# Math 32A Practice Midterm 2 Solutions

Joseph Breen

February something, 2020

**Name**: ________________________________________
**ID**: __________________________________________

---

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 50 | |

1. For each of the following statements, answer TRUE or FALSE. No justification required! Read carefully!!

   (a) (2 points) If $\int_0^1 \mathbf{r}'(t)\,dt = \mathbf{0}$, then $\int_0^1 \|\mathbf{r}'(t)\|\,dt = 0$.

   (b) (2 points) If $\|\mathbf{r}'(t)\| = 2$, then $\kappa(t) = \frac{1}{4}\|\mathbf{r}''(t)\|$.

   (c) (2 points) Suppose that $\mathbf{r}(t)$ is a regular parameterization (this just means that $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t$) for $-\infty < t < \infty$ such that $\kappa(t) = 0$, then

   $$\lim_{t\to\infty} \|\mathbf{r}(t)\| = \infty.$$

   (d) (2 points) The direction of steepest descent of a function $f$ is given by $-\nabla f$.

   (e) (2 points) If $\mathbf{r}(s)$ is an arc length parameterization, then $D_{\mathbf{r}'(s)}f(\mathbf{r}(s)) = \nabla f(\mathbf{r}(s)) \cdot \mathbf{r}'(s)$.

**Answers:**

(a) FALSE. The first integral represents change in position, while the second integral represents distance travelled. For a counter example, consider $\mathbf{r}(t) = \langle \cos(2\pi t), \sin(2\pi t)\rangle$.

(b) TRUE. One way to see this is to use the fact that since $\mathbf{r}'(t)$ has constant length, $\mathbf{r}'(t) \perp \mathbf{r}''(t)$. Thus,

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t)\|\,\|\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}''(t)\|}{4}.$$

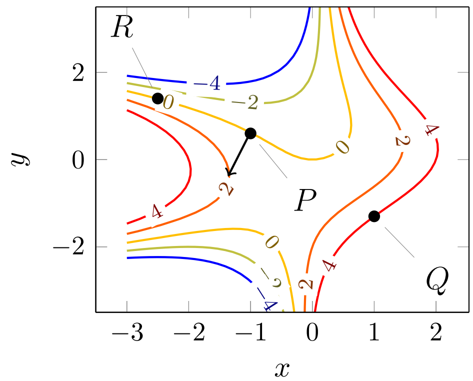
Another possibly more confusing way to see this is by constructing an arc length parameterization: $\mathbf{r}_1(s) := \mathbf{r}(s/2)$. Then $\kappa(s) = \|\mathbf{r}_1''(s)\| = \frac{1}{2^2}\|\mathbf{r}''(s/2)\|$ (by the chain rule). Thus, the curvature at the point $\mathbf{r}(s/2)$ is $\frac{1}{4}\|\mathbf{r}''(s/2)\|$ and so $\kappa(t) = \frac{1}{4}\|\mathbf{r}''(t)\|$.

(c) FALSE. The statement looks a little confusing, but here is what it's saying in words: if $\mathbf{r}(t)$ defines a curve which is always moving and with curvature 0, does the curve eventually travel off to infinity? It is tempting to think of this as being true because the image of $\mathbf{r}(t)$ must be a line, but it is possible for this to only trace out a bounded line segment. For example, $\mathbf{r}(t) = \langle \arctan t, 0\rangle$ satisfies the above hypothesis but traces out a bounded line segment.

(d) TRUE. The gradient $\nabla f$ gives the direction of steepest ascent, so $-\nabla f$ gives the direction of steepest descent.

(e) TRUE. Because $\mathbf{r}(s)$ is an arc length parameterization, $\mathbf{r}'(s)$ is a unit vector. Thus, we can get the above by the formula $D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u}$. Alternatively, the right hand side follows from the computation of $\frac{d}{ds}f(\mathbf{r}(s))$, which is valid because $\mathbf{r}(s)$ is an arc length parameterization.

2. Consider the following contour plot of a function $f(x, y)$.




(a) (2 points) Sketch the gradient vector $\nabla f(P)$ on the plot above. (Don't worry about drawing the correct length, just get the correct direction.)

**Solution.**

See the picture above.

(b) (2 points) Let $\mathbf{v} = \langle 0, 1 \rangle$. Is the directional derivative $D_{\mathbf{v}} f(Q)$ positive, negative, or zero?

**Solution.**

$D_{\mathbf{v}}(Q) < 0$. As one moves north from the point $Q$, the function decreases. Observe that this is exactly $f_y(Q)$.

(c) (2 points) Let $\mathbf{r}(t)$ be a parameterization for the level curve passing through the point $Q$. Suppose that $\mathbf{r}(0) = Q$. Let $\mathbf{u} = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|}$. Is the directional derivative $D_{\mathbf{u}} f(Q)$ positive, negative, or zero?

**Solution.**

This direction is tangent to a level curve, so $D_{\mathbf{u}} f(Q) = 0$.

(d) (2 points) Which number is larger: $\|\nabla f(P)\|$ or $\|\nabla f(R)\|$?

**Solution.**

$\|\nabla f(R)\|$ is likely to be larger because the level curves nearby $R$ are closer together than they are at $P$, which indicates a steeper ascent.

(e) (2 points) Which number is likely closer to 0: $f_x(P)$ or $f_y(P)$?

**Solution.**

$f_x(P)$ is likely closer to 0, as the vector $\langle 1, 0 \rangle$ (the direction of the $x$ partial derivative) is more close to tangent to the level curve than $\langle 0, 1 \rangle$.

3. Consider the function
$$f(x,y) = \frac{(x-1)x^3y}{(y-1)(x^2+y^2)}.$$

   (a) (2 points) What is the domain of $f$?

   **Solution.**
   The domain is
   $$\{(x,y) \mid y \neq 1,\ (x,y) \neq (0,0)\}.$$

   (b) (4 points) Evaluate the limit
   $$\lim_{(x,y)\to(0,0)} f(x,y)$$
   or show that it does not exist.

   **Solution.**
   We convert to polar coordinates:
   $$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{(x-1)x^3y}{(y-1)(x^2+y^2)} &= \lim_{r\to 0} \frac{(r\cos\theta-1)(r\cos\theta)^3(r\sin\theta)}{(r\sin\theta-1)r^2} \\ &= \lim_{r\to 0} \frac{r\cos\theta-1}{r\sin\theta-1}\cdot r^2\cos^3\theta\sin\theta \\ &= \frac{-1}{-1}\cdot 0 \\ &= 0.\end{aligned}$$

   (c) (4 points) Evaluate the limit
   $$\lim_{(x,y)\to(1,1)} f(x,y)$$
   or show that it does not exist.

   **Solution.**
   First, approach along the line $x=1$. Then
   $$\lim_{(1,y)\to(1,1)} \frac{(1-1)1^3y}{(y-1)(1^2+y^2)} = \lim_{y\to 1} 0 = 0.$$

   Next, approach along the line $y=x$. Note that this line does in fact pass through the point $(1,1)$. Note that one cannot approach along the line $y=0$ because of the domain.
   $$\lim_{(x,x)\to(1,1)} \frac{(x-1)x^3x}{(x-1)(x^2+x^2)} = \lim_{x\to 1}\frac{x^4}{2x^2} = \lim_{x\to 1}\frac{x^2}{2} = \frac{1}{2}.$$

   Since $0 \neq \frac{1}{2}$, the limit does not exist.

4. Consider the curve $y = \ln x$

(a) (8 points) Find the point of maximum curvature on the curve.

**Solution.**
Let $f(x) = \ln x$. Note that the domain of $f(x)$, and so the domain of $\kappa(x)$, is $x > 0$. Then

$$\kappa(x) = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{\frac{3}{2}}} = \frac{\left|-\frac{1}{x^2}\right|}{\left(1+\left[\frac{1}{x}\right]^2\right)^{\frac{3}{2}}} = \frac{1}{x^2\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}} = \frac{1}{\left(x^{\frac{4}{3}}\right)^{\frac{3}{2}}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}}$$
$$= \frac{1}{\left(x^{\frac{4}{3}}+x^{-\frac{2}{3}}\right)^{\frac{3}{2}}}.$$

Alternatively, one can use $\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$ with the parameterization $\mathbf{r}(t) = \langle t, \ln t, 0\rangle$.
Next, to maximize the curvature function we seek critical points:

$$\kappa'(x) = -\frac{3}{2}\cdot\frac{1}{\left(x^{\frac{4}{3}}+x^{-\frac{2}{3}}\right)^{\frac{5}{2}}}\cdot\left(\frac{4}{3}x^{\frac{1}{3}}-\frac{2}{3}x^{\frac{-5}{3}}\right).$$

Because the domain of $\kappa(x)$ is $x > 0$, the only critical point(s) occur when

$$\frac{4}{3}x^{\frac{1}{3}}-\frac{2}{3}x^{\frac{-5}{3}} = 0 \qquad \Rightarrow \qquad 2x - x^{-1} = 0 \qquad \Rightarrow \qquad x = \frac{1}{\sqrt{2}}.$$

Since

$$\lim_{x\to 0^+}\kappa(x) = 0 \quad \text{and} \quad \lim_{x\to\infty}\kappa(x) = 0$$

it follows that $x = \frac{1}{\sqrt{2}}$ yields the global maximum value of $\kappa(x)$. Thus, the point of maximum curvature on the curve $y = \ln x$ is

$$\left(\frac{1}{\sqrt{2}}, \ln\left(\frac{1}{\sqrt{2}}\right)\right).$$

(b) (2 points) Write down a function $g(x, y)$ such that the curve $y = \ln x$ is a level curve for $g$.

**Solution.**
Let $g(x, y) = y - \ln x$. Then the curve $y = \ln x$ is precisely the curve $g(x, y) = 0$.

5. Let $h(x,y) = x^2 + 2y^2$.

(a) (4 points) Find the point $(a,b)$ where the tangent plane to the graph of $h$ is parallel to the plane $3x - 5y + 2z = 0$.

**Solution.**
Let $H(x,y,z) = z - h(x,y) = z - x^2 - 2y^2$. Then the gradient vector

$$\nabla H(x,y,z) = \langle -2x, -4y, 1\rangle$$

is perpendicular to level surfaces of $H$. Since the graph of $h(x,y)$ is precisely the level surface $H(x,y,z) = 0$, $\nabla H(x,y,z)$ is a normal vector to the tangent plane to the graph at every point. Thus, we seek a point where $\langle -2x, -4y, 1\rangle$ is parallel to $\langle 3, -5, 2\rangle$, the normal vector to the given plane. Because $2 = 2(1)$ (matching up the $z$-components), we want

$$\begin{cases} 3 = 2(-2x) \\ -5 = 2(-4y) \end{cases} \qquad \Rightarrow \qquad \begin{cases} x = -\frac{3}{4} \\ y = \frac{5}{8}. \end{cases}$$

Thus, the point at which the tangent plane is parallel to the given plane is $\left(-\frac{3}{4}, \frac{5}{8}\right)$.

(b) (3 points) Find the maximum possible rate of change of $h$ at the point $(2,-1)$.

**Solution.**
The maximum possible rate of change of $h$ at the point $(2,-1)$ is

$$\|\nabla h(2,-1)\| = \left\| \langle 2x, 4y\rangle \,|_{(2,-1)} \right\| = \|\langle 4, -4\rangle\| = \sqrt{32}.$$

(c) (3 points) Find a unit direction vector $\mathbf{u}$ such that $D_{\mathbf{u}}h(2,-1) = 0$.

**Solution.**
We seek a unit vector $\mathbf{u}$ such that

$$D_{\mathbf{u}}h(2,-1) = \nabla h(2,-1) \cdot \mathbf{u} = 0.$$

By the previous part, $\nabla h(2,-1) = \langle 4, -4\rangle$. Thus, a unit vector $\mathbf{u}$ such that $\langle 4,-4\rangle \cdot \mathbf{u} = 0$ is

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$