Equivalent characterizations of Loewner energy

Recent work by Yilin Wang

Ben Johnsrude

18 Feb 2020

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- $\mathscr{C}$ is the collection of "simple chords in $\mathbb{H}$ from 0 to ∞"

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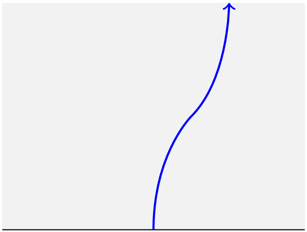

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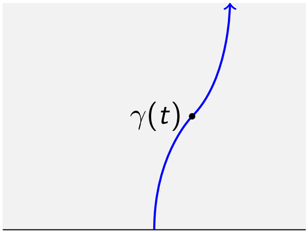

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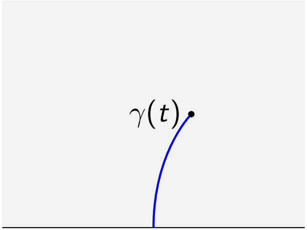

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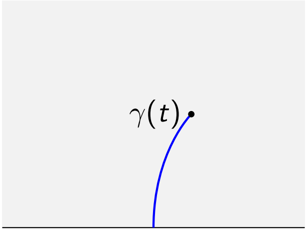

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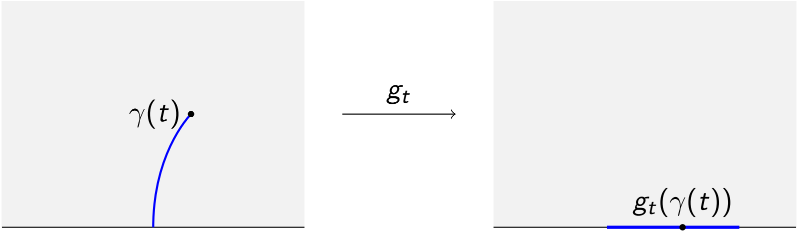

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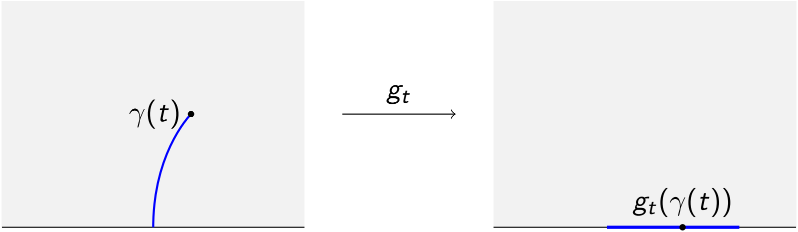

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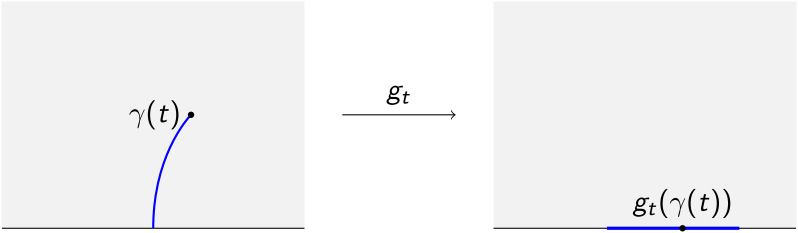

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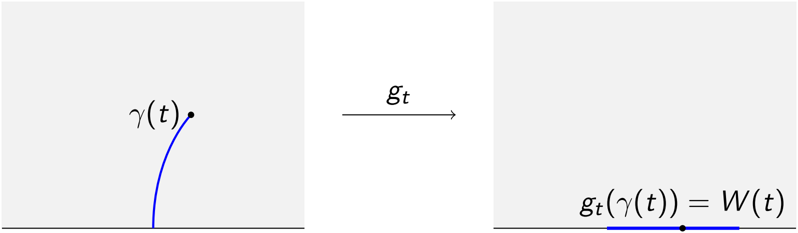

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- $(g_t)_{t\geq 0}$ are the *mapping-out functions*, $t \mapsto W(t)$ is the *driving function*

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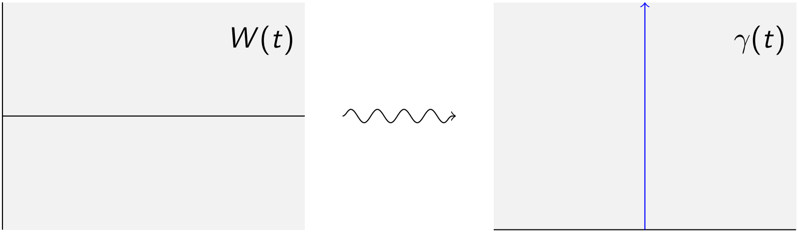

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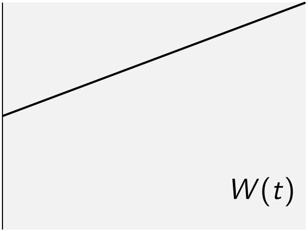

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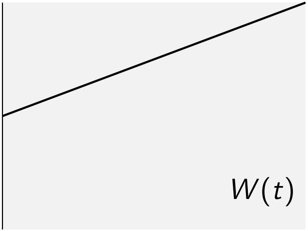

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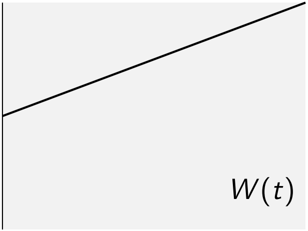

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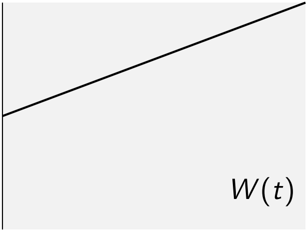

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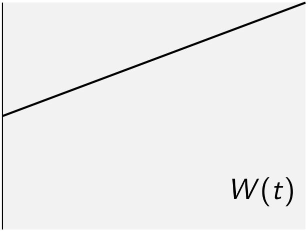

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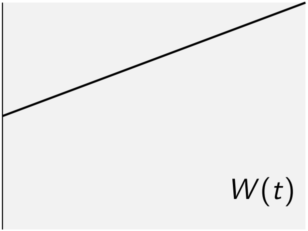

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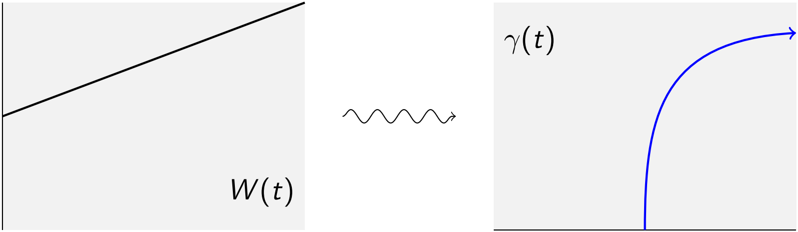

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- Regularity correspondence: for $0 < \alpha < 1, \alpha \neq 1/2$, have

$$\gamma \in C^{1+\alpha}(0, \infty) \iff W \in C^{1+\alpha-1/2}(0, \infty)$$

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$I(\gamma)$ is the *Loewner energy* of γ.

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"γ driven by finite-energy drivers form a skeleton of SLE_κ for small κ"

Guiding question

How may one characterize chords driven by finite-energy drivers?

Conformal invariance

For $\lambda > 0$,

$$I(\lambda\gamma) = I(t \mapsto \lambda W(\lambda^{-2} t))$$

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$\implies$ I is invariant under conformal automorphisms of $(\mathbb{H}, 0, \infty)$

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$$\phi : (\mathbb{H}, 0, \infty) \to (D, a, b) \text{ conformal}$$

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Well-defined!

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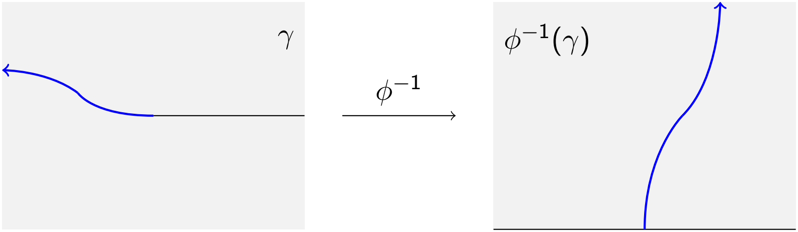

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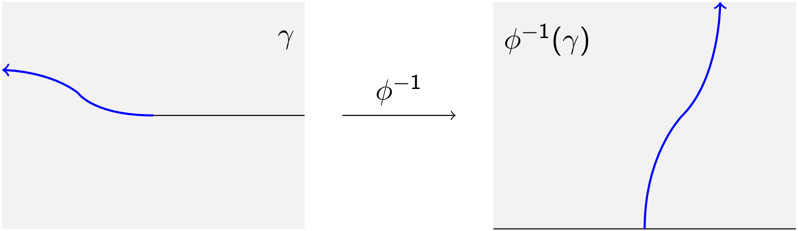
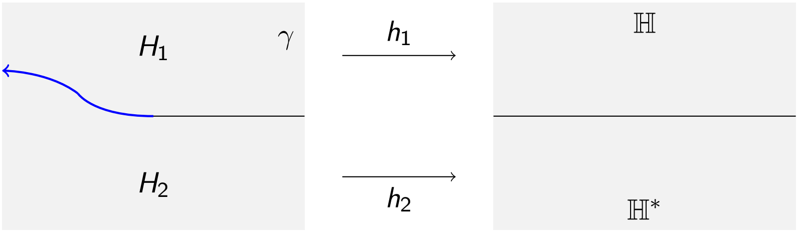


$$I_{\Sigma,0,\infty}(\gamma) = I_{\mathbb{H},0,\infty}(\sqrt{\gamma})$$

J-energy



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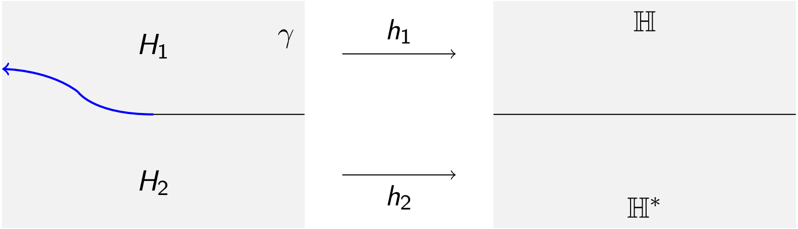

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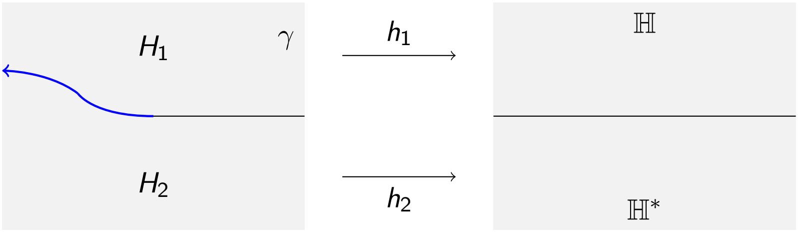

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$\longrightarrow$ Measure how much h_1, h_2 differ from an affine map

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$$\begin{aligned} h_j \text{ affine } &\iff \sigma_j \text{ constant} \\ &\iff \int_{H_j} |\nabla \sigma_j|^2 |dz|^2 = 0 \end{aligned}$$

J-energy

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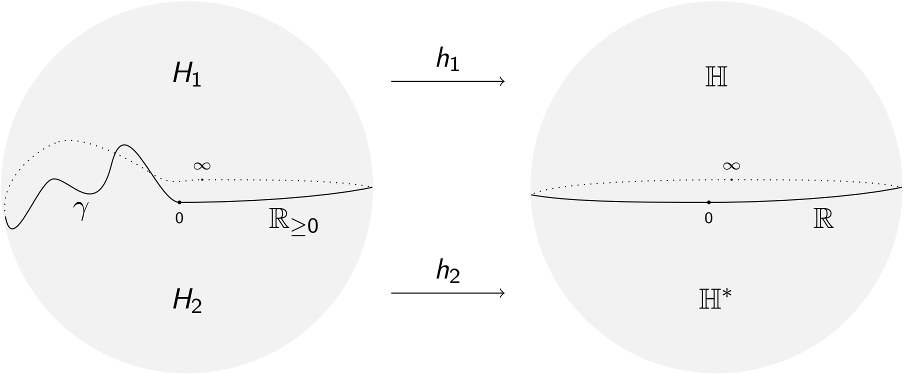
Proof: To be discussed in the sequel.


H_1
h_1
$\mathbb{H}$
∞
∞
γ
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0
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H_2
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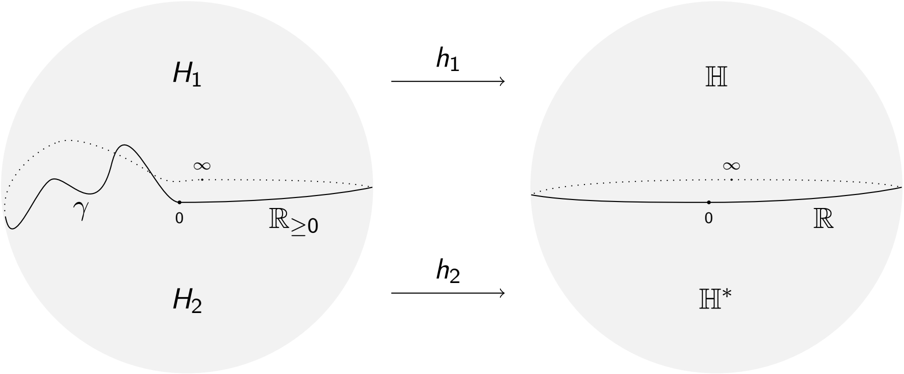

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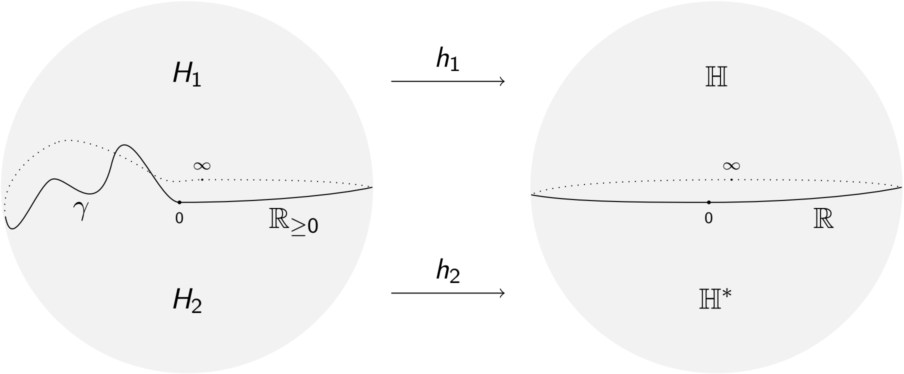
Loop version of J-energy



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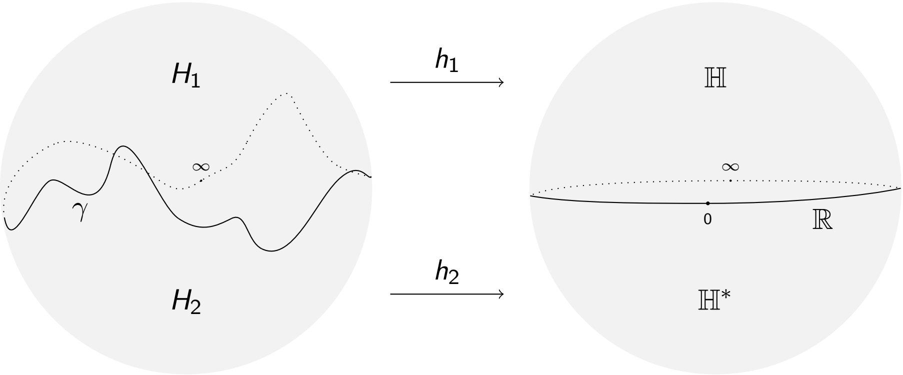

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Loop Loewner energy

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For $\gamma : [0,1] \to \widehat{\mathbb{C}}$ simple loop, set

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$\implies$ I^L is well-defined on the space of simple loops (i.e. loops not equipped with a parametrization)

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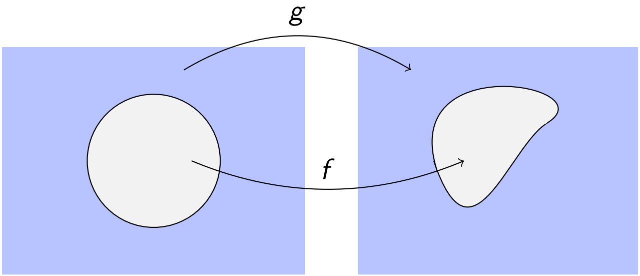
Idea: identify finite energy loops as a subspace of universal Teichmüller space!

Teichmüller space



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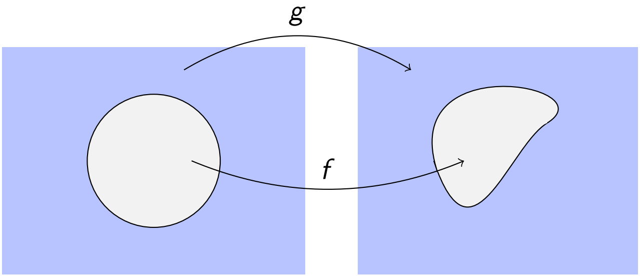

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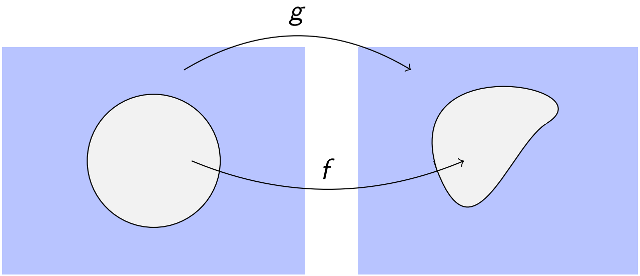

$$T(1) = \{\gamma \text{ quasicircle}\}/\text{Möb}$$
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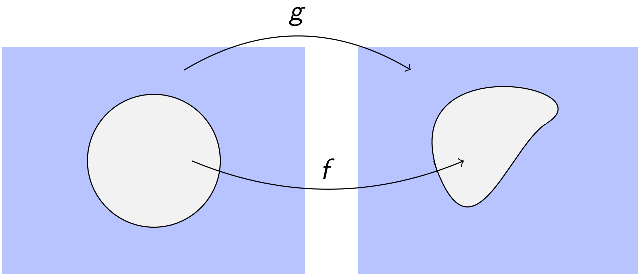

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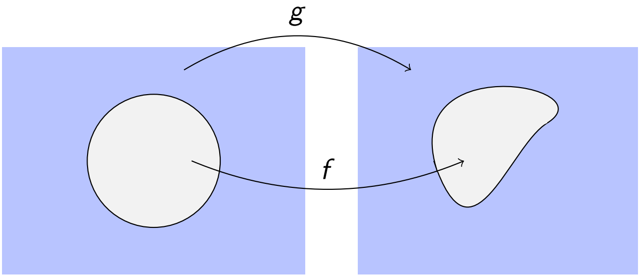
$$\simeq \left\{ f, g \text{ complementary conf. maps extending q.c.'ally to } \widehat{\mathbb{C}} \right\} / \sim$$

$T(1)$ admits natural structure of a Hilbert manifold (Takhtajan-Teo '06)

Teichmüller space



Theorem (Takhtajan-Teo '06) *Let γ be a simple bounded loop. TFAE:*

- $\gamma \in T_0(1)$ (connected component of $\{|z| = 1\}$)
- $\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 |dz|^2 < \infty$
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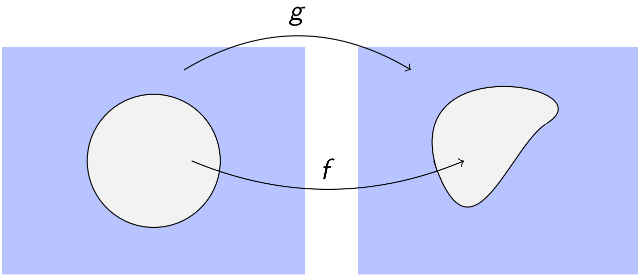

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  Such quasicircles γ are known as *Weil-Petersson class*.

Characterization of finite-energy loops

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Moreover,

$$I^L(\gamma) = \frac{1}{\pi}\mathbf{S}_1(\gamma)$$

where $\mathbf{S}_1$ is the universal Liouville action $T_0(1) \to \mathbb{R}$, a Kähler potential for the Weil-Petersson metric on $T_0(1)$.

End of Part 1

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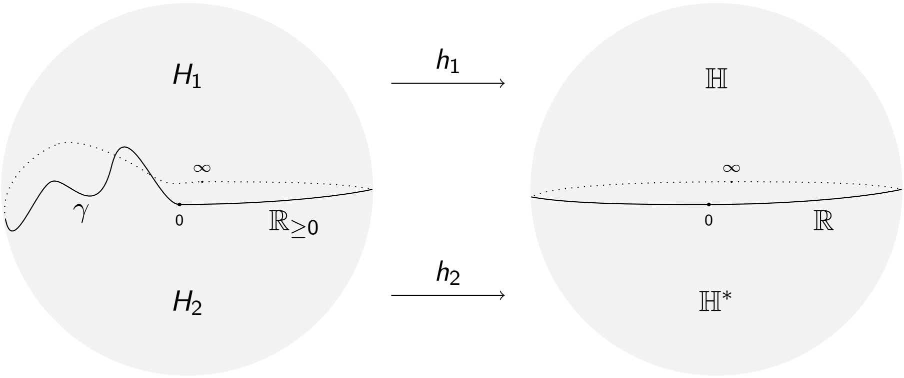

We now demonstrate some of the core steps in the proof of $I = J$ in the chordal case.

Step 1a: Additivity

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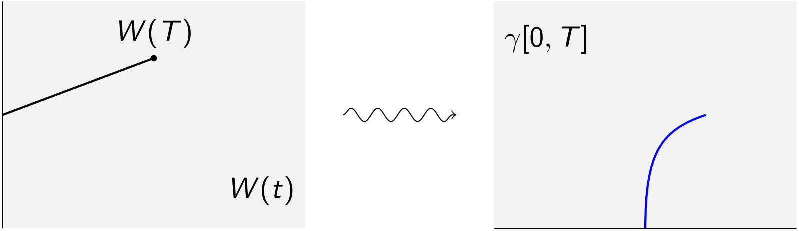
where $\overline{\gamma[0,T]}$ is the “completed” chord given by $\gamma[0,T]$ followed by the geodesic from $\gamma(T)$ to ∞.

Step 1a: I-Additivity



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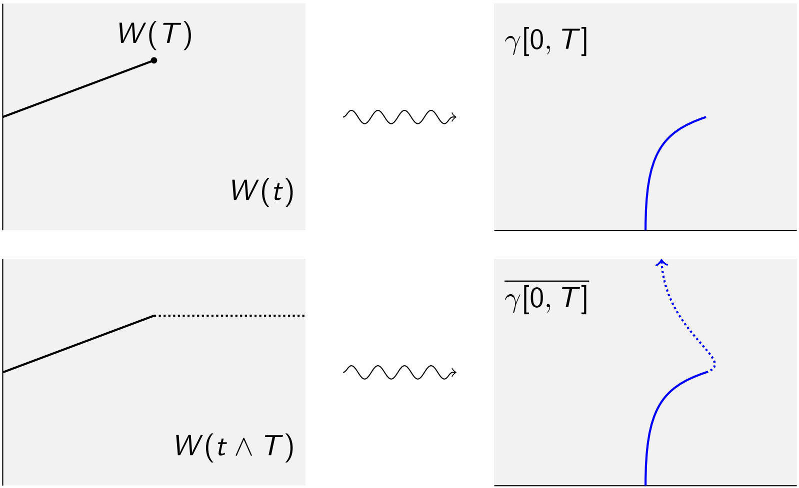

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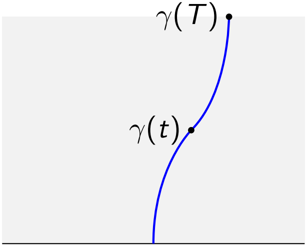

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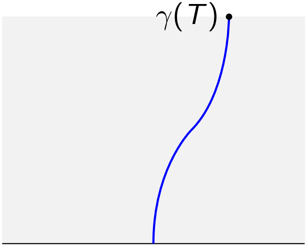

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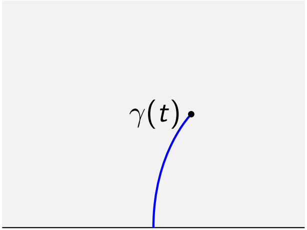

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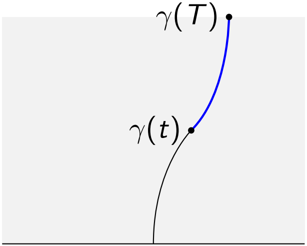

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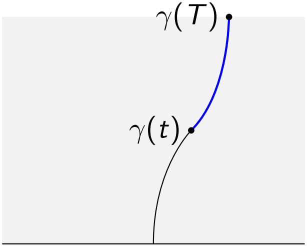

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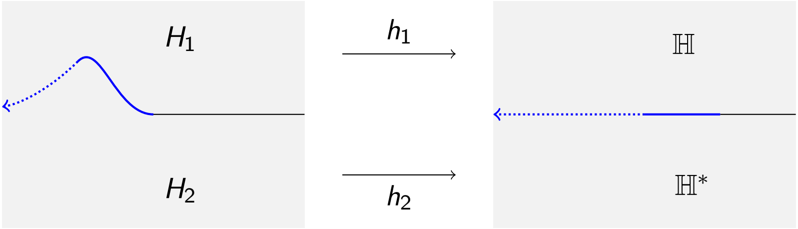

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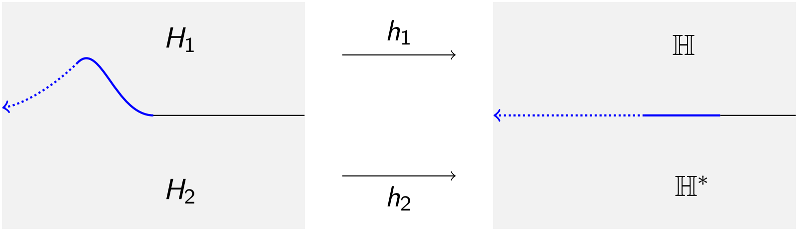

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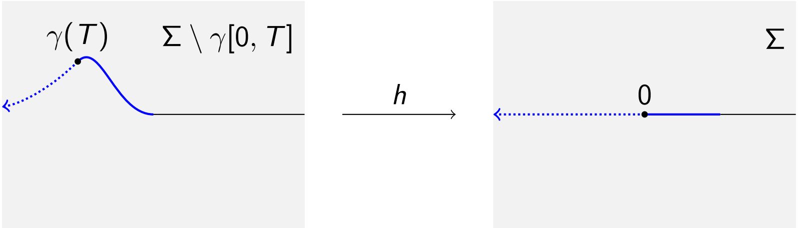

Note: one choice of $h = h_1 \cup h_2$ would be the mapping-out function

$$h : \Sigma \setminus \gamma[0, T] \to \Sigma, \quad h(\gamma(T)) = 0, \quad h(z) = z + O(1)$$

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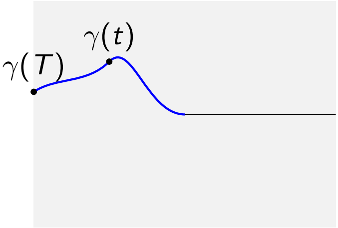
For $0 \leq t \leq T$,



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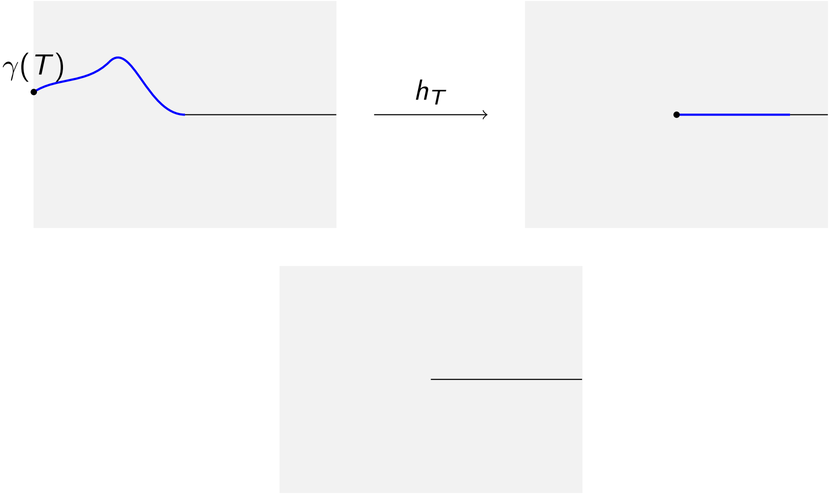

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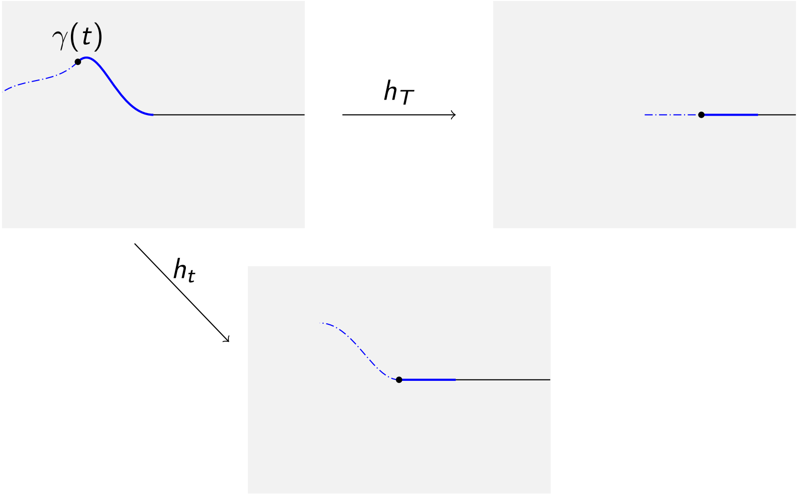

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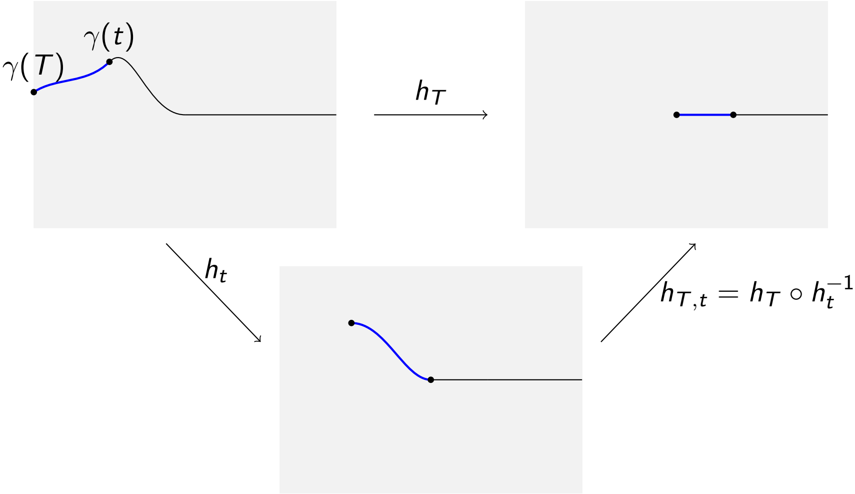

Step 1b: J-additivity

Proposition (J-additivity): *Let $\gamma[0, T]$ be a simple chord in $(\Sigma, 0, \infty)$ with finite Loewner energy. Then, for all $0 \leq s < t \leq T$,*

$$J(h_t) = J(h_s) + J(h_{t,s})$$

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Lemma: If a finite capacity curve $\gamma = \gamma[0, T]$ in $(\Sigma, 0, \infty)$ satisfies:

- $\gamma \cup \mathbb{R}_{\geq 0}$ is $C^{1,\alpha}$ for some $\alpha > 0$,
- $\sigma_{h_T} \in \mathscr{D}(\Sigma \setminus \gamma)$

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then, for all $g \in \mathscr{D}(\Sigma)$,

$$\int_{\Sigma \setminus \gamma} \nabla g \cdot \nabla \sigma_{h_T}(z) |dz|^2 = 0$$

Step 1b: J-additivity

Sketch of proof of Lemma: Denote $\Gamma = \overline{\gamma} \cup \mathbb{R}_{\geq 0}$ and H_1, H_2 the components of $\mathbb{C} \setminus \Gamma$.

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(for less smooth Γ, approximate by $h_{1,T}^{-1}(\mathbb{R} + i\varepsilon)$)

Step 1b: J-additivity

Integrating by parts:

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General formula relating σ_{h_T} to curvature:

$$\partial_n \sigma_{h_T}(z) = k(h_T(z)) e^{\sigma_{h_T}(z)} - k_0(z)$$

Along Γ, $h_T(z) \in \partial\mathbb{H}$ which has curvature 0, so $k(h_T(z)) = 0$.

Step 1b: J-additivity

General formula relating σ_{h_T} to curvature:

$$\partial_n \sigma_{h_T}(z) = k(h_T(z)) e^{\sigma_{h_T}(z)} - k_0(z)$$

Along Γ, $h_T(z) \in \partial\mathbb{H}$ which has curvature 0, so $k(h_T(z)) = 0$.

Along $\mathbb{R}_{\geq 0}$, $k_0(z) = 0$.

Step 1b: J-additivity

Thus

$$\int_{\Sigma\setminus\gamma} \nabla g \cdot \nabla \sigma_{h_T}(z)|dz|^2 = \int_{\Gamma_1 \cup \Gamma_2} g(z)\partial_n \sigma_{h_T}(z) dl(z)$$

Step 1b: J-additivity

Thus

$$\int_{\Sigma\setminus\gamma} \nabla g \cdot \nabla \sigma_{h_T}(z)|dz|^2 = \int_{\Gamma_1\cup\Gamma_2} g(z)\partial_n \sigma_{h_T}(z) dl(z)$$
$$= -\int_{\overline{\gamma}} (g|_{H_2}(z) - g|_{H_1}(z)) k_0(z) dl(z)$$

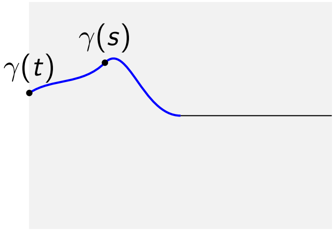
Thus

$$\begin{aligned}\int_{\Sigma\setminus\gamma} \nabla g \cdot \nabla \sigma_{h_T}(z)|dz|^2 &= \int_{\Gamma_1\cup\Gamma_2} g(z)\partial_n \sigma_{h_T}(z)dl(z) \\ &= -\int_{\overline{\gamma}} (g|_{H_2}(z) - g|_{H_1}(z))k_0(z)dl(z) \\ &= 0\end{aligned}$$

since g extends continuously across $\overline{\gamma}$.

Step 1b: J-additivity

Proof of proposition (J-additivity):



Step 1b: J-additivity

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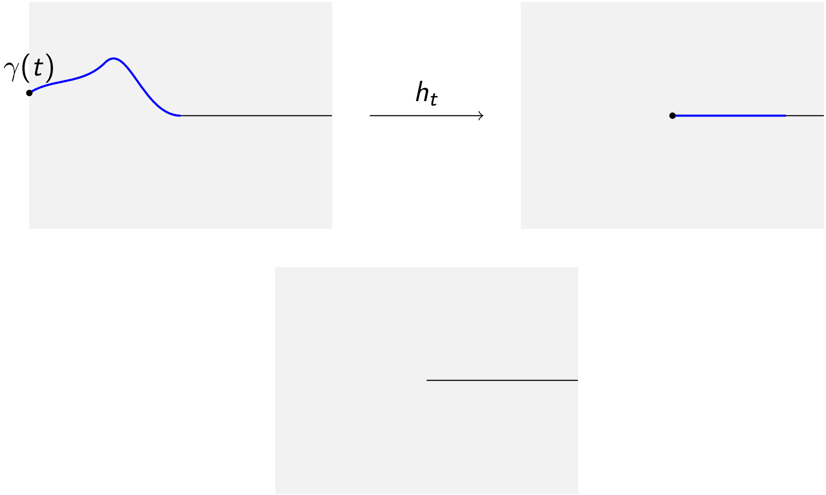

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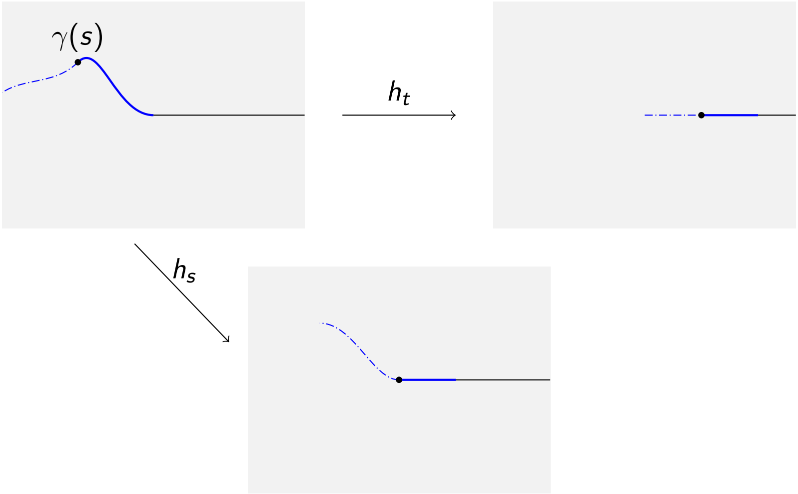

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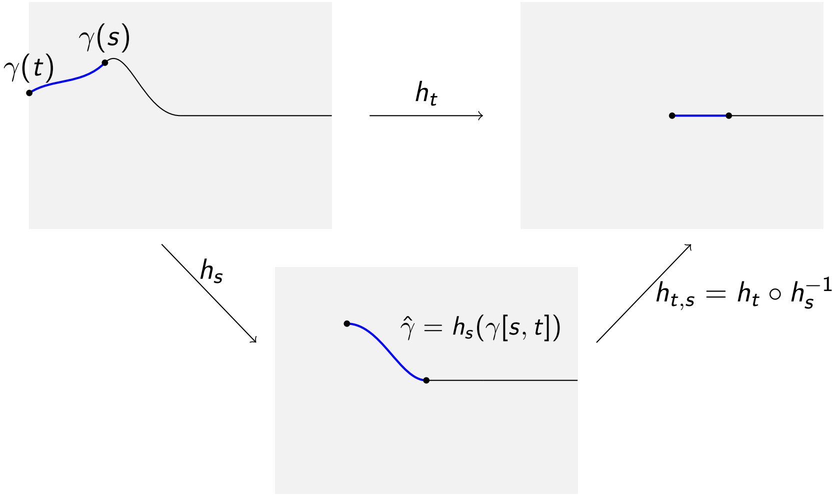

Step 1b: J-additivity

$$\sigma_{h_t}(z) = \log|h_t'| = \log|(h_{t,s} \circ h_s)'| = \sigma_{h_{t,s}}(h_s(z)) + \sigma_{h_s}(z)$$

Step 1b: J-additivity

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$$\pi J(h_t) = \pi J(h_s) + \int_{\Sigma \setminus \gamma} |\nabla(\sigma_{h_{t,s}} \circ h_s)(z)|^2 |dz|^2 + 2\int_{\Sigma \setminus \gamma} \nabla(\sigma_{h_{t,s}} \circ h_s) \cdot \nabla \sigma_{h_s}(z) |dz|^2$$

Step 1b: J-additivity

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Step 1b: J-additivity

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Step 1b: J-additivity

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&2\int_{\Sigma\setminus\gamma} \nabla(\sigma_{h_{t,s}} \circ h_s) \cdot \nabla\sigma_{h_s}(z)|dz|^2 \\
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\end{aligned}
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By assumption, $J(h_{t,s}) < \infty$, so $\sigma_{h_{t,s}} \in \mathscr{D}(\Sigma \setminus \hat{\gamma})$

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One may also compute

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so $\sigma_{h_s^{-1}} \in \mathscr{D}(\Sigma)$.

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By assumption, $J(h_{t,s}) < \infty$, so $\sigma_{h_{t,s}} \in \mathscr{D}(\Sigma \setminus \hat{\gamma})$
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so $\sigma_{h_s^{-1}} \in \mathscr{D}(\Sigma)$.
By the lemma,

$$-2 \int_{\Sigma \setminus \hat{\gamma}} \nabla \sigma_{h_{t,s}} \cdot \nabla \sigma_{h_s^{-1}}(z) |dz|^2 = 0$$

Step 1b: J-additivity

Thus

$$\pi J(h_t) = \pi J(h_s) + \int_{\Sigma\setminus\gamma} |\nabla(\sigma_{h_{t,s}} \circ h_s)(z)|^2 |dz|^2 + 2\int_{\Sigma\setminus\gamma} \nabla(\sigma_{h_{t,s}} \circ h_s) \cdot \nabla\sigma_{h_s}(z)|dz|^2$$

Step 1b: J-additivity

Thus

$$\begin{aligned}\pi J(h_t) &= \pi J(h_s) + \int_{\Sigma\setminus\gamma} |\nabla(\sigma_{h_{t,s}} \circ h_s)(z)|^2 |dz|^2 \\ &\quad + 2\int_{\Sigma\setminus\gamma} \nabla(\sigma_{h_{t,s}} \circ h_s) \cdot \nabla\sigma_{h_s}(z)|dz|^2 \\ &= \pi J(h_s) + \pi J(h_{t,s}) + 0\end{aligned}$$

as desired.

Step 2: $I = J$ for linear drivers

Proof that $I(\gamma[0, T]) = J(\gamma[0, T])$ when W is linear:

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Hence it suffices to show

$$I(\gamma[0,T]) \sim J(h_T) \quad \text{as } T \to 0$$

Step 2: $I = J$ for linear drivers

- $g_t(z) = \sqrt{h_t(z^2)} + \lambda t$ mapping-out function in $\mathbb{H}$ for $\sqrt{\gamma[0,t]}$

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- Solving produces

$$\sigma_t(z) = -\frac{\lambda t}{2}\left(\lambda t + \mathrm{Re}\left(\sqrt{h_t(z)}\right) - \mathrm{Re}\left(\sqrt{z}\right)\right)$$

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- As a consequence, for $t > 0$,

$$|\nabla \sigma_t(z)| = O\left(|z|^{-1/2}\right) \qquad \text{for } z \text{ small}$$

$$|\nabla \sigma_t(z)| = O\left(|z|^{-3/2}\right) \qquad \text{for } z \text{ large}$$

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Can apply Stokes:

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$$J(h_T) = \frac{1}{\pi} \int_{\Gamma_1 \cup \Gamma_2} \sigma_{h_T} \partial_n \sigma_{h_T}(z) dl$$

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Can apply Stokes:

$$\begin{aligned} J(h_T) &= \frac{1}{\pi}\int_{\Gamma_1\cup\Gamma_2} \sigma_{h_T}\partial_n\sigma_{h_T}(z)dl \\ &= \frac{1}{\pi}\int_{\gamma(t)\in\Gamma_1\cup\Gamma_2} \sigma_{h_T}\partial_n\sigma_{h_T}(\gamma(t))dl \end{aligned}$$

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&= \frac{1}{\pi}\int_{\gamma(t)\in\Gamma_1\cup\Gamma_2} \sigma_{h_T}\partial_n\sigma_{h_T}(\gamma(t))dl \\
&= \frac{\lambda^2}{4\pi}\int_0^T \left[\sqrt{h_T\left(\gamma(t)^-\right)} - \sqrt{h_T\left(\gamma(t)^+\right)}\right] \operatorname{Im}\left(\partial_t\sqrt{\gamma(t)}\right) dt
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Step 2: $I = J$ for linear drivers

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$$
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&= \frac{\lambda^2}{4\pi}\int_0^T \left[\sqrt{h_T\left(\gamma(t)^-\right)} - \sqrt{h_T\left(\gamma(t)^+\right)}\right] \operatorname{Im}\left(\partial_t \sqrt{\gamma(t)}\right) dt \\
&= \frac{\lambda^2}{4\pi}\int_0^T \left[\sqrt{h_{T-t}(0^-)} - \sqrt{h_{T-t}(0^+)}\right] \operatorname{Im}\left(\partial_t \sqrt{\gamma(t)}\right) dt
\end{aligned}
$$

using $h_T = h_{T-t} \circ h_t$, since W is linear.

Step 2: $I = J$ for linear drivers

$\sqrt{\gamma}$ satisfies the backward Loewner equation:

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Recall the asymptotics for t small:

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$\sqrt{\gamma}$ satisfies the backward Loewner equation:

$$\partial_t \sqrt{\gamma(t)} = -\frac{2}{\sqrt{\gamma(t)}} + \lambda$$

Recall the asymptotics for t small:

$$\sqrt{\gamma(t)} = 2i\sqrt{t} + O(t)$$

$$\sqrt{h_t(0^+)} = 2\sqrt{t} + O(t)$$
$$\sqrt{h_t(0^-)} = -2\sqrt{t} + O(t)$$

Step 2: $I = J$ for linear drivers

Thus, as $T \to 0$,

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Thus, as $T \to 0$,

$$J(h_T) = \frac{\lambda^2}{4\pi} \int_0^T \left[\sqrt{h_{T-t}(0^+)} - \sqrt{h_{T-t}(0^-)}\right] \operatorname{Im}\left(\partial_t \sqrt{\gamma(t)}\right) dt$$

Step 2: $I = J$ for linear drivers

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$$= \frac{\lambda^2}{\pi}(1 + O(\sqrt{T})) \int_0^T \frac{\sqrt{T-t}}{\sqrt{t}} dt$$

Step 2: $I = J$ for linear drivers

Thus, as $T \to 0$,

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Step 2: $I = J$ for linear drivers

Thus, as $T \to 0$,

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Step 2: $I = J$ for linear drivers

Since $T \mapsto J(h_T)$ is linear,

Step 2: $I = J$ for linear drivers

Since $T \mapsto J(h_T)$ is linear,

$$J(h_T) = \frac{\lambda^2}{2} T = I(\gamma[0, T])$$

the desired identity.

Step 3: $I = J$ for most drivers

Corollary: $I = J$ when γ is driven by a piecewise linear function.

Step 4: $I = J$ by approximation

For W of finite Dirichlet energy, pick step functions approximating W' in L^2.

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Step 4: $I = J$ by approximation

For W of finite Dirichlet energy, pick step functions approximating W' in L^2.
$\implies$ have piecewise linear drivers approximating W uniformly and in Dirichlet energy
$\implies$ ... $\implies$ $I = J$ for general finite-energy drivers
(tools: J is lower semicontinuous, I and J additive)