A Guide to the Limit Comparison Test

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The limit comparison test is the GOAT infinite series convergence test, but knowing when and how to use it effectively can be difficult. This guide explains the intuition, subtleties, and heuristics of the test and hopefully provides enough elucidating examples. The blue links below are hyperlinks to each section.

Contents

LCT only works with *positive* series

1 The statement of the limit comparison test

In order to use limit comparison, we have to know the statement. I'll provide the mathematical statement, but also how you should *think* about the statement.

Theorem 1.1 (Limit comparison test.)**.** *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with $a_n > 0$. Let $b_n > 0$ be a positive sequence.*

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite, positive number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or diverge.

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges. $\sum a_n \leq \sum b_n$

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

$\sum a_n \simeq \sum b_n$ $\sum a_n \geq \sum b_n$

WARNING: One thing to keep in mind about cases (ii) and (iii) above is that **you need to be careful if you do limit comparison and you get a limit of 0 or ∞**. In particular, let's say you pick a series $\sum b_n$ to compare with, and you know that $\sum b_n$ *diverges*. If you compute that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, you have argued that $\sum a_n \leq \sum b_n$. Since $\sum b_n$ diverges, this doesn't tell you anything! Similarly, if you pick a series $\sum b_n$ to compare with, you that $\sum b_n$ *converges*, and you compute that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, you have argued that $\sum a_n \geq \sum b_n$ and this doesn't tell you anything!

2 When and how to use limit comparison

In my mind, there are **three main scenarios** in which limit comparison is helpful, and then a fourth catch-all scenario. The first subsection gives a summary, and the following subsections below describe these scenarios in detail.

2.1 A quick summary

If you are given a series...

(I) ...of the form

$$\sum_{n=1}^{\infty} \frac{[\text{dominant term}] + [\text{fluff}]}{[\text{dominant term}] + [\text{fluff}]}$$

then run limit comparison against the series

$$\sum_{n=1}^{\infty} \frac{[\text{dominant term}]}{[\text{dominant term}]}.$$

$\ln n \rightsquigarrow n^{\text{small \#}}$

Typically this will be a p-series. Also, if you do this correctly your limit computation should give you $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and thus you are in scenario (i) of the limit comparison test.

(II) ...with a $\ln n$ that **you can't ignore**, then generate a b_n to compare with by replacing $\ln n$ with $n^{\text{small \#}}$. The small number depends on what else is in the series. **When you do this you will almost certainly be in scenario (ii) or (iii) of the limit comparison test**, so you need to be careful about your conclusion.

$\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$

(III) ...with something like $\sin\left(\frac{1}{n}\right)$, generate a b_n to compare with by replacing the $\sin\left(\frac{1}{n}\right)$ term with $\frac{1}{n}$. There are other useful approximations (based on Taylor polynomials) but in 31B, $\sin\left(\frac{1}{n}\right)$ is the mostly likely suspect.

(IV) ...and none of the above situations above apply. If you think that the series diverges, just run limit comparison against $\frac{1}{n}$. If you think the series converges, run limit comparison against $\frac{1}{n^2}$. If you're unsure, just try both!

2.2 Case I: Isolating obvious dominant behavior

This is the main situation in which limit comparison applies, and it is also the most straightforward. **Given a series that exhibits obvious dominant terms in the numerator and denominator, you can generate a b_n to compare with by isolating this dominant behavior**. This is best explained with an example: consider the series

$$\sum_{n=1}^{\infty} \frac{n^2 + 1 + \sin n}{\sqrt{n^7 + n^5 + 1}}.$$

My first thought when faced with the above series is that there is a lot of fluff that I can ignore. For example, in the numerator, there are three different terms: n^2, 1, and $\sin n$. As n grows to infinity, the most important term is n^2. (The 1 term does not grow at all, and $\sin(n)$ dances around in the interval $[-1, 1]$. Note that we are *not* in Case III described above because $\sin(n)$ is *much* different than $\sin\left(\frac{1}{n}\right)$.) Likewise, in the denominator there are three terms *under a square root*: n^7, n^5, and 1. Thus, the dominant behavior in the denominator is $\sqrt{n^7}$. All of this leads me to the following intuition:

$$\sum_{n=1}^{\infty} \frac{n^2+1+\sin n}{\sqrt{n^7+n^5+1}} \approx \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^7}} = \sum_{n=1}^{\infty} \frac{n^2}{n^{\frac{7}{2}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}.$$

Because $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series, this intuition suggests that $\sum_{n=1}^{\infty} \frac{n^2+1+\sin n}{\sqrt{n^7+n^5+1}}$ should also converge. **The limit comparison test is the way to formalize this intuition**! Indeed,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n^2+1+\sin n}{\sqrt{n^7+n^5+1}}}{\frac{n^2}{\sqrt{n^7}}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}+\frac{\sin n}{n^2}}{\sqrt{1+\frac{1}{n^5}+\frac{1}{n^7}}} \\
&= \frac{1+0+0}{\sqrt{1+0+0}} \\
&= 1.
\end{aligned}$$

Because 1 is a finite, positive number, we are in case (i) of the limit comparison test: $\sum_{n=1}^{\infty} \frac{n^2+1+\sin n}{\sqrt{n^7+n^5+1}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ either both converge or both diverge. Because the latter series converges, we have concluded by limit comparison that $\sum_{n=1}^{\infty} \frac{n^2+1+\sin n}{\sqrt{n^7+n^5+1}}$ converges as well!

2.3 Case II: Pesky logarithms

Logarithms introduce some subtleties in limit comparison. The main takeaway from this section is the following fact:

$$\ln n \ll n^a \qquad \text{for any } a > 0.$$

$\ln n$ grows *very* slowly

In fact,

$$(\ln n)^p \ll n^a \qquad \text{for any } a > 0 \text{ and any } p > 0.$$

All of this means that we can absorb the growth of a logarithm with a very small power of n. In particular, **if there is a logarithm that you can't ignore in a series, you should generate a candidate for comparison by replacing the $\ln n$ term with a very small power of n**. By doing this, you absorb the logarithmic growth and will likely be in scenario (ii) or (iii) of the limit comparison test.

It's easiest to explain this through an example. Consider the following series:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

Because the denominator grows like n^2 and the denominator grows much slower than n^2, my intuition suggests that this series should converge. Let's formalize this with limit comparison.

WARNING: In a situation like this, you *cannot* just ignore the $\ln n$ and run comparison against $\frac{1}{n^2}$. It will fail! Let's check:

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \ln n = \infty.$$

$\sum \frac{\ln n}{n^2} \geq \sum \frac{1}{n^2}$

Go back and read scenario (iii) of the limit comparison test carefully: because we got a limit of ∞, we can only make a conclusion if we know that the corresponding series in the denominator diverges. But $\sum_{n=1}^{\infty} \frac{1}{n^2}$. So the test is inconclusive!

This warning shows that even though logarithmic growth is insanely slow, you cannot simply ignore it. You do have to budget a small amount of growth to absorb the logarithm. This is where the heuristic

$$\ln n \rightsquigarrow n^{\text{small }\#}$$

comes into play. If we run this heuristic on our summand, we get

$$\frac{\ln n}{n^2} \rightsquigarrow \frac{n^{\text{small }\#}}{n^2} = \frac{1}{n^{2-(\text{small }\#)}}.$$

Because we want to show that our series converges, we want to make sure that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-(\text{small }\#)}}$$

converges. This means that we need $2 - (\text{small }\#) > 1$ and so $0 < \text{small }\# < 1$. In this case, we can pick small $\# = \frac{1}{2}$ and thus run limit comparison against $\frac{1}{n^{\frac{3}{2}}}$. Indeed,

since $\ln n \ll n^a$

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{2}}} = 0.$$

$\sum \frac{\ln n}{n^{\frac{3}{2}}} \leq \sum \frac{1}{n^{\frac{3}{2}}}$

Because we got a limit of 0 and we know that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges, we can successfully apply scenario (ii) of the limit comparison test to conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ also converges.

2.4 Case III: Trig functions and other oddities (Taylor polynomials)

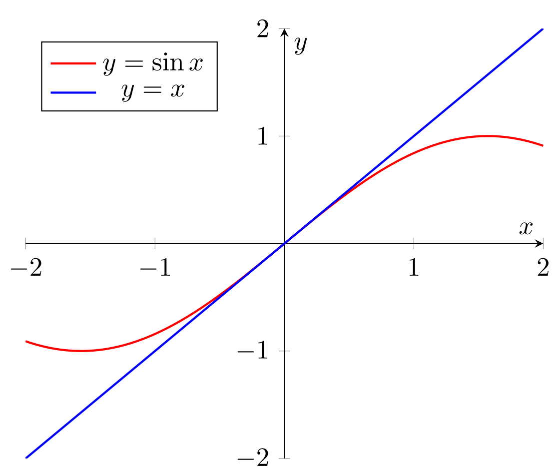
The main take away from this case is the following approximation:

$$\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}.$$

Where does this come from, and are there other useful approximations?

The first approximation: $\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$

Let's talk about the function $\sin x$ first, and then I will describe the more general situation. Consider the graph of $y = \sin x$. The tangent line at $x = 0$ is $y = x$, because $\sin(0) = 0$ and $\cos(0) = 1$. We know that close to 0, this tangent line should be a good approximation to the function. Indeed, we can graph both near 0 to confirm this:



It is clear that $\sin x \approx x$ **as long as** x **is close to** 0. The fraction $\frac{1}{n}$ is close to 0 for large values of n, so indeed $\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$. We can exploit this fact to generate useful candidates for limit comparison. For example, consider the series

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n}.$$

Based on the above discussion, my intuition tells me that

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n} \approx \sum_{n=1}^{\infty} \frac{\frac{1}{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

As usual, the formalization of this intuition is limit comparison. Let's compute:

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{\sin\left(\frac{1}{n}\right)}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \\ (L'H) &= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2}}{-\frac{1}{x^2}} = \cos(0) = 1. \end{aligned}$$

Since 1 is a finite, positive number, we are in scenario (i) of the limit comparison test. This implies that the series $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ behave the same, just as we suspected. Since the latter is a convergent p-series, the former converges as well!

Other useful approximations

Case III doesn't end with the approximation of $\sin\left(\frac{1}{n}\right)$. Because $\sin x \approx x$ for x small, we get a host of other similar approximations:

$$\sin\left(\frac{1}{n^2}\right) \approx \frac{1}{n^2}$$
$$\sin\left(\frac{1}{n^3}\right) \approx \frac{1}{n^3}$$
$$\sin\left(\frac{1}{\sqrt{n}}\right) \approx \frac{1}{\sqrt{n}}$$

and so on and so forth. Furthermore, by manipulating these approximations we can go even further:

$$\sin^2\left(\frac{1}{\sqrt{n}}\right) \approx \left(\frac{1}{\sqrt{n}}\right)^2 = \frac{1}{n}$$
$$\sqrt{\sin\left(\frac{1}{n^3}\right)} \approx \sqrt{\frac{1}{n^3}} = \frac{1}{n^{\frac{3}{2}}}.$$

In general, approximations like $\sin x \approx x$ for x small can give you intuition about the behavior of many different series.

WARNING: The approximation $\sin x \approx x$ **only holds when** x **is small**. In order to use this approximation, the thing inside of the sine function **needs to tend to** 0. In particular, if you see something like $\sin(n)$ in a series, **it would be wildly incorrect to claim that** $\sin(n) \approx n$.

The full story: Taylor polynomial approximations

More generally, we can capture the behavior of other sequences by using knowledge of Taylor polynomials. For example, recall that the degree 2 Taylor polynomial for $\cos x$ at 0 is

$$T_2(x) = 1 - \frac{x^2}{2}.$$

This means that for x near 0, we have the approximation $\cos x \approx 1 - \frac{x^2}{2}$. We can use this approximation just like we used the one for $\sin x$ above. In particular, given a series like

$$\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n^2}\right)\right)$$

we could identify the general behavior of the summand as

$$1 - \cos\left(\frac{1}{n^2}\right) \approx \frac{1}{2}\left(\frac{1}{n^2}\right)^2 = \frac{1}{2n^4}.$$

We could then use limit comparison to verify that $\sum_{n=1}^{\infty}\left(1 - \cos\left(\frac{1}{n^2}\right)\right)$ behaves the same as $\sum_{n=1}^{\infty} \frac{1}{2n^4}$, and thus converges.

As another quick example of generating such approximations, recall that the degree 1 Taylor polynomial for e^x at 0 is

$$T_1(x) = 1 + x.$$

Thus, we expect $e^x \approx 1 + x$ for x near 0. This leads to sequential approximations like

$$1 - e^{\frac{1}{n}} \approx -\frac{1}{n}$$

and so on and so forth.

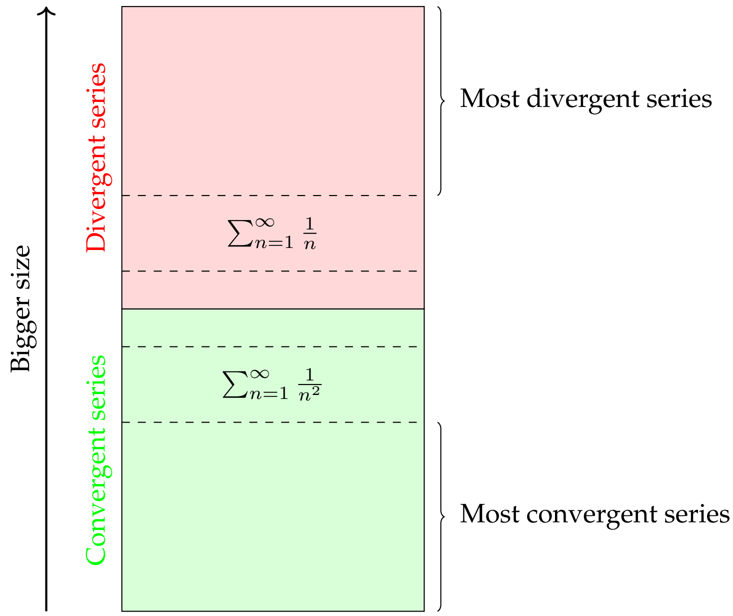
The general description is this: **given a series $\sum_{n=1}^{\infty} f(s_n)$ where f is some function and $s_n \to 0$, we can generate a candidate for limit comparison by considering $T_k(s_n)$, where $T_k(x)$ is a Taylor polynomial for f centered at 0.**

2.5 Case IV: A last ditch resort

If you are given an infinite series and none of the above cases apply (and nothing else seems to work), sometimes you can just run limit comparison against either $\frac{1}{n}$ or $\frac{1}{n^2}$. **This won't always work**, but many times it will. Explicitly:

- If you think the series $\sum_{n=1}^{\infty} a_n$ diverges, run limit comparison against $\frac{1}{n}$.
- If you think the series $\sum_{n=1}^{\infty} a_n$ converges, run limit comparison against $\frac{1}{n^2}$.

The reason why this is an effective catch-all strategy is that in the world of convergent series, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is one of the "biggest" infinite series, i.e., it converges pretty slowly. Likewise, $\sum_{n=1}^{\infty} \frac{1}{n}$ is one of the smallest divergent series, in that it diverges very slowly. Here's what I mean in a picture:



This relative size comparison is just a heuristic and not actually precise, but it is helpful nonetheless. The point is this: **if you want to conclude convergence via comparison, you need to compare with something bigger that also converges, and if you want to conclude divergence via comparison, you need to compare with something smaller that also diverges**. Because $\sum_{n=1}^\infty \frac{1}{n^2}$ is heuristically bigger than most convergent series, it's a decent candidate for a convergent comparison argument. Likewise with $\sum_{n=1}^\infty \frac{1}{n}$ and a divergence comparison argument.

3 Some examples

Here some worked out examples that illustrate the principles in this guide. I encourage you to try them yourself first to test your mastery of limit comparison!

Example 3.1. *Determine whether the series*

$$\sum_{n=1}^\infty \frac{n^2+\cos^2 n}{1+\sqrt{n^5+1}}$$

converges or diverges.

Solution. This series looks like a Case I infinite series, in that there are clearly dominant terms in both the numerator and denominator surrounded by some unimportant fluff that we can throw away. In particular, we expect

$$\frac{n^2+\cos^2 n}{1+\sqrt{n^5+1}} \approx \frac{n^2}{\sqrt{n^5}}$$

for large n. Thus, we run limit comparison with $b_n = \frac{n^2}{\sqrt{n^5}}$. Note that

$$\lim_{n\to\infty} \frac{\frac{n^2+\cos^2 n}{1+\sqrt{n^5+1}}}{\frac{n^2}{\sqrt{n^5}}} = \lim_{n\to\infty} \frac{1+\frac{\cos^2 n}{n^2}}{\frac{1}{\sqrt{n^5}}+\sqrt{1+\frac{1}{n^5}}} = \frac{1+0}{0+\sqrt{1+0}} = 1.$$

This last equality uses the fact that $0 \leq \left|\frac{\cos^2 n}{n^2}\right| \leq \frac{1}{n^2} \to 0$. Since 1 is a finite, positive number, we are in scenario (i) of the limit comparison test. Because the series

$$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

is a divergent p-series, it follows that $\sum_{n=1}^{\infty} \frac{n^2+\cos^2 n}{1+\sqrt{n^5+1}}$ diverges as well. □

Example 3.2. *Determine whether the series*

$$\sum_{n=1}^{\infty} \frac{1+n}{n^{\frac{7}{4}} \ln n}$$

converges or diverges.

Solution. Because the denominator is $n^{\frac{7}{4}} \ln n$, there is logarithmic growth that we cannot ignore, which suggests this is a Case II series. The numerator $1+n$ has fluff that we can ignore, and in the denominator we will use the heuristic $\ln n \rightsquigarrow n^{\text{small } \#}$:

$$\frac{1+n}{n^{\frac{7}{4}} \ln n} \rightsquigarrow \frac{n}{n^{\frac{7}{4}} n^{\text{small } \#}} = \frac{1}{n^{\frac{3}{4}+(\text{small } \#)}}.$$

Because we can choose (small $\#$) as small as we like, and the power of n in the denominator is $\frac{3}{4}+$(small $\#$), we expect this series to diverge (if we pick a sufficiently small number, the exponent will be ≤ 1). In particular, we can choose (small $\#$) $= \frac{1}{4}$ and thus run limit comparison against $\frac{1}{n}$.

Let's compute:

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1+n}{n^{\frac{7}{4}} \ln n}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{1+n}{n^{\frac{3}{4}} \ln n} = \lim_{x\to\infty} \frac{1+x}{x^{\frac{3}{4}} \ln x} \\ (L'H) &= \lim_{x\to\infty} \frac{1}{\frac{3}{4}x^{-\frac{1}{4}} \ln x + x^{-\frac{1}{4}}} = \infty. \end{aligned}$$

This last equality comes from the fact that $\frac{\ln x}{x^{\frac{1}{4}}} \to 0$ and $\frac{1}{x^{\frac{1}{4}}} \to 0$ as $x \to \infty$.

Because the limit is ∞, we are in scenario (iii) of the limit comparison test. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we can conclude that the series $\sum_{n=1}^{\infty} \frac{1+n}{n^{\frac{7}{4}} \ln n}$ diverges as well. □

Example 3.3. *Determine whether the series*

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^{\frac{3}{2}}}\right) \ln(n+2)$$

converges or diverges.

Solution. Because there is only one term of the summand, and it has both $\sin\left(\frac{1}{n^{\frac{3}{2}}}\right)$ and $\ln(n+2)$, this series is a mixture of Case II and Case III. We will thus use the heuristics $\sin x \approx x$ for x small and $\ln n \rightsquigarrow n^{\text{small } \#}$:

$$\sin\left(\frac{1}{n^{\frac{3}{2}}}\right) \ln(n+2) \rightsquigarrow \frac{1}{n^{\frac{3}{2}}} \cdot n^{\text{small } \#} = \frac{1}{n^{\frac{3}{2}-(\text{small } \#)}}.$$

Because we can choose (small #) as small as we like and the power of n in the denominator is $\frac{3}{2} - (\text{small } \#)$, we expect this series to converge (we can choose (small #) so small that the exponent is > 1). In particular, we can choose $(\text{small } \#) = \frac{1}{4}$ so that we will run limit comparison with $b_n = \frac{1}{n^{\frac{5}{4}}}$.

Let's compute the relevant limit. One remark: I am going to be absurdly clever with how I group terms and compute the limit, but it really comes from the heuristics above. Also, anyway you want to compute the limit should be fine!

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^{\frac{3}{2}}}\right)\ln(n+2)}{\frac{1}{n^{\frac{5}{4}}}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^{\frac{3}{2}}}\right)}{\frac{1}{n^{\frac{3}{2}}}} \cdot \frac{\ln(n+2)}{n^{\frac{1}{4}}}.$$

I will treat these two fractions separately. Note that

$$\lim_{n\to\infty} \frac{\ln(n+2)}{n^{\frac{1}{4}}} = \lim_{x\to\infty} \frac{\ln(x+2)}{x^{\frac{1}{4}}} = \lim_{x\to\infty} \frac{4x^{\frac{3}{4}}}{x+2} = 0$$

by L'Hopital's rule, and likewise

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^{\frac{3}{2}}}\right)}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x^{\frac{3}{2}}}\right)}{\frac{1}{x^{\frac{3}{2}}}} = \lim_{x\to\infty} \cos\left(\frac{1}{x^{\frac{3}{2}}}\right) = 1$$

by L'Hopital's rule. Putting these together,

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^{\frac{3}{2}}}\right)\ln(n+2)}{\frac{1}{n^{\frac{5}{4}}}} = 1 \cdot 0 = 0.$$

Thus, we are in scenario (ii) of the limit comparison test. Because we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{4}}}$ converges, we can conclude that the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^{\frac{3}{2}}}\right)\ln(n+2)$ converges as well. $\square$

Example 3.4. *Determine whether the series*

$$\sum_{n=1}^{\infty} \frac{1}{e^{\sqrt{n}}}$$

converges or diverges.

Solution. There is no fluff to throw away, no logarithmic growth, and no way to use Taylor polynomials to our advantage (note that $\sqrt{n} \to \infty$ so it would not make sense to use a Taylor polynomial of e^x, for example). This series will also deflect just about anything else you throw at it: it's not geometric and cannot be converted into one, cannot be compared in any obvious way to a geometric series, and both the ratio and root tests are inconclusive.

Because nothing else seems to work, this seems like a good Case IV infinite sum. My intuition tells me that the series should converge, just because it kind of looks geometric. Thus, let's run limit comparison against $\frac{1}{n^2}$ and hope that it works out. Let's compute:

$$\lim_{n\to\infty} \frac{\frac{1}{e^{\sqrt{n}}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{e^{\sqrt{n}}}.$$

It's possible to L'Hopital this limit (it would take four applications), but I'll do something lazier. Make a variable substitution $m = \sqrt{n}$. Then $n^2 = m^4$ and as $n \to \infty$, $m \to \infty$. The limit turns into

$$\lim_{n\to\infty} \frac{n^2}{e^{\sqrt{n}}} = \lim_{m\to\infty} \frac{m^4}{e^m}.$$

Because we already know that $m^a \ll b^m$ for any $a, b > 0$, it follows that

$$\lim_{m\to\infty} \frac{m^4}{e^m} = 0.$$

Thus, we are in scenario (ii) of the limit comparison test and since we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{e^{\sqrt{n}}}$ converges as well. □

Example 3.5. *Determine whether the series*

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{\sqrt{n}}} - 1}{n}$$

converges or diverges.

Solution. Because of the $e^{\frac{1}{\sqrt{n}}}$ term, we can approximate the behavior in the numerator using Taylor polynomials. In particular, recall that the degree 1 Taylor polynomial centered at 0 for e^x is $1 + x$. Thus, we expect $e^x - 1 \approx x$ for x small and thus $e^{\frac{1}{\sqrt{n}}} - 1 \approx \frac{1}{\sqrt{n}}$. Thus, we will run limit comparison against

$$\frac{\frac{1}{\sqrt{n}}}{n} = \frac{1}{n^{\frac{3}{2}}}.$$

Note that

$$\lim_{n\to\infty} \frac{\frac{e^{\frac{1}{\sqrt{n}}}-1}{n}}{\frac{\frac{1}{\sqrt{n}}}{n}} = \lim_{n\to\infty} \frac{e^{\frac{1}{\sqrt{n}}} - 1}{\frac{1}{\sqrt{n}}} = \lim_{x\to\infty} \frac{e^{\frac{1}{\sqrt{x}}} - 1}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} e^{\frac{1}{\sqrt{x}}} = e^0 = 1$$

by L'Hopital's rule. Since we got a limit of 1, we are in scenario (i) of the limit comparison test. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges, the series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{\sqrt{n}}}-1}{n}$ converges as well. □

Example 3.6. *Determine whether the series*

$$\sum_{n=1}^{\infty} \frac{2^n + \ln n}{e^n + \sin n}$$

converges or diverges.

Solution. Note that **even though there is a $\ln n$ in the numerator**, we are *not* in Case II. Because the $\ln n$ is a separate term from 2^n, and 2^n grows faster, this instance of $\ln n$ counts as fluff. Likewise, the $\sin n$ in the denominator does not indicate that we are in Case III or anything like that; it counts as fluff. This is purely a Case I infinite series.

We will run limit comparison against $\frac{2^n}{e^n}$. Note that

$$\lim_{n\to\infty} \frac{\frac{2^n+\ln n}{e^n+\sin n}}{\frac{2^n}{e^n}} = \lim_{n\to\infty} \frac{1+\frac{\ln n}{2^n}}{1+\frac{\sin n}{e^n}} = \frac{1+0}{1+0} = 1.$$

The limit in the numerator comes from the fact that $\ln n \ll n^a \ll b^n$ and the limit in the denominator is a result of $0 \le \left|\frac{\sin n}{e^n}\right| \le \frac{1}{e^n} \to 0$. Because we got a limit of 1, we are in scenario (i) of the limit comparison test. Because

$$\sum_{n=1}^{\infty} \frac{2^n}{e^n} = \sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n$$

is a convergent geometric series ($2 < e$), we can conclude that the series $\sum_{n=1}^{\infty} \frac{2^n+\ln n}{e^n+\sin n}$ converges as well. $\square$

Example 3.7. *Determine whether the series*

$$\sum_{n=1}^{\infty} \frac{\ln(\ln n)}{(n^5+1)^{\frac{1}{4}}}$$

converges or diverges.

Solution. Because of the logarithm mess in the numerator, we are definitely in Case II. Note that even though the term is $\ln(\ln n)$ instead of just $\ln n$, the same principles. In particular, our heuristics lead to:

$$\frac{\ln(\ln n)}{(n^5+1)^{\frac{1}{4}}} \rightsquigarrow \frac{n^{\text{small }\#}}{n^{\frac{5}{4}}} = \frac{1}{n^{\frac{5}{4}-(\text{small }\#)}}.$$

Because we can choose $(\text{small }\#)$ as small as we like, and the power of n in the denominator is $\frac{5}{4}-(\text{small }\#)$, we expect this series to converge (we can choose $(\text{small }\#)$ so small that the exponent remains larger than 1). In particular, we can pick small $\# = \frac{1}{8}$ and run limit comparison with $\frac{1}{n^{\frac{9}{8}}}$. Indeed,

$$\lim_{n\to\infty} \frac{\frac{\ln(\ln n)}{(n^5+1)^{\frac{1}{4}}}}{\frac{n^{\frac{1}{8}}}{n^{\frac{5}{4}}}} = \lim_{n\to\infty} \frac{\ln(\ln n)}{n^{\frac{1}{8}}} \cdot \frac{1}{\left(1+\frac{1}{n^5}\right)^{\frac{1}{4}}} = 0 \cdot \frac{1}{(1+0)^{\frac{1}{4}}} = 0.$$

The limit computation uses the fact that $\ln n \ll n^a$ and so $\ln(\ln n) \ll \ln n \ll n^a$. Since we got a limit of 0, we are in scenario (ii) of the limit comparison test. Because we know that

$$\sum_{n=1}^{\infty} \frac{n^{\frac{1}{8}}}{n^{\frac{5}{4}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{8}}}$$

converges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln(\ln n)}{(n^5+1)^{\frac{1}{4}}}$ converges as well. $\square$