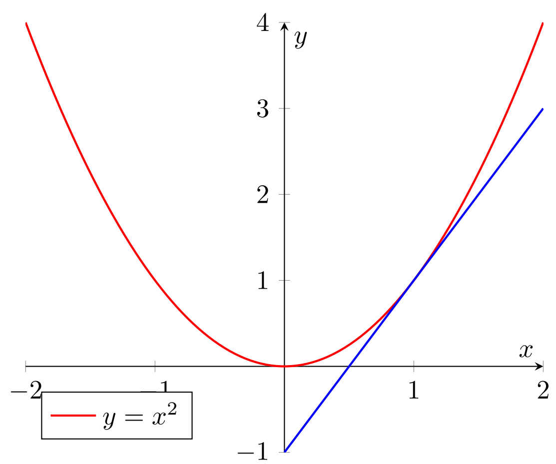
The general goal of this note is to describe how to find tangent planes to a surface via the gradient vector. Rather than jumping into an example and show you what to do, I'm going to motivate the perspective with a low-dimensional analogue. An important theme in multivariable calculus is that **every single topic is a direct extension of something familiar in one variable calculus**. As such, it can be insightful to reinterpret ideas like the gradient in a low-dimensional setting.

1 Computing a tangent line

Before computing a tangent plane, I'm going to begin with a problem that may seem a little silly:

Example 1.1. *Consider the curve $y = x^2$. Find the equation of the tangent line to this curve at the point* $(1, 1)$.



You should be thinking, objecting, or wondering something along the lines of: *Joe, I already know how to do this; I learned this in single variable calculus. Why are you wasting my time computing a boring tangent line?* The idea is the following: we can certainly compute the tangent line using single variable calculus, but **we can completely change the perspective of the problem and use the gradient vector of a multivariable function**. This will be the low-dimensional analogy to computing tangent planes via the gradient.

1.1 The first perspective: single variable calculus

First, we'll compute the tangent line like any normal person would and just use single variable calculus techniques. No multivariable calculus here! Recall that the equation for a tangent line to f through the point $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a).$$

Here $f(x) = x^2$ and $a = 1$. Since $f(1) = 1$ and $f'(1) = 2x\ |_1 = 2$, the equation of the tangent line we seek is

$$y = 1 + 2(x - 1).$$

Done!

1.2 The second perspective: multivariable calculus

This is where things get a little wild, and arguably needlessly complicated. I don't disagree, but the point is for you to see how the ideas of multivariable calculus are being used in low-dimensions so that you can more easily understand the analogous situation in higher dimensions.

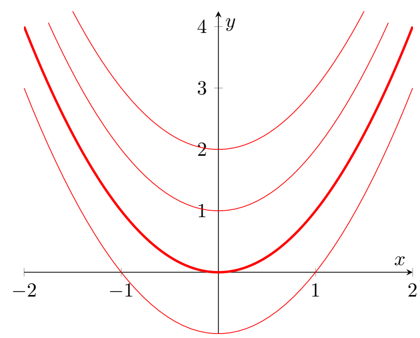
The change in perspective is that instead of viewing the curve $y = x^2$ as the graph of a single variable function $f(x)$, we will view the curve $y = x^2$ **as the level curve of a multivariable function $F(x, y)$.** In particular, let

$$F(x, y) := y - x^2.$$

Then the curve $y = x^2$, which rewritten is the curve $y - x^2 = 0$, is precisely a level curve of F with value 0:

$$F(x, y) = 0 \quad \rightsquigarrow \quad y - x^2 = 0.$$

Here is a contour plot of the function F:



I've plotted the level curves $F(x, y) = c$ for $c = -1, 0, 1, 2$. The thicker red line is the level curve $F(x, y) = 0$, and this is exactly the curve $y = x^2$ that we care about.

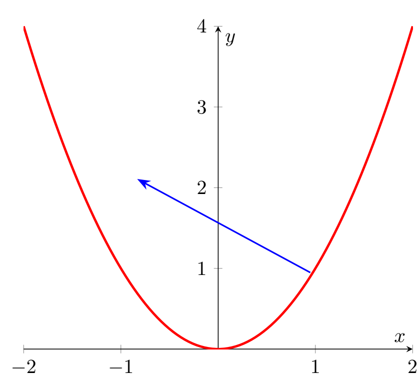
Next, **the main fact we need about the gradient vector is that it is perpendicular to level things**. In particular, the vector

$$\nabla F(x, y) = \langle F_x(x, y), F_y(x, y)\rangle = \langle -2x, 1\rangle$$

is perpendicular to the level curve through the point (x, y). We care about the point $(1, 1)$, so let's compute the gradient vector at that point:

$$\nabla F(1, 1) = \langle -2, 1\rangle.$$

Indeed, if I plot this vector on the contour plot above, we get:



Next, how do we use this vector to find the *tangent* line? Recall that to find the equation of a *plane*, you need a normal vector $\langle a, b, c\rangle$ and a point (x_0, y_0, z_0). With this information the plane is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

It turns out that the same exact thing works for lines! You should think about why this works to see if you really understand the above equation. But given a normal vector $\langle a, b\rangle$ to the line and a point (x_0, y_0) on the line, the equation of the line is

$$a(x - x_0) + b(y - y_0) = 0.$$

In our problem, the line passes through the point $(1, 1)$ and has normal vector $\langle -2, 1\rangle$ (the gradient vector of F at that point), so the equation of the tangent line is:

$$-2(x - 1) + 1(y - 1) = 0 \qquad \rightsquigarrow \qquad y = 1 + 2(x - 1).$$

Thankfully, this is exactly the line we got in the first part!

2 Computing a tangent plane

Having done the lower dimensional example above, let's tackle a tangent plane computation using two different perspectives:

Example 2.1. *Consider the surface* $z = x^2 + y^2$*. Find the equation of the tangent plane at the point* $(1, 1, 2)$.

2.1 The first perspective: linearization / tangent plane formula

This solution is the analogue of the first perspective solution above. In 15.4 we learn that the tangent plane to the graph of a function $f(x, y)$ at the point (a, b) is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We can view the surface $z = x^2 + y^2$ as the graph of the function $f(x, y) = x^2 + y^2$. Since $f(1, 1) = 2$ and

$$f_x(1, 1) = (2x)\,|_{(1,1)}= 2 \quad \text{and} \quad f_y(1, 1) = (2y)\,|_{(1,1)}$$

it follows that the equation of the tangent plane we seek is

$$z = 2 + 2(x - 1) + 2(y - 1).$$

2.2 The second perspective: level surfaces

This solution is the analogue of the second perspective solution above. Instead of viewing the surface $z = x^2 + y^2$ as the graph of the function $f(x, y) = x^2 + y^2$, we can alternatively **view it as a level surface of a three variable function**. In particular, let

$$F(x, y, z) = z - x^2 - y^2.$$

Then the surface $z = x^2 + y^2$, which when rewritten is the surface $z - x^2 - y^2 = 0$, is precisely the level surface $F(x, y, z) = 0$. We know that the gradient vector ∇F is perpendicular to level surfaces, so we can find a normal vector to the tangent plane we seek by computing $\nabla F(1, 1, 2)$. We have

$$\nabla F(x, y, z) = \langle -2x, -2y, 1\rangle$$

and so a normal vector to the tangent plane is

$$\nabla F(1, 1, 2) = \langle -2, -2, 1\rangle.$$

Thus, the tangent plane equation is

$$-2(x-1) + -2(y-1) + 1(z-2) = 0 \qquad \rightsquigarrow \qquad z = 2 + 2(x-1) + 2(y-1).$$

Exactly the same as what we found above!

3 One more tangent plane example

So what's the point? Maybe it's kind of cool that we can compute the same thing using two different perspectives, but why bother? If I can always use the first perspective, why should I worry about the second perspective? **Sometimes, one perspective is much better suited for a given problem.** Being able to tackle a math problem with a variety of viewpoints is in immeasurably important skill to have! For example,

Example 3.1. *Consider the surface* $x^2 \sin z + yx + \cos(yz) = 2$. *Find the equation of the tangent plane at the point* $(1, 1, 0)$.

Solution. This is an example where you can't really treat the surface as the graph of a two variable function, so we kind of have to take the second perspective.[1] It is much more natural to view the surface defined above as the level surface of a three variable function. In particular, let

$$F(x, y, z) = x^2 \sin z + yx + \cos(yz).$$

The surface we care about is the level surface $F(x, y, z) = 2$. Thus, to get a normal vector for the tangent plane, we can compute the gradient vector $\nabla F(1, 1, 0)$. Since

$$\nabla F(x, y, z) = \big\langle 2x \sin z + y, x - z\sin(yz), x^2 \cos z - y\sin(yz)\big\rangle$$

we have

$$\nabla F(1, 1, 0) = \langle 1, 1, 1\rangle.$$

This is a normal vector for the plane we seek. Thus, the equation of the tangent plane is

$$1(x-1) + 1(y-1) + 1(z-0) = 0 \quad \rightsquigarrow \quad x + y + z = 2.$$

Easy! This would be a disaster if you tried to solve for one of the variables in terms of the others. □

[1]This is a bit of a lie, locally near the point we care about we could treat it like a function but this is not the point. It's much easier to use the second perspective.