Example. Count (but do not evaluate) the integral \( \iiint_E xy \, dv \) into spherical coordinates, where \( E \) is the region defined by:

\[
\begin{align*}
\frac{x^2+y^2+z^2}{r^2} &= 1 \\
\frac{x^2+y^2+(z-1)^2}{r^2} &< 1
\end{align*}
\]

Solution.

Bounds for \( \varphi \): lower bound for \( \varphi \) is cut out by \( x^2+y^2+z^2 = 1 \) \( \rightarrow \) \( \rho^2 = 1 \) \( \rightarrow \) \( \rho = 1 \).

Upper bound is determined by \( x^2+y^2+(z-1)^2 = 1 \) \( \rightarrow \) \( x^2+y^2+z^2 - 2z+1 = 1 \) \( \rightarrow \) \( x^2+y^2+z^2 = 2z \).

\( \rho^2 = 2 \rho \cos \varphi \) \( \rightarrow \) \( \varphi = \cos \theta \).

Bounds for \( \theta \): lower value determined by \( y = |x| \),

\( \sin \theta = | \cos \varphi | \) \( \rightarrow \) \( \theta = \pi/4, 3\pi/4 \).

Bounds for \( \varphi \): lower bound for \( \varphi \) is 0.
The upper half for $E$ is determined by the intersection of $p = 1$ and $p = 2\cos\phi$.

$$\delta = 1 = 2\cos\phi \implies \cos\phi = \frac{1}{2} \implies \phi = \frac{\pi}{3}$$

$$\int \int \int_{E} x \, dV = \int_{0}^{\frac{\pi}{3}} \int_{0}^{2\cos\phi} \int_{0}^{\sqrt{2\sin\phi}} r \sin\phi \, r \, dr \, d\phi \, dz$$

Jacobian

**Example.** Let $E$ be the upper half of the ball of radius 3 centered at the origin. Find the centroid of $E$.

**Solution.**

Centroid of $E$ is the average point of $E$

i.e. average of each coord

$$\text{Centroid of } E = (\bar{x}, \bar{y}, \bar{z}) = \left( \frac{\iiint_{E} x \, dV}{\text{Vol} \, E}, \frac{\iiint_{E} y \, dV}{\text{Vol} \, E}, \frac{\iiint_{E} z \, dV}{\text{Vol} \, E} \right)$$

Recall that \(\text{vol} \, (\text{ball of radius } R) = \frac{4}{3} \pi R^3\).

So \(\text{Vol} \, (E) = \frac{1}{2} \cdot \frac{4}{3} \pi (3)^3 = 18\pi\).
Note: $x$ is odd wrt $x$, $E$ is symmetric across $x=0$. By symmetry, $\iiint_E x \, dV = 0$, so $\bar{x} = 0$.

By the same argument, $\bar{y} = 0$.

Finally, 
$$
\bar{z} = \frac{1}{18\pi} \int_0^{2\pi} \int_0^\frac{\pi}{2} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
$$

$$
= \frac{a}{8}
$$

So, centroid $(E) = (0, 0, a/8)$.