Math 32A Practice Midterm 1 Solutions

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Name: ______________________________
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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. For each of the following statements, answer TRUE or FALSE. No justification required! Read carefully!!

   (a) (2 points) Suppose that $\mathbf{v}$ and $\mathbf{w}$ are parallel vectors and that $\mathbf{p} \neq \mathbf{q}$ are different vectors. Then $\mathbf{r}_1(t) = \mathbf{p} + t\mathbf{v}$ and $\mathbf{r}_2(t) = \mathbf{q} + t\mathbf{w}$ are parameterizations for different lines.

   (b) (2 points) A parameterization for the unit circle is $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2) \rangle$.

   (c) (2 points) If $\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \mathbf{0}$, then $\|\mathbf{a} \times \mathbf{b}\|$ is a positive number.

   (d) (2 points) If $\mathbf{a} \cdot \mathbf{b} > 0$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

   (e) (2 points) The area of the triangle spanned by vectors $\mathbf{a}$ and $\mathbf{b}$ is the same as the area of the triangle spanned by vectors $\mathbf{a}$ and $\mathbf{b} - \mathbf{a}$.

Answers:

(a) FALSE. Consider $\mathbf{r}_1(t) = \langle 0, 0 \rangle + t\langle 1, 0 \rangle$ and $\mathbf{r}_2(t) = \langle 1, 0 \rangle + t\langle 1, 0 \rangle$. Both of these lines are parameterizations for the x-axis.

(b) TRUE. Note that

$$\sin^2(t^2) + \cos^2(t^2) = 1$$

and so the above parametrization satisfies the equation $x^2 + y^2 = 1$. (Furthermore, t^2 will range over all angles between 0 and 2π, so the entire unit circle is in fact traced out.)

(c) FALSE. Let $\mathbf{a} = \mathbf{0}$.

(d) FALSE. Let $\mathbf{a} = \langle 1, 0, 0 \rangle$ and $\mathbf{b} = \langle 1, 1, 0 \rangle$.

(e) TRUE. This can be seen by drawing a picture of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{b} - \mathbf{a}$, or observing that

$$\mathbf{a} \times (\mathbf{b} - \mathbf{a}) == \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{a} = \mathbf{a} \times \mathbf{b}.$$

2. Let $\mathbf{r}_1(t) = \langle 1+t, 1-t, 2t\rangle$ and $\mathbf{r}_2(t) = \langle t-1, 2t-6, 3t-7\rangle$ represent the paths of two particles p_1 and p_2.

   (a) (5 points) Do the paths of p_1 and p_2 cross? Do the particles intersect each other?

   Solution.
   To determine if the paths cross, we seek a solution to $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some parameters t and s. The resulting system of equations is

$$\langle 1+t, 1-t, 2t\rangle = \langle s-1, 2s-6, 3s-7\rangle \qquad \Rightarrow \qquad \begin{cases} 1+t = s-1 \\ 1-t = 2s-6 \\ 2t = 3s-7 \end{cases}.$$

   The first equation implies $s = t+2$. Plugging this into the second equation yields

$$1-t = 2(t+2)-6 \qquad \Rightarrow \qquad 1+6-4 = 3t \quad \Rightarrow \quad t = 1.$$

   Since $s = t+2$ by the first equation, we have $s = 3$. It remains to check if these two values for t and s satisfy the third equation. Since

$$2(1) = 3(3) - 7$$

   it follows that $t = 1$ and $s = 3$ is a solution to the system of equations. Thus, the paths do cross.
   Since $s \neq t$, the particles do not collide.

   (b) (5 points) Suppose that p_1 and p_2 begin their journey at $t = 0$. Which particle is the first to reach the plane $x - 2y = 9$?

   Solution.
   To determine which particle reaches the plane $x - 2y = 9$ first, we will find the time of intersection for each line. Consider $\mathbf{r}_1(t) = \langle 1+t, 1-t, 2t\rangle$ first. This line intersects the plane when

$$(1+t) - 2(1-t) = 9 \qquad \Rightarrow \qquad 3t + 1 - 2 = 9 \qquad \Rightarrow \qquad t = \frac{10}{3}.$$

   The line $\mathbf{r}_2(t) = \langle t-1, 2t-6, 3t-7\rangle$ intersects the plane when

$$(t-1) - 2(2t-6) = 9 \qquad \Rightarrow \qquad -3t - 1 + 12 = 9 \qquad \Rightarrow \qquad t = \frac{2}{3}.$$

   Since the first particle reaches the plane at times $t = \frac{5}{3}$ and the second particle reaches the plane at time $t = \frac{2}{3}$, the second particle reaches the plane first.

3. Consider the plane P given by $x + 2y + 3z = 1$.

   (a) (4 points) Find a parameterization of the line passing through the origin which is perpendicular to the plane P.

   Solution.

   Because the line we seek is perpendicular to the plane, we can use a normal vector to the plane as a direction vector for the line. For example, $\langle 1, 2, 3\rangle$ is a normal vector to the plane and hence a direction vector for our line. Because the line passes through the origin, a point on the line is $(0, 0, 0)$. Thus a parameterization for the line we seek is

   $$\mathbf{r}(t) = \langle 0, 0, 0\rangle + t\langle 1, 2, 3\rangle.$$

   (b) (6 points) Find an equation of a plane which contains the line you found in the previous part and is perpendicular to the plane P.

   Solution.

   Because the plane we seek is perpendicular to the plane P, the normal vector $\langle 1, 2, 3\rangle$ must be parallel to the plane we seek. (Alternatively, since the plane we seek contains the line from part (a), the direction vector $\langle 1, 2, 3\rangle$ must be parallel to the plane we seek.)

   Because we want to find *any* plane which contains the above line and is perpendicular to the plane, we can compute a normal vector by taking the cross product of $\langle 1, 2, 3\rangle$ with *any* (non parallel) vector we choose! For example, a normal vector for our plane could be

   $$\langle 1, 2, 3\rangle \times \langle 1, 0, 0\rangle = \langle 0, 3, -2\rangle.$$

   Because the plane has to contain the line from part (a), it in particular contains the point $(0, 0, 0)$. Thus, the equation of a plane satisfying these conditions is

   $$0(x - 0) + 3(y - 0) - 2(z - 0) = 0 \qquad \Rightarrow \qquad 3y - 2z = 0.$$

   Like I said, this is not the only correct solution! There are infinitely many correct answers. As long as the normal vector you choose is perpendicular to the vector $\langle 1, 2, 3\rangle$ and contains the point $(0, 0, 0)$, your answer will be correct.

4. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors such that $\|\mathbf{a}+\mathbf{b}\| = \sqrt{2+\sqrt{2}}$.

(a) (4 points) Compute $\|\mathbf{a}-\mathbf{b}\|$.

Solution.
Note that $\|\mathbf{a}+\mathbf{b}\|^2 = 2+\sqrt{2}$. On the other hand,

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\ &= \|\mathbf{a}\|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + \|\mathbf{b}\|^2 \\ &= 1^2 + 2(\mathbf{a}\cdot\mathbf{b}) + 1^2 = 2 + 2(\mathbf{a}\cdot\mathbf{b}).\end{aligned}$$

It follows that $2+2(\mathbf{a}\cdot\mathbf{b}) = 2+\sqrt{2}$ and so $\mathbf{a}\cdot\mathbf{b} = \frac{1}{\sqrt{2}}$.
Next, note that

$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= \|\mathbf{a}\|^2 - 2(\mathbf{a}\cdot\mathbf{b}) + \|\mathbf{b}\|^2 = 2 - 2(\mathbf{a}\cdot\mathbf{b}) = 2 - 2\frac{1}{\sqrt{2}} = 2-\sqrt{2}.\end{aligned}$$

Thus,

$$\|\mathbf{a}-\mathbf{b}\| = \sqrt{2-\sqrt{2}}.$$

(b) (2 points) What is the angle between $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$?

Solution.
Note that

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \|\mathbf{a}\|^2 - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \|\mathbf{b}\|^2 = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = 1^2 - 1^2 = 0.$$

Since $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 0$, the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are perpendicular, and thus the angle between them is $\frac{\pi}{2}$.

(c) (4 points) Compute $\|\mathbf{a}\times\mathbf{b}\|$.

Solution. Recall that $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{a}\cdot\mathbf{b} = \frac{1}{\sqrt{2}}$ from the first part, and $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta = 1\cdot 1\cdot\cos\theta$, it follows that $\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and so $\theta = \frac{\pi}{4}$. Thus,

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta = 1\cdot 1\cdot\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$