A Geometry Problem

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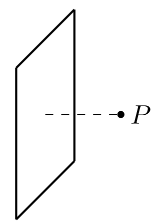
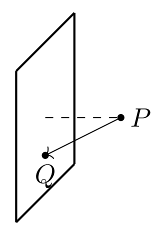
Find the distance from the point $(1, 2, 3)$ to the plane $3x - y + 2z = 3$.

Solution 1: Vectors. In my opinion, the most natural way to think about this problem is in terms of *vector geometry.* To find the shortest distance between a point a plane, we need to travel from the point to the plane *in the direction of the normal vector to the plane:*



So here's what we can do: pick *any* point Q on the plane and connect P to Q with the vector $\overrightarrow{PQ} = Q - P$:



Then if we find the absolute value of the component of this vector $\overrightarrow{PQ}$ in the direction of the normal vector $\mathbf{n}$ (which is in the direction of the dotted line, which is the direction we need to travel in), this will give us the shortest distance. If this isn't intuitive, all we're doing is projecting $\overrightarrow{PQ}$ onto $\mathbf{n}$ and then taking the length.

So our point P is $(1, 2, 3)$, and the normal vector to the plane is $\langle 3, -1, 2\rangle$. To get a point Q on the plane, we can pick any point that satisfies the equation $3x - y + 2z = 3$. Anything will work, but an easy point is $Q = (1, 0, 0)$. Thus, our vector $\overrightarrow{PQ}$ is:

$$\overrightarrow{PQ} = (1, 0, 0) - (1, 2, 3) = \langle 0, -2, -3\rangle$$

Now, the component of $\overrightarrow{PQ}$ in the $\mathbf{n}$ direction is:

$$\frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\,\langle 0, -2, -3\rangle \cdot \langle 3, -1, 2\rangle\,|}{\sqrt{3^2 + (-1)^2 + (-2)^2}} = \frac{|-4|}{\sqrt{14}} = \frac{4}{\sqrt{14}}$$

Hence, the shortest distance from the point P to the plane is $\frac{4}{\sqrt{14}}$.

$\square$

Solution 2: Lines. An alternate (but very similar way) to approach this problem is to find the equation of a line that go from P in the direction we want to travel. Then, we can calculate where the line intersects the plane, and figure out the distance from there.

As before, the shortest distance from P to the plane is in the direction of the normal vector. So we can write the equation of a line passing through P and going in the direction we want by:

$$\mathbf{r}(t) = P + t\mathbf{n}$$

Expanding this out gives us:

$$\langle x(t),\, y(t),\, z(t)\rangle = \langle 1,\, 2,\, 3\rangle + t\,\langle 3,\, -1,\, 2\rangle = \langle 1+3t,\, 2-t,\, 3+2t\rangle$$

Next, we can figure out where the line intersects the point on the plane. To do this, take the expressions for x, y, and z, and plug them into the plane equation:

$$\begin{aligned} 3x - y + 2z &= 3 \\ 3(1+3t) - (2-t) + 2(3+2t) &= 3 \\ 7 + 14t &= 3 \\ t &= -\frac{2}{7} \end{aligned}$$

So we know that the line intersects the plane when $t = -\frac{2}{7}$. This happens at the point:

$$\left(1 + 3\left(-\frac{2}{7}\right), 2 - \left(-\frac{2}{7}\right), 3 + 2\left(-\frac{2}{7}\right)\right) = \left(\frac{1}{7}, \frac{16}{7}, \frac{17}{7}\right)$$

Now, we can use the distance formula to find the distance from that point to the point P:

$$\begin{aligned} d &= \sqrt{\left(1 - \frac{1}{7}\right)^2 + \left(2 - \frac{16}{7}\right)^2 + \left(3 - \frac{17}{7}\right)^2} \\ &= \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{4}{7}\right)^2} \\ &= \sqrt{\frac{16}{14}} \\ &= \frac{4}{\sqrt{14}} \end{aligned}$$

□

Solution 3: Minimizing a Function. We can think of this problem in terms of finding the minimum value of a certain function (after all, we want to find the *minimum distance* from a point to the plane). So consider the following: we can describe the distance from *any point* to P in the following way:

$$d(x, y, z) = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$

But we don't want the distance from *any* point; we only care about points on the plane:

$$3x - y + 2z = 3$$

So what we can do is solve for one of the variables in the above equation, and plug the result into the distance equation. Since the coefficient on y is -1, that's what I will solve for (but it doesn't actually matter):

$$y = 3x + 2z - 3$$

Which gives us distance as a function of two variables:

$$d(x,z) = \sqrt{(x-1)^2 + ((3x+2z-3)-2)^2 + (z-3)^2} = \sqrt{(x-1)^2 + (3x+2z-5)^2 + (z-3)^2}$$

The value we want is the minimum value of this function!

To make computations easier, lets minimize the square of the distance, d^2. This doesn't change *where* the minimum occurs, so we're allowed to do this. I'm going to rename the function $d^2 = f$:

$$f(x,z) = (x-1)^2 + (3x+2z-5)^2 + (z-3)^2$$

To find the minumum value of this function, we need to find critical points:

$$\begin{aligned} f_x(x,z) &= 0 & f_z(x,z) &= 0 \\ 2(x-1) + 2(3x+2z-5)(3) &= 0 & 2(3x+2z-5)(2) + 2(z-3) &= 0 \\ 20x + 12z - 32 &= 0 & 12x + 10z - 26 &= 0 \end{aligned}$$

After dividing by 2, this gives us the system of equations:

$$\begin{aligned} 10x + 6z &= 16 \\ 6x + 5z &= 13 \end{aligned}$$

We could solve for z in the second equation:

$$z = \frac{1}{5}(13 - 6x)$$

and then plug this into the first equation:

$$10x + 6\left(\frac{1}{5}(13 - 6x)\right) = 16$$

Solving this gives us $x = \frac{1}{7}$. This means that

$$z = \frac{1}{5}\left(13 - 6\left(\frac{1}{7}\right)\right) = \frac{17}{7}$$

Plugging this into the distance equation gives us:

$$\begin{aligned} d\left(\frac{1}{7}, \frac{17}{7}\right) &= \sqrt{\left(\left(\frac{1}{7}\right) - 1\right)^2 + \left(3\left(\frac{1}{7}\right) + 2\left(\frac{17}{7}\right) - 5\right)^2 + \left(\frac{17}{7} - 3\right)^2} \\ &= \frac{4}{\sqrt{14}} \end{aligned}$$

□

Solution 4: Minimizing with Lagrange Multipliers. A slightly different way to do this problem is to set it up the same way as the last problem, but minimizing the distance function using Lagrange multipliers.

As before, we can describe the (square of) the distance from the point $P = (1, 2, 3)$ by:

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$$

But we are constrained by the equation:

$$3x - y + 2z = 3$$

Hence, by the method of Lagrange multipliers, if we call $g(x, y, z) = 3x - y + 2z$, the minimum of the function f will occur when:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

for some number λ. Expanding this out gives:

$$\langle 2(x-1),\ 2(y-2),\ 2(z-3) \rangle = \lambda \langle 3,\ -1,\ 2 \rangle$$

Giving us the system of equations:

$$2x - 2 = 3\lambda$$

$$2y - 4 = -\lambda$$

$$2z - 6 = 2\lambda$$

Keep in mind that we also have our constraint equation $3x - y + 2z = 3$. Hence, one way to solve this system is to solve for each variable x, y, and z in terms of λ and plug the expressions into the constraint:

$$x = \frac{3\lambda + 2}{2}$$
$$y = \frac{4 - \lambda}{2}$$
$$z = \lambda + 3$$

Plugging this gives us the equation:

$$3\left(\frac{3\lambda + 2}{2}\right) - \left(\frac{4 - \lambda}{2}\right) + 2(\lambda + 3) = 3$$

Solving this equation for λ gives us $\lambda = -\frac{4}{7}$. Now that we know this, we can figure out what x, y, and z are:

$$x = \frac{3\left(-\frac{4}{7}\right) + 2}{2} = \frac{1}{7}$$
$$y = \frac{4 - \left(-\frac{4}{7}\right)}{2} = \frac{16}{7}$$
$$z = \left(-\frac{4}{7}\right) + 3 = \frac{17}{7}$$

As we've seen a couple times now, plugging this point into the distance function gives us a minimum distance of $\frac{4}{\sqrt{14}}$. □