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Functoriality and the Artin Conjecture

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Contents

This article contains an exposition of the proof of the Artin conjecture for two-dimensional Galois representations of tetrahedral and octahedral type. The proofs, given in Section 3, are carried out by applying some general theorems about cuspidal representations on $GL(2)$ and $GL(3)$ to a particular situation. As such, they provide good illustrations of how the automorphic formalism works. On the other hand, it should be noted that the arguments ultimately rely on some fortunate but accidental features of the low-dimensional situation. For an interesting mathematical and historical discussion of the base change problem and its relation to the Artin conjecture, we recommend the introduction to [L2].

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The first two sections provide background material, much of which has already been covered in [Kn]. More precisely, Section 1 contains a review of Artin L-functions. In Section 2, we state some general conjectures about automorphic representations of $GL(n)$, emphasizing the analogy between irreducible Galois representations and cuspidal representations. We also state the general theorems needed for the proofs in Section 3.

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Notation. Throughout this article, F denotes a number field, $\overline{F}$ an algebraic closure of F, and $Gal(\overline{F}/F)$ the absolute Galois group of F. We view $Gal(\overline{F}/F)$ as a topological group relative to the Krull topology. All finite extensions of E will implicitly be assumed to be subfields of $\overline{F}$. We write $\mathbf{A}_F$ and I_F, respectively, for the adele ring and idele group attached to F. As usual, we identify F (resp. F^*) with its image in $\mathbf{A}_F$ (resp., I_F) under the diagonal embedding. A **Hecke character** is an idele class character, that is, a continuous homomorphism from I_F to $\mathbb{C}^*$ trivial on F^*. For each place v of F, let F_v denote the completion of F relative to v. We also fix algebraic closures $\overline{F}_v$ of F_v for each place v.

I. Artin L-Functions

1. Definitions

We use the term **Galois representation** to denote a continuous homomorphism

$$\sigma : Gal(\overline{F}/F) \longrightarrow GL(V)$$

where V is a finite-dimensional complex vector space. Recall that σ is continuous if and only if σ factors through the projection $Gal(\overline{F}/F) \to Gal(E/F)$ for some finite extension E/F. The determinant $\det(\sigma)$ of σ is the complex-valued character $x \to \det(\sigma(x))$. It is identified with a Hecke character of I_F via the Artin isomorphism of class field theory. The relevant part of class field theory is reviewed in Sections 4–5 of [Kn].

The Artin L-function $L(s,\sigma)$ attached to σ is an Euler product

$$L(s,\sigma) = \prod_v L(s,\sigma_v).$$

Here our convention is that v runs over all places of F, archimedean and non-archimedean; by contrast archimedean places are not included in the definition in Section 5 of [Kn]. To define the local factor $L(s,\sigma_v)$, choose an embedding $i'_v : \overline{F} \to \overline{F}_v$. This gives rise to an embedding of Galois groups

$$i_v : Gal(\overline{F}_v/F_v) \longrightarrow Gal(\overline{F}/F)$$

via restriction. The composition $\sigma_v = \sigma \circ i_v$ is a continuous representation of $Gal(\overline{F}_v/F_v)$. It depends on the choice of i'_v, but different choices of i'_v lead to conjugate embeddings i_v. The equivalence class of σ_v is therefore well-defined and depends only on v.

In the nonarchimedean case, let k_v and $\overline{k}_v$ denote the residue fields of F_v and $\overline{F}_v$, respectively. Then $Gal(\overline{F}_v/F_v)$ acts on $\overline{k}_v$ and we have an exact sequence

$$1 \longrightarrow I_v \longrightarrow Gal(\overline{F}_v/F_v) \longrightarrow Gal(\overline{k}_v/k_v) \longrightarrow 1,$$

where I_v is the inertia subgroup. Set $q_v = Card(k_v)$. A **Frobenius element** Fr_v is an element of $Gal(\overline{F}_v/F_v)$ whose image in $Gal(\overline{k}_v/k_v)$ is the automorphism $x \to x^{q_v}$. The action of $\sigma_v(Fr_v)$ on the subspace V^{I_v} of inertial invariants in V is independent of the choice of Fr_v and we define the local factor at v by:

$$L(s, \sigma_v) = \det(1 - q^{-s}\sigma_v(Fr_v)|V^{I_v})^{-1}$$

The representation σ is said to be **unramified** at v if $\sigma_v(I_v) = 1$. In this case, the element $\sigma_v(Fr_v)$ is independent of the choice of Fr_v. The **Frobenius class** attached to v is the *conjugacy class* $\{\sigma_v(Fr_v)\}$ of $\sigma_v(Fr_v)$ in $GL(V)$. The Frobenius class is independent of the choice of embedding i_v and thus depends only on v. Furthermore, it is a **semisimple** conjugacy class, i.e., it consists of diagonalizable elements. Indeed, since Image(σ) is a finite group, $\sigma_v(Fr_v)$ is a linear transformation of finite order, hence diagonalizable. Furthermore, its eigenvalues $z_1, \ldots, z_n$ are roots of unity. Identifying $GL(V)$ with $GL_n(\mathbb{C})$, we have

$$\sigma_v(Fr_v) \sim \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$$

where $\sim$ denotes conjugacy. In this case, the definition yields

$$L(s, \sigma_v) = \prod_{j=1}^{n} (1 - q_v^{-s} z_j)^{-1}$$

Observe that $\det(\sigma_v(Fr_v)) = z_1 \cdots z_n$. Under the local Artin isomorphism sending Fr_v to a uniformizing element $\varpi_v \in F_v^*$, $\det(\sigma_v)$ is identified with the unramified character of F_v^* defined by $x \to z^{val(x)}$, where $z = z_1 \cdots z_n$ and $val(x)$ is the v-adic valuation on F_v^*.

If v is archimedean, then $F_v \approx \mathbb{R}$ or $\mathbb{C}$. In the first case, $Gal(\overline{F}_v/F_v) \approx Gal(\mathbb{C}/\mathbb{R}) = \{1, c\}$, where c denotes complex conjugation. The eigenvalues of $\sigma_v(c)$ are ± 1. We set

$$L(s, \sigma_v) = (\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}))^k (\pi^{-\frac{s+1}{2}}\Gamma(\tfrac{s+1}{2}))^\ell$$

where k (resp., ℓ) are the number of $+1$ (resp., -1) eigenvalues of $\sigma_v(c)$. If $F_v \approx \mathbb{C}$, then $Gal(\overline{F}_v/F_v)$ is the trivial group and we set

$$L(s, \sigma_v) = (2(2\pi)^{-s}\Gamma(s))^n$$

where $n = \dim(\sigma)$.

With these definitions, it is clear that the correspondence $\sigma \to L(s, \sigma)$ is **additive** in the sense that

$$L(s, \sigma \oplus \tau) = L(s, \sigma)L(s, \tau)$$

for any two Galois representations σ and τ.

For any finite set of places S, the partial L-function is defined as the Euler product

$$L_S(s, \sigma) = \prod_{v \notin S} L(s, \sigma_v).$$

In particular, $L(s, \sigma) = L_S(s, \sigma)$ for $S = \phi$.

Example 1. Let σ be the trivial representation and let S be the set of archimedean places. Then $L_S(s, \sigma) = \prod_{v<\infty}(1 - q_v^{-s})^{-1}$ is the Dedekind zeta function $\zeta_F(s)$ of F.

2. The Artin Conjecture

As mentioned above, the eigenvalues of $\sigma_v(Fr_v)$ are roots of unity. It follows easily that the Euler product for $L(s,\sigma)$ converges absolutely in the half plane $\mathrm{Re}(s) > 1$. The following theorem combines results of Hecke, Artin, and Brauer.

Theorem 1. *$L(s,\sigma)$ extends analytically to a meromorphic function on the complex plane $\mathbb{C}$.*

The Artin L-function $L(s,\sigma)$ also satisfies a functional equation of the form $L(s,\sigma) = \varepsilon(s,\sigma)L(1-s,\sigma^*)$ where σ^* is the contragredient representation to σ and $\varepsilon(s,\sigma)$ is the so-called "epsilon factor" [T].

We can now state the famous

Artin Conjecture. *If σ is irreducible and nontrivial, then $L(s,\sigma)$ can be analytically continued to an entire function of s.*

3. The Frobenius Classes Determine σ

Let $S(\sigma)$ be the set of places v such that either v is archimedean or σ is ramified at v. Thus σ_v is unramified if and only if $v \notin S(\sigma)$. The continuity of σ implies that $S(\sigma)$ is a finite set. It is useful to emphasize the following point:

> *a Galois representation σ defines a family of semisimple conjugacy classes $\{\sigma_v(Fr_v)\}$ in $GL_n(\mathbb{C})$ indexed by $v \notin S(\sigma)$.*

The following basic theorem asserts that this collection determines σ uniquely.

Theorem 2. *Let σ_1 and σ_2 be Galois representations of dimension n such that $\sigma_{1v}(Fr_v) \sim \sigma_{2v}(Fr_v)$ for almost all v. Then $\sigma_1 \simeq \sigma_2$.*

Proof. This theorem is an immediate consequence of the Tchebotarev density theorem [La], [N]. Choose an extension E/F such that both σ_1 and σ_2 factor through the projection $\pi : Gal(\overline{F}/F) \to Gal(E/F)$. The Tchebotarev density theorem implies that every conjugacy class in $Gal(E/F)$ is of the form $\{\pi \circ \sigma_v(Fr_v)\}$ for infinitely many primes v of F (in fact, it says that the density of primes for which $\{\pi \circ \sigma_v(Fr_v)\}$ is a given class c in $Gal(E/F)$ is the correct one, namely $|c|/N$ where $N = |Gal(E/F)|$). In particular, if $\sigma_{1v}(Fr_v) \sim \sigma_{2v}(Fr_v)$ for almost all v, then the characters of σ_1 and σ_2 are equal and hence are equivalent. $\square$

We shall see below that cuspidal representations also give rise to collections of conjugacy classes in $GL_n(\mathbb{C})$ and are determined by them in an analagous fashion (Sec. 9).

4. Change of Field: Induction and Restriction

Artin L-functions behave well with respect to induction and restriction. Let E/F be a finite extension and let ρ be a Galois representation of $Gal(\overline{F}/E)$. We write $Ind_E^F(\rho)$ for the representation of $Gal(\overline{F}/F)$ **induced** from ρ. Recall that $Ind_E^F(\rho)$ is the representation of $Gal(\overline{F}/F)$ by right translation on the space of all functions $f : Gal(\overline{F}/F) \to V$ such that $f(xy) = \rho(x)f(y)$ for all $x \in Gal(\overline{F}/E)$ and $y \in Gal(\overline{F}/F)$ (here V is the space on which ρ acts).

If σ is a representation of $Gal(\overline{F}/F)$, we write σ_E for the **restriction** of σ to $Gal(\overline{F}/E)$. Let

$$R_{E/F} = Ind_E^F(1)$$

denote the representation of $Gal(\overline{F}/F)$ induced from the trivial representation of $Gal(\overline{F}/E)$.

Proposition 3. *Let E/F be a finite extension.*

1. *The formation of L-functions is invariant under induction. More precisely, if ρ is a representation of $Gal(\overline{F}/E)$ and $\sigma = Ind_E^F(\rho)$, then $L(s,\sigma) = L(s,\rho)$.*
2. *Let σ be a representation of $Gal(\overline{F}/F)$. Then $L(s,\sigma_E) = L(s,\sigma \otimes R_{E/F})$.*

Proof. Part (1) is proved in [La] (see also [Kn], Section 5). It is equivalent to the assertion

$$L(s,\sigma_v) = \prod_{w|v} L(s,\rho_w)$$

for all places v of F. Part (2) follows (1) and the general **projection formula** for induced representations: if ρ is a representation of a group G and ρ_H is its restriction to a subgroup H of finite index, then $Ind_H^G(\rho_H) \simeq Ind_H^G(1) \otimes \rho$. In our case, this formula yields $Ind_E^F(\sigma_E) = \sigma \otimes R_{E/F}$. □

II. Cuspidal Representations

Let G denote the algebraic group $GL(n)$ and Z the subgroup of scalar matrices. For any commutative ring R with identity, $G(R) = GL_n(R)$ and $Z(R) \simeq R^*$. If v is a place of F, we write G_v for $GL_n(F_v)$.

To describe the cuspidal representations of $GL_n(\mathbf{A}_F)$, we fix a unitary Hecke character ξ of I_F and regard it as a character of $Z(F)\backslash Z(\mathbf{A}_F)$ in the obvious way:

$$\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \longrightarrow \xi(\lambda).$$

Let $L^2(\xi)$ be the Hilbert space of measurable functions $\varphi(g)$ on $GL_n(F)\backslash GL_n(\mathbf{A}_F)$ such that $\varphi(zg) = \xi(z)\varphi(g)$ for all $z \in Z(\mathbf{A}_F)$ and

$$\int_{GL_2(F)Z(\mathbf{A}_F)\backslash GL_2(\mathbf{A}_F)} |\varphi(g)|^2 dg < \infty.$$

Then $G(\mathbf{A}_F)$ acts on $L^2(\xi)$ by right translation ρ. The center $Z(\mathbf{A}_F)$ acts on $L^2(\xi)$ via ξ.

In general, if π is an irreducible admissible representation of $G(\mathbf{A}_F)$ or G_v, the center ($Z(\mathbf{A}_F)$ or Z_v) acts by a character ω_π called the **central character** of π. We view ω_π as a character of I_F or F_v^* in the two cases. Observe that if $n = 1$, then $G = Z$ and $L^2(\xi)$ is a one-dimensional vector space consisting of multiples of the function ξ.

5. Subspace of Cuspidal Functions

To define the subspace $L_o^2(\xi)$ of cuspidal functions ([GGP], [H], [JL]) let N be a standard unipotent subgroup of G attached to a partition $n = n_1 + \cdots + n_r$. By definition, N is the subgroup of elements of G with identity matrices of size $n_j \times n_j$

along the diagonal, arbitrary entries above them and zeroes below. For example, the partition $5 = 2 + 3 + 2$ corresponds to the subgroup of matrices of the form

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{I_2}} & * & * & * & * & * \\ & & * & * & * & * & * \\ 0 & 0 & & & & * & * \\ 0 & 0 & & I_3 & & * & * \\ 0 & 0 & & & & * & * \\ 0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{I_2}} \\ 0 & 0 & 0 & 0 & 0 & & \end{pmatrix}$$

The **constant term** of an element $\varphi \in L^2(\xi)$ relative to N is the function

$$\varphi_N(g) = \int_{N(F)\backslash N(\mathbf{A}_F)} \varphi(ng)dn.$$

We say that φ is **cuspidal** if, for all standard unipotent subgroups other than $\{e\}$, $\varphi_N(g) = 0$ for almost all g. Let $L_o^2(\xi)$ be the subspace of cuspidal functions. It is clearly invariant under ρ. We denote the restriction of ρ to $L_o^2(\xi)$ by ρ_o. See Section 7 of [Kn] for a discussion of the historical origins of "constant term" and "cuspidal."

According to a general theorem of Gelfand-Graev-Piatetskii-Shapiro [H], ρ_o decomposes as a direct sum of irreducible unitary representations of $G(\mathbf{A}_F)$. This result holds for any reductive group in place of $GL(n)$. The irreducible constituents ρ_o are called **cuspidal representations** (with central character ξ). Note that for any π with central character ξ, the contragredient representation π^* has central character ξ^{-1}.

We denote by $\mathcal{A}_F(n, \xi)$ the set of all cuspidal representation of $G(\mathbf{A}_F)$ with central character ξ and $\mathcal{A}_F(n)$ the set of all cuspidal representations (with any unitary central character). In particular, $\mathcal{A}_F(1)$ is simply the set of unitary Hecke characters of the idele group I_F.

6. Multiplicity-One Theorem

It is a basic fact that each representation occurring in the decomposition of the representation ρ_o of $GL_n(\mathbf{A})$ appears with multiplicity one. This result is due to Jacquet-Langlands [JL] for $n = 2$ and to Shalika [S] for general n. Accordingly, we have

$$\rho_o \simeq \bigoplus_{\pi \in \mathcal{A}_F(n,\xi)} \pi.$$

This multiplicity-one result need not hold for groups other than $GL(n)$. For example, it is false for $SL(n)$ if $n > 3$ [Bl]. See [BR] for a discussion of multiplicity questions.

7. Unramified Representations

We recall some facts about unramified representations of the groups G_v. Assume that v is nonarchimedean. Recall that an irreducible admissible representation π_v of G_v is called **unramified** if the space of π_v contains a nonzero vector fixed under the maximal compact subgroup $GL_n(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of integers of F_v. Fortunately, the unramified representations are easily classified. Let $B \subset G$ be the standard Borel subgroup of upper-triangular matrices in G, that is, the subgroup of matrices (b_{ij}) in G such that $b_{ij} = 0$ for $i > j$. An unramified character χ of B_v is a character of the form

$$\chi\left(\begin{pmatrix} a_1 & \cdots & * \\ & \ddots & * \\ & & a_n \end{pmatrix}\right) = \prod z_j^{val(a_j)}$$

where $z_1, ..., z_n$ are nonzero complex numbers. The representations $Ind_{B_v}^{G_v}(\chi)$ for χ unramified are called **unramified principal series representations**. Here, $Ind_{B_v}^{G_v}(\chi)$ denotes normalized induction, so that $Ind_{B_v}^{G_v}(\chi)$ is unitary if χ is unitary. For $\sigma \in S_n$, let $\sigma\chi$ denote the character defined by the collection $z_{\sigma(1)}, ..., z_{\sigma(n)}$.

Theorem 4.

1. *Let χ be an unramified character of B_v.. Then $Ind_{B_v}^{G_v}(\chi)$ has a unique constitutent $\pi(\chi)$ that is unramified.*
2. *Every unramified representation is of the form $\pi(\chi)$ for some unramified χ.*
3. *Let χ_1, χ_2 be unramified characters of B_v. Then $\pi(\chi_1) = \pi(\chi_2)$ if and only $\sigma\chi_1 = \chi_2$ for some $\sigma \in S_n$.*

See [Ca] for more details about these facts.

This theorem allows us to attach a conjugacy class of semisimple elements $\{g(\pi_v)\}$ in $GL_n(\mathbb{C})$ to each unramified representation π_v. Namely, we set

$$g(\pi_v) = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$$

if $\pi_v = \pi(\chi)$ and χ is defined as above by $z_1, ..., z_n$. The contragredient π_v^* of $\pi_v = \pi(\chi)$ is the representation $\pi(\chi^{-1})$. Therefore $g(\pi_v^*) = g(\pi_v)^{-1}$.

An unramified character of Z_v (which we identify with F_v^*) is of the form $x \to z^{val(x)}$ for some $z \in \mathbb{C}^*$. The center Z_v acts on $Ind_{B_v}^{G_v}(\chi)$ (and hence also on $\pi(\chi)$) by the character $x \to (z_1 \cdots z_n)^{val(x)}$. Therefore the central character of $\pi(\chi)$ is the unramified character corresponding to $z = \det(g(\pi_v))$. The classification can be restated as follows.

Theorem 5. *Let v be a finite place. Then there is a bijection $\pi_v \to \{g(\pi_v)\}$ between the set of isomorphism classes of unramified representations of $GL_n(F_v)$ and the set of semisimple conjugacy classes in $GL_n(\mathbb{C})$. The central character of π_v is the character of F_v^* defined by $x \to (\det(g(\pi_v)))^{val(x)}$.*

Definition 1. The conjugacy class $\{g(\pi_v)\}$ of the unramified representation π_v of $GL_n(F_v)$ is called the **Langlands class** of π_v.

Sometimes it is useful to restate the classification in terms of Galois representations. A representation σ of $Gal(\overline{F}_v/F_v)$ is called **unramified** if it factors through the projection to $Gal(\overline{k}_v/k_v)$. Such a representation is uniquely determined by the image $\sigma(Fr_v)$, which is independent of the choice of Frobenius element Fr_v. We associate to π_v the unique unramified representation

$$\rho(\pi_v) : Gal(\overline{F}_v/F_v) \longrightarrow GL_n(\mathbb{C}).$$

such that $\rho(\pi_v)(Fr_v) = g(\pi_v)$. By the previous theorem, this gives a *bijection* between the set of irreducible unramified representations of $GL_n(F_v)$ and the set of unramified n-dimensional representations of $Gal(\overline{F}_v/F_v)$ such that the image of Fr_v is semisimple. Of course, this is the easy part of the local Langlands correspondence described in Section 8 of [Kn].

8. The L-Function of a Cuspidal Representation

A cuspidal representation $\pi \in \mathcal{A}_F(n)$ can be decomposed as a restricted tensor product over all places of F

$$\pi = \bigotimes{}' \pi_v.$$

For each v, π_v is an irreducible admissible representation of G_v. We refer to [F] for the precise definitions and theorems. For *almost all* nonarchimedean places v, π_v is unramified. Let $S(\pi)$ be the complement of the set of places such that π_v is unramified. Note that $S(\pi)$ contains the archimedean places. The L-function of π is an Euler product over all places v of F :

$$L(s,\pi) = \prod_v L(s,\pi_v).$$

For $v \notin S(\pi)$, the local factor is defined by the formula

$$L(s,\pi_v) = \det(1 - q_v^{-s} g(\pi_v))^{-1} = \prod_{j=1}^{n} (1 - q_v^{-s} z_j)^{-1}.$$

The general definition of $L(s,\pi_v)$ for all places v (i.e., including $v \in S(\pi)$) is given in [J1], [GJ]. We shall not need this definition in the sequel. For any finite set of places S, the L-function $L_S(s,\pi)$ of π is defined as an Euler product

$$L_S(s,\pi) = \prod_{v \notin S} L(s,\pi_v).$$

For the correspondence between these definitions and the definitions for more general reductive groups, see Section 9 of [Kn].

9. Convergence of the Euler Product

The convergence in some half-plane of the Euler product $L(s,\pi)$ attached to $\pi \in \mathcal{A}_F(n)$ is a consequence of the *unitarity* of the local components π_v. More precisely, it can be shown there exists a real number t (depending only on n) such that for all $v \notin S(\pi)$, the eigenvalues $\{z_j\}$ of $g(\pi_v)$ satisfy $|z_j| \leq q^t$ (see [Bo] for a discussion of this point). This implies the absolute convergence of $L(s,\pi)$ in the half-plane $\mathrm{Re}(s) > t+1$. The following theorem is due to Hecke and Jacquet-Langlands [JL] for $n = 2$ and to Godement-Jacquet for general n [J1], [GoJ]. For an exposition in this volume, see [J2].

Theorem 6. *Let $\pi \in \mathcal{A}_F(n)$. Then $L(s,\pi)$ has an analytic continuation to an entire function of s.*

The L-function also satisfies a function equation of the form $L(s,\pi) = \varepsilon(s,\pi)L(1-s,\pi^*)$ [J1]. As mentioned above, convergence of the Euler product defining an Artin L-function in a half-plane follows from the fact that the eigenvalues of $\sigma_v(Fr_v)$ are roots of unity and hence have absolute value one for $v \notin S(\sigma)$. For cuspidal representations of $GL(n)$, we have the so-called **generalized Ramanujan conjecture**:

Conjecture 7. *Let $\pi \in \mathcal{A}_F(n)$. Then the eigenvalues of $g(\pi_v)$ have absolute value one for all $v \notin S(\pi)$.*

Unlike the case of Galois representations, however, the eigenvalues of $g(\pi_v)$ are not roots of unity in general (cf. Sec. 11 below).

Conjecture 7 reduces to the classical Ramanujan-Petersson conjecture for modular forms when π is a cuspidal representation of $GL(2)_{/\mathbb{Q}}$ such that the component π_∞ lies in the discrete series or limit of discrete series. This was proved by Deligne [De] for π_∞ discrete series and Deligne-Serre [DS] for π_∞ limit of discrete series (cf. [R]).

10. Strong Multiplicity-One Theorem

The family of Langlands classes attached to a cuspidal representation is analogous to the family of Frobenius classes attached to a Galois representation. In particular,

> *a cuspidal representation $\pi \in \mathcal{A}_F(n)$ defines a family of semisimple conjugacy classes $\{g(\pi_v)\}$ in $GL_n(\mathbb{C})$ indexed by $v \notin S(\pi)$.*

Example 2. If $n = 1$, then π is a Hecke character of I_F. If π_v is unramified, the Langlands class $g(\pi_v)$ lies in $GL_1(\mathbb{C}) = \mathbb{C}^*$. In fact, we have $g(\pi_v) = \pi_v(\varpi_v)$ where $\varpi_v \in F_v^*$ is a prime element.

The analogue of the Theorem 2 is the so-called **strong multiplicity-one** theorem due to Jacquet-Shalika [JS].

Theorem 8 (Strong Multiplicity-One)**.** *Let π_1, $\pi_2 \in \mathcal{A}_F(n)$ be cuspidal representations such that $g(\pi_{1v}) \sim g(\pi_{2v})$ for almost all v. Then $\pi_1 = \pi_2$.*

11. The Langlands-Artin Conjecture

In a foundational article published in 1970, Langlands [L1] stated a collection of conjectures known under the general heading **functoriality conjecture**. They imply, as a special case, a relation between Galois representations and cuspidal representations. While transparently simple to state, it has remarkably far-reaching ramifications. So far, it has been established only in a limited number of special cases, some of which are explained in Part III below.

Langlands-Artin Conjecture. *Let σ be an irreducible Galois representation of $Gal(\overline{F}/F)$ of dimension n. Then there exists a cuspidal representation π in $\mathcal{A}_n(F)$ such that for almost all places v, $\sigma_v(Fr_v) \sim g(\pi_v)$.*

By the strong multiplicity-one theorem, there is at most one cuspidal representation π satisfying $\sigma_v(Fr_v) \sim g(\pi_v)$ for almost all v. We write $\pi(\sigma)$ for this representation, if it exists.

Remarks.

1. Suppose that $\pi = \pi(\sigma)$ exists. We may view the determinant character $\det(\sigma)$ as a character of I_F via the Artin map. Then $\omega_\pi = \det(\sigma)$. Indeed, for almost all places v of F, the local components of ω_π and $\det(\sigma)$ are the characters $x \to (\det g(\pi_v))^{val(x)}$ and $x \to (\det \sigma_v(Fr_v))^{val(x)}$, respectively. By the strong multiplicity-one theorem for Hecke characters, we obtain $\omega_\pi = \det(\sigma)$.

2. The condition $\sigma_v(Fr_v) \sim g(\pi_v)$ is equivalent to the equality of local L-functions: $L(s, \sigma_v) = L(s, \pi_v)$.
3. The Langlands-Artin conjecture implies the Artin conjecture. This implication uses Theorem 8.8 of [Kn] and Theorem 6.
4. Suppose that σ is reducible, say $\sigma = \bigoplus_{j=1}^{N} \sigma_j$ with $\dim(\sigma_j) = n_j$. Then $L(s, \sigma) = \prod_{j=1}^{N} L(s, \sigma_j)$ and hence

$$L(s, \sigma) = \prod_{j=1}^{N} L(s, \pi(\sigma_j)).$$

The map $\sigma \longrightarrow \pi(\sigma)$ is a special case of the global Langlands correspondence, which is pieced together from the local Langlands correspondences described in Section 8 of [Kn]). We emphasize that the global correspondence is certainly not surjective, even if it exists. Indeed, if π is of the form $\pi(\sigma)$, the elements $g(\pi_v)$ must be of finite order for almost all π. To say more about the image of the map $\sigma \longrightarrow \pi(\sigma)$, recall that for archimedean v, π_v corresponds to an n-dimensional representation $\rho(\pi_v)$ of the Weil group $W_{\mathbb{C}/\mathbb{R}}$. One conjectures that if $\pi = \pi(\sigma)$, then π_v and σ_v correspond under the Langlands correspondence for *all* places v of F (and not just at the unramified places). This would imply that if v is archimedean, then the representation $\rho(\pi_v)$ is equivalent to the pullback of σ_v via the projection $W_{\mathbb{C}/\mathbb{R}} \rightarrow Gal(\mathbb{C}/\mathbb{R})$. In other words, if $\pi = \pi(\sigma)$, then the archimedean components of π are conjecturally of a very special type, corresponding to Weil group representations that factor through $Gal(\mathbb{C}/\mathbb{R})$. It is sometimes conjectured that the image of $\sigma \longrightarrow \pi(\sigma)$ is *precisely* the set of cuspidal representations whose archimedean components correspond to Weil group representations factoring through $Gal(\mathbb{C}/\mathbb{R})$. This assertion, however, appears to be independent of the general functoriality conjecture.

Consider the case of $GL(2)_{/\mathbb{Q}}$. The cuspidal representations π are divided into two classes, according as the archimedean component π_∞ is "holomorphic" of some weight $k \geq 1$ or not. If π_∞ is holomorphic, then π corresponds to a classical newform of weight $k \geq 1$. According to the Deligne-Serre theorem [DS], $\pi = \pi(\sigma)$ for some Galois representation σ if $k = 1$. If $k > 1$, the Langlands classes of π are not of finite order. In this case, π is associated to a "compatible family" of ℓ-adic representations, but it does not correspond to a complex Galois representation [De]. On the other hand, the nonholomorphic cuspidal representations of $GL(2)_{\mathbb{Q}}$ are attached to classical "Maass forms" on the upper half-plane with eigenvalue λ for the Laplacian [G]. It is conjectured that such a π is of the form $\pi(\sigma)$ for some irreducible Galois representation σ if and only if $\lambda = \frac{1}{4}$. The cuspidal π with $\lambda \neq \frac{1}{4}$ have no apparent connection with Galois theory and one even speculates that almost all of the Langlands classes of such π have transcendental eigenvalues.

12. Tensor Structure

The category of Galois representations has several operations defined on it: tensor product, induction, restriction, etc. A fundamental problem is to determine whether analogous operations exist on the set of cuspidal representations. This would follow from the Langlands-Artin conjecture for the *subset* of cuspidal representations $\pi \in \mathcal{A}_F(n)$ in the image of the map $\sigma \longrightarrow \pi(\sigma)$. The general Langlands conjectures predict that these operations exist on *all* of $\mathcal{A}_F(n)$. In this section, we describe this conjecture and some of its consequences in greater detail.

Consider the tensor product $\sigma_1 \otimes \sigma_2$ of two irreducible Galois representations. If $\sigma_1 \otimes \sigma_2$ is irreducible, then the three (conjectural) cuspidal representations $\pi(\sigma_1), \pi(\sigma_2)$, and $\pi(\sigma_1 \otimes \sigma_2)$ are related by

$$g(\pi_v(\sigma_1)) \otimes g(\pi_v(\sigma_2)) \sim g(\pi_v(\sigma_1 \otimes \sigma_2))$$

for almost all places v. Recall that $\sim$ denotes conjugacy within the general linear group. It is reasonable to think of $\pi(\sigma_1 \otimes \sigma_2)$ as a kind of product of $\pi(\sigma_1)$ and $\pi(\sigma_2)$. We shall write it as $\pi(\sigma_1) \boxtimes \pi(\sigma_2)$. In general, $\sigma_1 \otimes \sigma_2$ need not be irreducible, but will decompose as a direct sum of irreducibles, say $\sigma_1 \otimes \sigma_2 \simeq \bigoplus \tau_j$. In this case

$$g(\pi_v(\sigma_1)) \otimes g(\pi_v(\sigma_2)) \sim \bigoplus g(\pi_v(\tau_j)),$$

where the right-hand side is to be interpreted as a matrix in block diagonal form with blocks made up of the matrices $g(\pi_v(\tau_j))$. By analogy, we conjecture that the tensor product of two cuspidal representations exists, even if they are not attached to Galois representations. More precisely,

Definition 2. Let $\pi \in \mathcal{A}_F(n)$ and $\pi' \in \mathcal{A}_F(m)$. Suppose we are given cuspidal representations $\pi_j \in \mathcal{A}_F(n_j)$ for $j = 1, \ldots, r$ such that $\sum_{j=1}^{r} n_j = mn$. We shall write

$$\pi \boxtimes \pi' = \bigoplus_{j=1}^{r} \pi_j$$

if $g(\pi_v) \otimes g(\pi'_v) \sim \bigoplus_{j=1}^{r} g(\pi_{jv})$ for almost all v (where the conjugacy is understood to occur in $GL_{nm}(\mathbb{C})$).

Then we have the following conjecture, which is a piece of the general Langlands functoriality conjecture.

Conjecture 9. *The tensor product $\pi \boxtimes \pi'$ exists for any two cuspidal representations $\pi \in \mathcal{A}_F(n)$ and $\pi' \in \mathcal{A}_F(m)$.*

It is convenient to form the additive monoid

$$\mathcal{A}_F = \bigoplus \mathcal{A}_F(n),$$

consisting of formal sums $\sum_{j=1}^{N} \pi_j$ where $\pi_j \in \mathcal{A}_F(n_j)$ for some n_j. Define

$$\deg : \mathcal{A}_F \longrightarrow \mathbb{Z}$$

by setting $\deg(\pi) = n$ for $\pi \in \mathcal{A}_F(n)$ and extending to $\mathcal{A}_F$ by additivity. For any element $\pi = \sum_{j=1}^{N} \pi_j$ in $\mathcal{A}_F$ of degree n, let $\{g(\pi)\}$ denote the conjugacy class of the element $\bigoplus_{j=1}^{N} g(\pi_j)$ in $GL_n(\mathbb{C})$.

The tensor operation $\boxtimes$ defines a distributive multiplication on $\mathcal{A}_F$. We may also conjecture that other operations of linear algebra exist on $\mathcal{A}_F$, such as exterior or symmetric powers. Recall that a homomorphism $r : GL_n(\mathbb{C}) \to GL_m(\mathbb{C})$ is called an algebraic representation if the entries of $r(g)$ are polynomial functions of the entries in g and $\det(g)^{-1}$. More generally, the functoriality conjecture predicts the following.

Conjecture 10. *Let $r : GL_n(\mathbb{C}) \to GL_m(\mathbb{C})$ be an algebraic representation of $GL_n(\mathbb{C})$. Then for all $\pi \in \mathcal{A}_F(n)$, there exists $\pi' \in \mathcal{A}_F(m)$ such that*

$$r(g(\pi_v)) \sim g(\pi'_v)$$

for almost all v.

13. The Adjoint Lifting

We now discuss a special case in which this last conjecture is known. The group $GL_2(\mathbb{C})$ acts by conjugation on the three-dimensional space of 2×2 matrices of trace zero. This defines the three-dimensional adjoint representation,

$$Ad : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C}).$$

The following theorem was proved by Gelbart-Jacquet [GJ], generalizing a method introduced by Shimura for the study of symmetric square L-functions of modular forms.

Theorem 11. *Let $\pi \in \mathcal{A}_F(2)$. Then there exists an element, denoted $Ad(\pi)$, of degree 3 in $\mathcal{A}_F$ such that $Ad(g(\pi_v)) \sim g(Ad(\pi)_v)$ for almost all v.*

In particular,

$$\text{if} \quad g(\pi_v) \sim \begin{pmatrix} a & \\ & b \end{pmatrix}, \quad \text{then} \quad g(Ad(\pi)_v) \sim \begin{pmatrix} a/b & & \\ & 1 & \\ & & b/a \end{pmatrix}.$$

We can describe precisely when $Ad(\pi)$ is cuspidal in terms of automorphic induction defined below in Section 15. The element $Ad(\pi)$ is cuspidal if and only if π is not of the form $AI_E^F(\theta)$ for some quadratic extension E/F and some Hecke character θ of E. This is not exactly the description given in [GJ]. However, the results of Labesse-Langlands [LL] show that this description is equivalent with the description given in [GJ].

14. A Theorem of Jacquet and Shalika

The analogy between cuspidal representations of $GL(n)$ and Galois representations can be used very effectively to *predict* results that ought to be true. For example, if σ and τ are n-dimensional Galois representations (or representations of any finite group) with σ irreducible, then σ is isomorphic to τ if and only if $\sigma \otimes \sigma^*$ is isomorphic to $\tau \otimes \sigma^*$, where σ^* is the contragredient to σ. Indeed, $\sigma \otimes \sigma^* \approx Hom(\sigma, \sigma)$ and $\tau \otimes \sigma^* \approx Hom(\sigma, \tau)$. The image of the identity in $Hom(\sigma, \sigma)$ under an isomorphism $\sigma \otimes \sigma^* \to \tau \otimes \sigma^*$ yields an isomorphism of σ with τ. The analogue of this result for cuspidal representations of $GL(n)$ is the following theorem [JS].

Theorem 12 (Jacquet-Shalika)**.** *Let $\pi, \pi' \in \mathcal{A}_F$ be elements of degree n with π cuspidal. If $g(\pi_v) \otimes g(\pi_v^*) \sim g(\pi'_v) \otimes g(\pi_v^*)$ for almost all v, then $\pi = \pi'$.*

15. Induction and Restriction Revisited

Let E/F be a finite extension of degree ℓ. We now define operations that correspond to induction and restriction of Galois representations. The induction operation is called **automorphic induction** and the restriction operation is called **base change**:

$$AI_E^F : \mathcal{A}_E \longrightarrow \mathcal{A}_F \qquad \text{(automorphic induction)}$$
$$BC_{E/F} : \mathcal{A}_F \longrightarrow \mathcal{A}_E \qquad \text{(base change).}$$

Before proceeding to the definitions, we define a Galois action on $\mathcal{A}_F(n)$ (and by extension, on $\mathcal{A}_F$). First consider what happens in the Galois case. Let η be an automorphism of E/F. For any Galois representation σ of $Gal(\overline{F}/E)$, we define $\eta(\sigma)$ by $\eta(\sigma)(x) = \sigma(\eta^{-1}x\eta)$. If Fr_w is a Frobenius element of a place w of E, then $\eta^{-1}Fr_w\eta$ is a Frobenius element of the place $\eta^{-1}(w)$. In particular, $\eta(\sigma)(Fr_w) = \sigma(Fr_{\eta^{-1}(w)})$.

We define the conjugate $\eta(\Pi)$ of a representation Π of $GL_n(\mathbf{A}_E)$ to be the representation sending g to $\Pi(\eta^{-1}(g))$, where $\eta^{-1}(g)$ the matrix obtained by applying η^{-1} to the entries of g. This definition also makes sense in the local case for representations of $GL_n(E_w)$ if η is a Galois automorphism of E_w/F_w. If Π is a cuspidal representation of $GL_n(\mathbf{A}_E)$, then $\eta(\Pi)$ is again cuspidal since it is realized on the space of functions of the form $f(\eta(x))$, where $f(x)$ belongs to the subspace of $L_o(\xi)$ on which Π is realized. Furthermore, if $\Pi = \bigotimes \Pi_w$ (product over places w of E), then $\eta(\Pi) = \bigotimes \eta(\Pi_w)$. The next lemma shows that the Langlands classes of $\eta(\Pi)$ are a permutation of the Langlands classes of Π. It follows in particular that the action of Galois on $\mathcal{A}_E(n)$ is compatible with the correspondence $\sigma \to \pi(\sigma)$ in the sense that if $\pi(\sigma)$ exists, then $\pi(\eta(\sigma))$ exists and $\pi(\eta(\sigma)) = \eta(\pi(\sigma))$.

Lemma 13. *Let η be a Galois automorphism of E and let Π be a cuspidal representation of $GL_n(\mathbf{A}_E)$. Assume that $\Pi_{\eta^{-1}(w)}$ is unramified, where $\eta^{-1}(w)$ is the conjugate of w under η. Then $\eta(\Pi_w)$ is also unramified and $g(\eta(\Pi_w)) \sim g(\Pi_{\eta^{-1}(w)})$.*

Proof. Let $w' = \eta^{-1}(w)$. Then $val_w(x) = val_{w'}(\eta^{-1}(x))$ for $x \in E$, and η^{-1} induces isomorphisms $E_w \xrightarrow{\sim} E_{w'}$ and $G_w \xrightarrow{\sim} G_{w'}$. Let χ be a character of the Borel subgroup $B_{w'}$ and set $I' = Ind_{B_{w'}}^{G_{w'}}(\chi)$. Then $\eta(I')$ is isomorphic to the representation $I = Ind_{B_w}^{G_w}(\chi \circ \eta^{-1})$. Indeed, the map $f(x) \to f(\eta(x))$ for f in the induced space of I induces an isomorphism of $\eta(I)$ with I'. If χ is unramified and $\Pi_{w'}$ is isomorphic to the unique unramified constituent of I', then $\eta(\Pi_w)$ is isomorphic to the unramified constituent of I. Suppose that χ sends an upper-triangular matrix with diagonal entries $a_1, ..., a_n$ to $\prod z_j^{val_{w'}(a_j)}$. Then $\{z_1, ..., z_n\}$ is the set of eigenvalues of $g(\Pi_{w'})$. But then $\chi \circ \eta^{-1}$ sends an upper-triangular matrix with diagonal entries $a_1, ..., a_n$ to $\prod z_j^{val_w(a_j)}$ and therefore $g(\eta(\Pi_w)) \sim g(\Pi_{\eta^{-1}(w)})$. $\square$

We now define an operation of automorphic induction so as to correspond to induction of Galois representations. Let $\pi \in \mathcal{A}_E(n)$. By analogy with Artin L-functions, automorphic induction should preserve L-functions. An element $\Pi \in \mathcal{A}_F$ of degree $\deg(\Pi) = n\ell$ is said to be **automorphically induced** by π if

$$L(s, \Pi_v) = \prod_{w|v} L(s, \pi_w)$$

for almost all places v of F. The product is over places of E dividing v. There is at most one Π satisfying this condition for almost all v by the strong multiplicity-one theorem. We write $\Pi = AI_E^F(\pi)$ if such a Π exists. This definition is compatible with the Langlands correspondence in the following sense: if $\pi = \pi(\rho)$ for some Galois representation ρ of $Gal(\overline{F}/E)$, then

$$AI_E^F(\pi(\rho)) = \pi(Ind_E^F(\rho)).$$

We cannot expect Π to be cuspidal in all cases; this corresponds to the fact that $Ind_E^F(\rho)$ may be reducible even if ρ is irreducible.

Assume that v is unramified in E and that π_w is unramified for all w dividing v. In this case, Π_v is unramified and we describe $g(\Pi_v)$ explicitly as follows. It suffices to determine the eigenvalues of $g(\Pi_v)$. Set $d(w) = [E_w : F_v]$. We claim that $g_v(\Pi)$ is an element in $GL_{n\ell}(\mathbb{C})$ whose set of eigenvalues is the union over w dividing v of all $(d(w))^{th}$ roots of eigenvalues of $g(\pi_w)$. This gives a total of $n\big(\sum_{w|\ell} d(w)\big) = n\ell$ eigenvalues as required, since $\sum_{w|\ell} d(w) = \ell$. To check this assertion, fix w dividing v and set $d = d(w)$. Let $\zeta = \exp(\frac{2\pi i}{d})$ and let $\{z_{wj} : 1 \leq j \leq n\}$ be the set of eigenvalues of $g(\pi_w)$. Then $q_v = q_w^{\frac{1}{d}}$ and

$$L(s, \pi_w) = \prod_{j=1}^{n} (1 - z_{wj} q_w^{-s})^{-1} = \prod_{j=1}^{n} \prod_{k=1}^{d} (1 - z_{wj}^{\frac{1}{d}} \zeta^k q_v^{-s})^{-1}.$$

Therefore $\prod_{w|v} L(s, \pi_w)$ is equal to $\det(1 - q_v^{-s} g(\Pi_v))^{-1}$ with $g(\Pi_v)$ as described.

The following theorem is proved in [AC].

Theorem 14. *Let E/F be a cyclic extension. Then the automorphic induction map $AI_E^F : \mathcal{A}_E \to \mathcal{A}_F$ exists. Suppose that π is cuspidal. Then $AI_E^F(\pi)$ is cuspidal unless there exists a nontrivial element $\tau \in Gal(E/F)$ such that $\tau(\pi)$ is isomorphic to π.*

The condition for $AI_E^F(\pi)$ to be cuspidal is parallel to the condition for a representation induced from a cyclic extension to be irreducible. Indeed, if ρ is a Galois representation of $Gal(\overline{F}/E)$, then $Ind_E^F(\rho)$ be irreducible if and only if ρ is not isomorphic to any of its conjugates under $Gal(E/F)$.

Example 3. Assume E/F is quadratic and let $\theta \in \mathcal{A}_E(1)$ be a Hecke character. The existence of the automorphic representation of degree two $\pi = A_E^F(\theta)$ was proved in [JL], [ST] using the theory of the Weil representation. In the classical case $F = \mathbb{Q}$, π was constructed at the level of modular forms by Hecke and Maass. Observe that if v splits into two places w and w' in E, then

$$g(\pi_v) = \begin{pmatrix} \theta(\varpi_w) & 0 \\ 0 & \theta(\varpi_{w'}) \end{pmatrix}$$

where $\varpi_w, \varpi_{w'}$ denote prime elements at w, w'. If v is unramified in E and remains prime, then

$$g(\pi_v) = \begin{pmatrix} 0 & 1 \\ \theta(\varpi_w) & 0 \end{pmatrix}$$

where w is the unique place of E dividing v, since the eigenvalues of this matrix are the two square roots of $\theta(\varpi_w)$. Furthermore, $\pi = A_E^F(\theta)$ is cuspidal if and only if $\theta \neq \theta^\tau$, where τ is conjugation of E/F. This is equivalent to the condition that θ not be of the form $\theta' \circ N_{E/F}$.

Example 4. Let E/F is a cyclic cubic extension and let $\theta \in \mathcal{A}_E(1)$ be a Hecke character. The existence of the automorphic representation $AI_E^F(\theta)$ of degree three is due to Jacquet, Piatetski-Shapiro, and Shalika [JPS2].

16. Base Change

The operation of base change for automorphic representations corresponds to restriction of representations of Galois groups. Let E/F be a finite extension, let v be a place of F that is unramified in E, and let $Fr_v \in Gal(\overline{F}/F)$ be a Frobenius element. If w is a place of E dividing v, then $Fr_v^{d(w)}$ is a Frobenius element for w, where $d(w) = [E_w : F_v]$ is the relative degree. It follows that if σ is a Galois representation of $Gal(\overline{F}/F)$ unramified at v, then $\sigma(Fr_v)^{d(w)} \sim \sigma_E(Fr_w)$. We use this observation to define the base change lift of a cuspidal representation.

Definition 3. Let $\pi \in \mathcal{A}_F(n)$ and let $\Pi \in \mathcal{A}_E$ be an element of degree n. Then Π is said to be a **base change lift** of π if for almost all places v of F we have

$$g(\Pi_w) \sim g(\pi_v)^{d(w)}$$

for all places w of E dividing v.

By the strong multiplicity one theorem, the base change lift is unique if it exists. We denote it by π_E or $BC_{E/F}(\pi)$.

Example 5. Suppose that $n = 1$, so that π is a Hecke character of I_F. Let us check that π_E exists for any extension E/F, and that $\pi_E = \pi \circ N_{E/F}$ where $N_{E/F} : I_E \to I_F$ is the global norm map on ideles. Let ϖ be a prime element in F attached to a place v that is unramified in E. Then ϖ is also a prime element in E attached to any place w of E dividing v. By definition, $g(\pi_v) = \pi_v(\varpi)$ and $N_{E_w/F_v}(\varpi) = \varpi^{d(w)}$. Since the global norm induces the local norm maps on the idele components, we see that the Langlands class at w attached $\pi \circ N_{E/F}$ is $\pi_v(\varpi)^{d(w)}$ as required.

Example 6. Suppose that $\Pi = BC_{E/F}(\pi)$ where E/F is quadratic. Then $g(\Pi_w) \sim g(\pi_v)$ or $\sim g(\pi_v)^2$, according as v splits or remains prime in E. More generally if the degree $\ell = [E : F]$ is prime, then $g(\Pi_w) \sim g(\pi_v)$ or $\sim g(\pi_v)^\ell$, according as v splits or remains prime in E.

The base change lift is conjectured to exist in all cases, but it need not be cuspidal (just as the restriction of an irreducible Galois representation may become reducible upon restriction to a subgroup of finite index). The first pioneering work on the base change problem was done by Saito and Shintani [Sh]. A complete theory of base change for $GL(2)$ and cyclic extensions of prime degree was developed by Langlands ([L2]). It was generalized to $GL(m)$ for $m > 2$ by Arthur and Clozel ([AC]). Before stating the general theorem, we note some properties of base change that follow readily from the definition:

1. Base change lifting is transitive: let $F \subset E \subset K$ is a sequence of number fields and let $\pi \in \mathcal{A}_F(n)$. Assume that $BC_{K/F}(\pi)$ and $BC_{E/F}(\pi)$ exist. Then $BC_{K/E}(BC_{E/F}(\pi))$ exists and

$$BC_{K/F}(\pi) = BC_{K/E}(BC_{E/F}(\pi)).$$

2. Base change is compatible with twisting in the following sense: let χ be a Hecke character of I_F. Then
$$BC_{E/F}(\pi \otimes \chi) = BC_{E/F}(\pi) \otimes \chi_E,$$
where $\chi_E = \chi \circ N_{E/F}$.
3. Base change is compatible with the Langlands correspondence in the following sense: if $\pi = \pi(\sigma)$ for some Galois representation σ of $Gal(\overline{F}/F)$, then $\pi_E = \pi(\sigma_E)$.

Theorem 15. *Assume that E/F is a cyclic extension of prime degree ℓ.*

(a) (Existence) *For all automorphic representations $\pi \in \mathcal{A}_F(n)$, the base change lifting π_E exists. Furthermore, π_E cuspidal unless ℓ divides n and $\pi \otimes \omega \simeq \pi$ for some nontrivial character ω of $I_F/F^* N_E(I_E)$.*
(b) (Description of fibers) *Let π, $\pi' \in \mathcal{A}_F(n)$. Then $\pi_E = \pi'_E$ if and only if there exists a character ψ of $I_F/F^* N_E(I_E)$ such that $\pi = \pi' \otimes \psi$.*
(c) (Descent) *Let $\Pi \in \mathcal{A}_E(n)$. Then there exists $\pi \in \mathcal{A}_F(n)$ such that $\Pi = \pi_E$ if and only if $\eta(\Pi) = \Pi$ for all $\eta \in Gal(E/F)$.*

Remarks.

1. Part (a) of this theorem clearly remains true for any extension K/F that can be obtained by successive cyclic extensions of prime degree, that is, for any *solvable* extension.
2. Let $\eta \in Gal(E/F)$. Then $\eta(\Pi) = \Pi$ if and only if $\eta(\Pi_w) = \Pi_w$ for almost all places w of E by the strong multiplicity-one theorem. Assume that w divides the place v of F. If v remains prime in E, then $\eta(w) = w$ and the condition $\eta(\Pi_v) = \Pi_v$ is automatically satisfied. On the other hand, if v splits completely, we may identify the groups G_w for w dividing v. In this case, the condition $\eta(\Pi) = \Pi$ implies that the local components Π_w for w dividing v are all isomorphic. For example, if $\ell = 2$, then $\eta(\Pi) = \Pi$ if and only if $\Pi_w \approx \Pi_{w'}$ whenever w, w' are two places lying above a split prime v of F.

III. Special Cases of the Artin Conjecture

We shall consider an *irreducible* two-dimensional Galois representation
$$\rho : Gal(\overline{F}/F) \longrightarrow GL_2(\mathbb{C}).$$
As before, if L is a finite extension of F, we write ρ_L for the restriction of ρ to $Gal(\overline{F}/L)$. We say that $\pi(\rho)$ **exists** if there exists a cuspidal representation $\pi(\rho)$ of $GL_2(\mathbf{A}_F)$ satisfying the Langlands-Artin conjecture.

The group $GL_2(\mathbb{C})$ acts by conjugation (the adjoint representation) on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of 2×2 matrices of trace zero. Choosing a basis of $\mathfrak{sl}_2(\mathbb{C})$, we obtain a three-dimensional representation which we denote by Ad :
$$Ad : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C})$$
The symmetric bilinear form $Tr(AB)$ is invariant under the adjoint action, and the image of Ad is isomorphic to the complex orthogonal group $SO_3(\mathbb{C})$ defined relative to this bilinear form. The irreducible two-dimensional representations ρ are classified according to the image of $Ad \circ \rho$ in $SO_3(\mathbb{C})$. As is well-known, a

finite subgroup of $SO_3(\mathbb{C})$ is either cyclic, dihedral, or isomorphic to the one of the symmetry groups of the Platonic solids:

1. **tetrahedral group**, isomorphic to A_4
2. **octahedral group**, isomorphic to S_4
3. **icosahedral group**, isomorphic to A_5

We shall say that ρ is of cyclic, dihedral, tetrahedral, ... type if Image($Ad \circ \rho$) is of the corresponding type. We check below that $\pi(\rho)$ exists when ρ is of cyclic or dihedral type. Our main goal is to prove that $\pi(\rho)$ exists also if ρ is of tetrahedral or octahedral type. The Langlands-Artin conjecture is still open for icosahedral Galois representations, although it has been verified in some special cases [Bu].

17. Dihedral Representations

We first check that $\pi(\rho)$ exists if ρ is of cyclic or dihedral type. We use the following lemma.

Lemma 16. *Let $\tau : G \to GL_2(\mathbb{C})$ be an irreducible two-dimensional representation of a finite group G. Then τ is of cyclic or dihedral type if and only if τ is induced from a character χ of a subgroup H of index two. Furthermore, $\chi \neq \chi^g$ where g is any element in $G - H$.*

PROOF. The representation $Ad \circ \tau$ stabilizes a line if and only if τ is of cyclic or dihedral type since there are no irreducible three-dimensional representations of cyclic or dihedral groups. We claim that $Ad \circ \tau$ stabilizes a line if and only if τ preserves a symmetric bilinear form up to multiples (i.e., Image(τ) lies in the similitude group of a symmetric bilinear form). To check the claim, let $Sym : GL_2(\mathbb{C}) \to GL_3(\mathbb{C})$ be the representation on the symmetric tensors of degree two. It is easy to check that $Sym \approx Ad \otimes \nu$ where ν is the character $\nu(g) = \det(g)$, and therefore $Sym \circ \tau$ stabilizes a line if and only if τ^* preserves a symmetric bilinear form up to multiples. Since the map $g \to \det(g)^t g^{-1}$ is an inner automorphism of $GL_2(\mathbb{C})$, two-dimensional representations have the property that $\tau^* \approx \tau \otimes \det(\tau)^{-1}$. This shows that $Sym \circ \tau^* \approx Sym \circ \tau \otimes \det(\tau)^{-1}$, and the claim follows.

The similitude group of the standard form $x_1 y_2 + x_2 y_1$ is the normalizer $N(T)$ of the diagonal subgroup $T \subset GL_2(\mathbb{C})$. Since all symmetric bilinear forms are equivalent over $\mathbb{C}$, we conclude that $Ad(\tau)$ is cyclic or dihedral if and only if Image(τ) is conjugate to a subgroup of $N(T)$. We may assume that Image(τ) $\subset N(T)$. Now τ is irreducible, and so Image(τ) is not contained in T. Since $[N(T) : T] = 2$, we see that $H = \{g \in G : \tau(g) \in T\}$ is a subgroup of index two in G. The restriction τ_H is isomorphic to a direct sum of two distinct characters χ_1 and χ_2 of H. Any element $g \in G - H$ must interchange the χ_1 and χ_2 eigenspaces and it follows easily that $\tau \xrightarrow{\sim} Ind_H^G \chi_j$ for $j = 1$ or 2. □

Now we can prove

Theorem 17. *Assume that ρ is of cyclic or dihedral type. Then $\pi(\rho)$ exists.*

PROOF. Applying the lemma to Galois representations, we see that if ρ is cyclic or dihedral, then there is a quadratic extension E/F such that $\rho = Ind_E^F \theta$ for some character θ of $Gal(\overline{F}/E)$. The irreducibility of ρ implies that $\theta \neq \theta^\sigma$, where σ is conjugation relative to E/F . By class field theory, θ may be identified with an element of $\mathcal{A}_E(1)$ such that $\theta \neq \theta^\sigma$. The representation $AI_E^F(\theta)$ exists and is cuspidal by Example 3 (special case of Theorem 14), and we have $\pi(\rho) = AI_E^F(\theta)$. Indeed, the Langlands classes of $AI_E^F(\theta)$ coincide with the Frobenius classes of $Ind_E^F\theta$ for almost all v. $\square$

18. Tetrahedral Representations

In this section we prove the following theorem due to Langlands [L2].

Theorem 18 (Langlands). *Assume that ρ is of tetrahedral type. Then $\pi(\rho)$ exists.*

We begin with some preliminary remarks. The group A_4 has a unique irreducible representation of dimension three $\rho_{tet} : A_4 \to GL_3(\mathbb{C})$, defined via the action of A_4 on the tetrahedron in $\mathbb{R}^3$. Let us describe this representation in more detail. The six edges of the tetrahedron break up into three pairs of opposite edges. For each pair, consider the line passing through the centers of opposite edges. The three lines obtained in this way are mutually orthogonal and may be taken as the axes in $\mathbb{R}^3$. Furthermore, they are permuted by the action of A_4, yielding a map from A_4 to S_3 whose image has order 3. This defines an exact sequence

$$1 \longrightarrow V \longrightarrow A_4 \longrightarrow \mathbb{Z}/3 \longrightarrow 1$$

where $V = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We observe that ρ_{tet} is induced from the subgroup V. Indeed, V stabilizes each of the three axes and acts on them by distinct nontrivial characters. Frobenius reciprocity implies that $\rho_{tet} \approx Ind_V^{A_4}\theta$, where θ is any one of the three nontrivial characters of V. Note that the exact sequence above defines an action of $\mathbb{Z}/3$ on V and that the three nontrivial characters are permuted transitively by this action.

Now assume that ρ is of tetrahedral type. The composition of $Ad \circ \rho$ with the projection to $\mathbb{Z}/3$ yields a surjective map $Gal(\overline{F}/F) \to \mathbb{Z}/3$ whose kernel is of the form $Gal(\overline{F}/E)$, where E/F is a cyclic cubic extension. By the remarks in the previous paragraph, $Ad \circ \rho$ is isomorphic to $Ind_E^F\theta'$, where θ' is a character of order two of $Gal(\overline{F}/E)$. Furthermore, θ' is not fixed by either of the two nontrivial elements of $Gal(E/F)$. These observations allows us to conclude that $\pi(Ad \circ \rho)$ exists and is cuspidal. Indeed, θ' may be viewed as a Hecke character of finite order of the ideles I_F. According to Example 4 (special case of Theorem 14), we may automorphically induce θ' to obtain an element $AI_E^F(\theta')$ of $\mathcal{A}_F$ of degree 3. It is cuspidal since θ' is not fixed by any nontrivial element of $Gal(E/F)$. Finally, $AI_E^F(\theta') = \pi(Ad \circ \rho)$ by the compatibility of automorphic induction with the Langlands correspondence. This proves the first statement in the following lemma.

Lemma 19. *If ρ is of tetrahedral type, then the cuspidal representations $\pi(Ad(\rho))$ and $\pi(\rho_E)$ exist.*

PROOF. The representation ρ_E is irreducible. Indeed, if it were not, then it would decompose as a direct sum of two invariant lines. These lines must be permuted by $Gal(\overline{F}/F)$ under the action of ρ since $Gal(\overline{F}/E)$ is normal in $Gal(\overline{F}/F)$. Since $[E : F] = 3$, this action would have to be trivial and ρ itself would be reducible. Therefore ρ_E is irreducible. Furthermore, $Ad(\rho_E)$ is dihedral of order 4 by construction. The existence of $\pi(\rho_E)$ follows from Theorem 17. $\square$

Next, we prove the following proposition.

Proposition 20. *Let ρ be of tetrahedral type. Suppose that $\pi \in \mathcal{A}_F(2)$ has the following three properties:*

(i) $BC_{E/F}(\pi) = \pi(\rho_E)$
(ii) $\omega_\pi = \det(\rho)$
(iii) $Ad(\pi) = \pi(Ad(\rho))$.

Then $\pi = \pi(\rho)$.

PROOF. Let v be a finite place of F outside of $S(\pi) \cup S(\rho)$. Suppose that

$$g(\pi_v) \sim \begin{pmatrix} a & \\ & b \end{pmatrix}, \qquad \rho(Fr_v) \sim \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}.$$

We must show that $g(\pi_v) \sim \rho(Fr_v)$, i.e., that $\{a, b\} = \{\alpha, \beta\}$ as unordered sets. Let w be a place of E dividing v and let $d(w) = [E_w : F_v]$. Our hypotheses give us the following information:

(a) $BC_{E/F}(\pi) = \pi(\rho_E)$ implies $g(\pi_v)^{d(w)} \sim \rho(Fr_v)^{d(w)}$
(b) $\omega_\pi = \det(\rho)$ implies $ab = \alpha\beta$ (cf. Remark 1 in Sec. 11)
(c) $Ad(\pi) = \pi(Ad(\rho))$ implies $\{a/b, 1, b/a\} = \{\alpha/\beta, 1, \beta/\alpha\}$.

If $d(w) = 1$, then (a) already gives what we want. Otherwise, $d(w) = 3$. In this case, (a) and (b) imply that we may choose the labelling so that $a = \zeta\alpha$ and $b = \zeta^2\beta$ for some cube root of unity ζ. Thus we have

$$g(\pi_v) \sim \begin{pmatrix} \zeta\alpha & \\ & \zeta^2\beta \end{pmatrix}, \qquad \rho(Fr_v) \sim \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}.$$

If $\zeta = 1$, we are done. If not, (c) implies that $\zeta^{-1}\alpha/\beta = \beta/\alpha$ and hence that $\alpha/\beta = \pm\zeta^2$. If $\alpha/\beta = \zeta^2$, both matrices have eigenvalues $\{\beta, \zeta^2\beta\}$ and we are done. It remains to rule out the possibility $\alpha/\beta = -\zeta^2$. However, if $\alpha/\beta = -\zeta^2$, then

$$Ad(\rho(Fr_v)) \sim \begin{pmatrix} -\zeta^2 & & \\ & 1 & \\ & & -\zeta \end{pmatrix},$$

and this is an element of order 6. This is not possible since A_4 does not have an element of order 6! □

To conclude the proof of Theorem 18, we must show that a cuspidal representation π satisfying the conditions of Proposition 20 exists. We have seen that $\pi(\rho_E)$ exists. Furthermore, we clearly have $\eta(\rho_E) \simeq \rho_E$ for all $\eta \in Gal(\overline{F}/F)$. The same relation $\eta(\pi(\rho_E)) \simeq \pi(\rho_E)$ therefore also holds by the compatibility mentioned in Section 16. The descent part of the base change theorem (Theorem 15) implies that there exists $\pi \in \mathcal{A}_F(2)$ such that $BC_{E/F}(\pi) = \pi(\rho_E)$. According to the description of the fibers of the base change map, π is unique up to twisting by a character of the cyclic group of order three $I_F/F^*N_{E/F}(I_E)$. Let us show that there exists a unique choice of π for which $\omega_\pi = \det(\rho)$. The relation $BC_{E/F}(\pi) = \pi(\rho_E)$ implies that $\omega_\pi \circ N_{E/F} = \det(\rho) \circ N_{E/F}$; so in any case $\omega_\pi\omega = \det(\rho)$ for some character ω of $I_F/F^*N_{E/F}(I_E)$. Since

$$\omega_{\pi\otimes\omega^2} = \omega_\pi\omega^4 = \omega_\pi\omega,$$

we may (and shall) choose π so that $\det(\pi) = \det(\rho)$.

Now set

$$\Pi_1 = Ad(\pi), \qquad \Pi_2 = \pi(Ad(\rho))$$

It remains to show that condition (iii) of Proposition 20 is satisfied, i.e., that $\Pi_1 = \Pi_2$. It will suffice, of course, to prove that $g(\Pi_{1v}) \sim g(\Pi_{2v})$ for almost all finite places v of F such π_v and ρ_v are unramified. This is obvious if v splits in E, since $g(\pi_v) \sim \rho(Fr_v)$ in that case, but there does not seem to be any elementary way to conclude that $g(\Pi_{1v}) \sim g(\Pi_{2v})$ if v remains prime. Therefore, Langlands uses the result of Jacquet-Shalika (stated as Theorem 12 above) at this stage in the argument. To apply it, we must check that

$$g_v(\Pi_1) \otimes g_v(\Pi_2^*) = g_v(\Pi_2) \otimes g_v(\Pi_2^*)$$

for almost all places v. This is clear if v splits; so assume that v remains prime in E. Then the image of a Frobenius element $Fr_v \in Gal(\overline{F}/F)$ in A_4 has order 3, and hence

$$g(\Pi_{2v}) \sim Ad(\rho)(Fr_v) \sim \begin{pmatrix} 1 & & \\ & \zeta & \\ & & \zeta^2 \end{pmatrix},$$

where $\zeta \neq 1$ is a cube root of unity. In other words, $\rho(Fr_v) \sim \begin{pmatrix} \alpha & \\ & \zeta\alpha \end{pmatrix}$ for some α. We also have $g(\Pi_{2v}^*) \sim g(\Pi_{2v})$ since $g(\Pi_{2v})$ is conjugate to its inverse. Since $g(\pi_v)^3 \sim \rho(Fr_v)^3$ and $\det(g(\pi_v)) \sim \det(\rho(Fr_v))$, we can conclude that

$$g(\pi_v) \sim \begin{pmatrix} \alpha & \\ & \zeta\alpha \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \zeta^2\alpha & \\ & \zeta^2\alpha \end{pmatrix}.$$

Therefore

$$g(\Pi_{1v}) \sim \begin{pmatrix} 1 & & \\ & \zeta & \\ & & \zeta^2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Although the second possibility would spell disaster if it really occurred, we do not have to rule it out in advance because

$$g_v(\Pi_1) \otimes g_v(\Pi_2^*) = g_v(\Pi_2) \otimes g_v(\Pi_2^*)$$

in both cases, as is easily checked. With this stroke of luck, the proof of Theorem 18 is complete! □

19. Octahedral Representations

We shall now prove that $\pi(\rho)$ exists also for octahedral representations, following the argument of J. Tunnell [Tu]. Certain octahedral cases had previously been established by Langlands [L2]. The improvement due to Tunnell was made possible by the following theorem of Jacquet, Piatetski-Shapiro, and Shalika [JPS1].

Theorem 21. *Let K/F be a nonnormal cubic extension. Then for all π in $\mathcal{A}_F(2)$, the base change lifting $B_{K/F}(\pi)$ exists and is cuspidal.*

Assuming this result, we shall prove

Theorem 22 (Langlands-Tunnell). *Let ρ be a Galois representation of octahedral type. Then $\pi(\rho)$ exists.*

Let N/F be the S_4-extension defined by the kernel of $Ad(\rho)$. The group S_4 has three 2-Sylow subgroups of order 8. The conjugation action of S_4 on this set of 3 subgroups defines a epimorphism $\varphi : S_4 \to S_3$ and exact sequence

$$1 \longrightarrow V \longrightarrow S_4 \xrightarrow{\varphi} S_3 \longrightarrow 1.$$

where $V = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Let M be the fixed field of V. We define two subfields K and E of M as follows. Let E be the quadratic extension defined by the sign character of S_4 (obtained by pull-back from the sign character of S_3). Fix a 2-Sylow subgroup H containing V, and let K/F be the (non-Galois) cubic extension fixed by H. We have the following diagram of fields:

$$\begin{array}{ccc} & N & \\ & | & \\ & M & \\ / & & \backslash \\ K & & E \\ \backslash & & / \\ & F & \end{array}$$

Note that ρ_E is of tetrahedral type and ρ_K is dihedral (since $Gal(N/K)$ is isomorphic to the dihedral group D_8). Therefore $\pi(\rho_E)$ and $\pi(\rho_K)$ both exist by Theorems 22 and 17, respectively. In the next lemma, we make use of quadratic base change $BC_{E/F}$ and the cubic base change $BC_{K/F}$ whose existence is guaranteed by Theorem 21.

Lemma 23. *Suppose that π is a cuspidal representation of $GL_2(\mathbf{A}_F)$ such that $\pi_E = \pi(\rho_E)$ and $\pi_K = \pi(\rho_K)$. Then $\pi = \pi(\rho)$.*

PROOF. Let v be a place of F at which both π and ρ are unramified, and suppose that

$$g(\pi_v) \sim \begin{pmatrix} a & \\ & b \end{pmatrix}, \qquad \rho(Fr_v) \sim \begin{pmatrix} a' & \\ & b' \end{pmatrix}.$$

Of course, if v splits in E or if there exists a prime of K of relative degree one dividing v, then we have $g(\pi_v) \sim \rho(Fr_v)$. Otherwise, we may conclude that $g(\pi_v)^2 \sim \rho(Fr_v)^2$ and $g(\pi_v)^3 \sim \rho(Fr_v)^3$. If $g(\pi_v)$ and $\rho(Fr_v)$ have an eigenvalue in common, then they are conjugate. Indeed, if $a = a'$, then $b^2 = b'^2$ and $b^3 = b'^3$ and hence $b = b'$. Suppose that $g(\pi_v)$ and $\rho(Fr_v)$ are not conjugate. Then they have no eigenvalue in common, and so we assume that $a' = -a$. The relation $g(\pi_v)^3 \sim \rho(Fr_v)^3$ forces $b' = \eta a$, where $\eta^3 = 1$ but $\eta \neq 1$. This gives

$$\rho(Fr_v) \sim \begin{pmatrix} -a & \\ & \eta a \end{pmatrix} \quad \text{and} \quad Ad(\rho)(Fr_v) \sim \begin{pmatrix} -\eta^2 & & \\ & 1 & \\ & & -\eta \end{pmatrix},$$

which implies that $Ad(\rho)(Fr_v)$ has order 6. This is not possible since S_4 has no elements of order 6. We conclude that $g(\pi_v) \sim \rho(Fr_v)$, as claimed. $\square$

To prove Theorem 22, we shall construct a π satisfying the conditions of the previous lemma. We have $\rho_E \simeq \tau(\rho_E)$, where τ is conjugation of E/F since ρ_E extends to ρ, and therefore $\tau(\pi(\rho_E)) = \pi(\rho_E)$. By the base change theorem (Theorem

15), $\pi(\rho_E)$ descends to a cuspidal representation of $GL_2(\mathbf{A}_F)$ in two distinct ways. Let π_1 and π_2 be the two cuspidal representations such that $BC_{E/F}(\pi_j) = \pi(\rho_E)$. Then $\pi_1 = \pi_2 \otimes \omega_{E/F}$. It will suffice to check $BC_{K/F}(\pi_j) = \pi(\rho_K)$ for one of $j = 1$ and $j = 2$, since this π_j will satisfy the conditions of the lemma.

As observed in Lemma 19 and its proof, ρ_M is irreducible and $\pi(\rho_M)$ exists. The cuspidal representation $\pi(\rho_M)$ is in the image of the base change lifting from K since $\pi(\rho_K)$ clearly lifts to $\pi(\rho_M)$. Theorem 15 implies that $\pi(\rho_M) = BC_{M/K}(\pi')$ for precisely two cuspidal representations π' of $GL_2(\mathbf{A}_K)$ and these two representation differ by a twist by $\omega_{M/K}$. The two cuspidal representations are therefore $\pi(\rho_K)$ and $\pi(\rho_K) \otimes \omega_{M/K}$.

We claim that $BC_{K/F}(\pi_1)$ and $BC_{K/F}(\pi_2)$ also lift to $\pi(\rho_M)$. Indeed, by the transitivity of base change and the compatibility of base change with the Langlands correspondence, we have

$$BC_{M/F}(\pi_j) = BC_{M/K}(BC_{K/F}(\pi_j)) = BC_{M/K}(\pi(\rho_K)) = \pi(\rho_M).$$

Since $\pi_1 = \pi_2 \otimes \omega_{E/F}$ and $\omega_{M/K} = \omega_{E/F} \circ N_{M/E}$, we have

$$BC_{K/F}(\pi_1) = BC_{K/F}(\pi_2 \otimes \omega_{E/F}) = BC_{K/F}(\pi_2) \otimes \omega_{M/K}$$

because of the compatibility of base change with twisting. We must therefore have $BC_{K/F}(\pi_j) = \pi(\rho_K)$ for some j. $\square$

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