Small cap decoupling for the parabola with logarithmic constant

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April 28, 2023

Abstract

We note that the subpolynomial factor for the $\ell^q L^p$ small cap decoupling constants for the truncated parabola $\mathbb{P}^1 = \{(t,t^2) : |t| \leq 1\}$ may be controlled by a suitable power of $\log R$. This is achieved by considering a suitable amplitude-dependent wave envelope estimate, as was introduced in a recent paper of Guth and Maldague to demonstrate a small cap decoupling for the $(2+1)$ cone; we demonstrate that the version for $\mathbb{P}^1$ may be taken with a loss controlled by a power of $\log R$ as well.

1 Introduction

In this note, we record that the methods of [10] suffice to derive small cap decoupling estimates for functions with Fourier support in the R^{-1}-neighborhood of the truncated parabola $\mathbb{P}^1 = \{(x,x^2) : |x| \leq 1\}$ with constant of the form $(\log R)^C$.

Small cap decouplings were introduced in [4]; we recall the formulation here. For large parameters $R > 1$, set $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ to be the R^{-1}-neighborhood of the truncated parabola. Consider a Schwartz function $f : \mathbb{R}^2 \to \mathbb{C}$ such that $\operatorname{supp}(\hat{f}) \subseteq \mathcal{N}_{R^{-1}}(\mathbb{P}^1)$, where $\hat{}$ denotes the Fourier transform. Let $\beta \in [\frac{1}{2}, 1]$. Partition $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ into a collection $\Gamma_\beta(R^{-1})$ of sets γ, which are the intersections of $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ with sets of the form $[c, c+R^{-\beta}] \times \mathbb{R}$; one may note that such γ are approximately boxes of dimensions $R^{-\beta} \times R^{-1}$, in the sense that for each γ we may find a box B such that $B \subseteq \gamma \subseteq CB$ for a universal constant C, where CB denotes dilation about the center of B. Set

$$f_\gamma(x) = \int_\gamma \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

to be the Fourier projection of f onto γ. Here and elsewhere all integrals will be with respect to Lebesgue measure. If $p, q \in [1,\infty)$, set $D_{p,q}(R;\beta)$ to be the infimal constant such that

$$\|f\|^p_{L^p(\mathbb{R}^2)} \leq D_{p,q}(R;\beta) \left(\sum_{\gamma \in \Gamma_\beta(R^{-1})} \|f_\gamma\|^q_{L^p(\mathbb{R}^2)} \right)^{p/q}.$$

The landmark paper [2] demonstrated the estimate $D_{p,2}(R;\frac{1}{2}) \lesssim_\varepsilon R^\varepsilon$ for all $\varepsilon > 0$ and $2 \leq p \leq 6$. The authors of [8] provided the improved estimate $D_{6,2}(R;\frac{1}{2}) \lesssim (\log R)^C$ for a suitable constant $C > 0$; the authors of [7] sharpened this upper bound to $C_\varepsilon (\log R)^{12+\varepsilon}$ for a bilinear variant over $\mathbb{Q}_p$, implying a matching discrete restriction estimate (over $\mathbb{R}$) with very good logarithmic constant. In another direction, the authors of [4] introduced the constants $D_{p,q}(R;\beta)$ for $\beta \in (\frac{1}{2}, 1]$, and showed that $D_{p,p}(R;\beta) \lesssim_\varepsilon R^{p\beta(\frac{1}{2}-\frac{1}{p})+\varepsilon}$ for all $\varepsilon > 0$ and $2 \leq p \leq 2 + \frac{2}{\beta}$ (Theorem 3.1). Each of these bounds is sharp up to the subpolynomial factors.

Our goal will be to show the following:

Theorem 1.1 (Small cap decoupling with logarithmic losses)**.** *There exists a constant $E_1 = E_1(p) > 0$ depending only on p such that the following holds. Let $p, q \geq 1$ satisfy $\frac{3}{p} + \frac{1}{q} \leq 1$, $R > 2$, and $\beta \in [\frac{1}{2}, 1]$. Then the small cap decoupling constant satisfies*

$$D_{p,q}(R;\beta) \lesssim (\log R)^{E_1}\Big(1 + R^{\beta(p-\frac{p}{q}-1)-1} + R^{p\beta(\frac{1}{2}-\frac{1}{q})}\Big). \tag{1.1}$$

Our methods permit one to take $E_1 = 30 + 3p$.

This formulation of the decoupling estimate, with instead a factor of $C_\varepsilon R^\varepsilon$ in place of the logarithmic factor, was originally proven in [5] (Corollary 5). For each triple (p, q, β), the dominating term on the right-hand side in 1.1 may be realized by a particular choice of f with large R, as demonstrated in [5] (Section 2), up to the subpolynomial factor. Thus the power-law terms are each separately sharp in the regime where they dominate.

In [1] (Remark 2), it was demonstrated using number theory methods that $D_{6,2}(R; \frac{1}{2}) \gtrsim (\log R)R$. It is not currently known if there is any other p, β with $2 \leq p \leq 2 + \frac{2}{\beta}$ such that the subpolynomial factor is unbounded in R.

Our estimate 1.1 is derived by first proving a version of an auxiliary wave envelope estimate, which is precisely stated in Theorem 1.2. We will write $|S|$ to denote the Lebesgue measure of sets S.

Theorem 1.2 (Wave envelope estimate)**.** *There exists a constant $E_2 > 0$ such that the following holds for all $R \gg 1$. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be Schwartz with Fourier support in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$. Then, for any $\alpha > 0$,*

$$\alpha^4|\{x : |f(x)| > \alpha\}| \lesssim (\log R)^{E_2} \sum_{\substack{R^{-1/2} \leq s \leq 1 \\ s \text{ dyadic}}} \sum_{\tau:\ell(\tau)=s} \sum_{U \in \mathcal{G}_\tau} |U|^{-1}\|S_U f\|^4_{L^2(\mathbb{R}^2)}.$$

Our methods permit one to take $E_2 = 31$.

Here we use the following notation: $U_{\tau,R}$ is a rectangle of dimensions $R \times sR$, with long edge in the direction of the normal vector to $\mathbb{P}^1$ at the center of τ, centered at 0; the set $\mathcal{G}_\tau$ is the subset of the tiling of $\mathbb{R}^2$ by translated copies of $U_{\tau,R}$ for which the following holds:

$$C(\log R)^4|U|^{-1}\int_U \sum_{\theta\subseteq\tau}|f_\theta|^2 \geq \frac{\alpha^2}{(\#\tau)^2} \tag{1.2}$$

for suitable choice of $C, D > 0$. Here $\#\tau$ denotes the number of τ of a particular length for which $f_\tau \not\equiv 0$. Lastly, we use $S_U f$ to denote the restricted square function $\left(\sum_{\theta\subseteq\tau}|f_\theta|^2\right)\big|_U$; one may observe that the quantities s and R may be read off of the dimensions of U, and τ is then uniquely determined from the direction of U's long edge, so this definition is well-formed.

Wave envelope estimates were introduced in [9] for the purpose of proving the reverse square function estimate for the cone in $\mathbb{R}^3$ (Theorem 1.3). In [10], these wave envelope estimates were refined to include only those envelopes corresponding to “high-amplitude” components of the various square functions. The latter paper demonstrated that the wave envelope estimate could also be used to derive the small cap results of [5]. Our argument closely follows that of [10], but with various technical refinements to facilitate a logarithmic constant in the wave envelope estimate (e.g. a gentler sequence of scales R_k).

We make use of the following notation. For $A, B > 0$, we say $A \lesssim B$ if $A \leq CB$ for a suitable constant C which may vary from line to line, which does not depend on any variable parameters in the problem unless explicitly indicated. We also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The expression $O(B)$ will be used to denote a quantity which is $\lesssim B$. We also note from the outset that we slightly redefine the notation $⨍$ to something better suited to our purposes than its usual meaning; see the pruning section below.

Throughout the paper, given a parallelogram B, we will write c_B for the center of B. For a scalar $\lambda > 0$, we will write λB for the box with the same center c_B but with sidelengths increased by the factor λ. We will also use an asterisk $*$ to denote a dual of parallelograms, that is, B^* is the parallelogram centered at 0 with the same edge directions but inverse lengths; note that this differs slightly from the notion of the polar of a convex set, which is sometimes chosen to play essentially the same role as our B^*.

The remainder of the paper is organized as follows. In section 2, we first give an overview of the argument, then provide the pruning and lemmas needed in the proof of Theorem 1.2. In section 3, we prove Theorem 1.2. In section 4, we show that Theorem 1.2 proves Theorem 1.1.

2 Infrastructure for proving Theorem 1.2

2.1 Overview of the argument

We first recall the general intuition behind the shape of the right-hand side of Theorem 1.2, without considering the amplitude dependence. Consider a Schwartz function $f : \mathbb{R}^2 \to \mathbb{C}$ with Fourier support contained in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$. The L^4 square function estimate for $\mathbb{P}^1$ implies that

$$\int |f|^4 \lesssim \int |\sum_\theta |f_\theta|^2|^2.$$

By Plancherel,

$$\int |\sum_\theta |f_\theta|^2(x)|^2 = \int |\widehat{\sum_\theta |f_\theta|^2}(\xi)|^2.$$

We study the latter integral by considering the contributions from different regimes of values of $|\xi|$. Since each f_θ has Fourier support contained in the cap θ of size $\sim R^{-1/2} \times R^{-1}$, the support of the latter integral is contained in the ball of radius $2R^{-1/2}$ centered at the origin, so we only need to consider frequency contributions below this magnitude.

On the other hand,

$$\int_{|\xi|<R^{-1}} \Big|\widehat{\sum_\theta |f_\theta|^2}(\xi)\Big|^2 \lesssim \int \Big|\sum_\theta |f_\theta|^2 * (R^{-2} w_{B_R})\Big|^2$$

for a suitable weight w_{B_R} which is ~ 1 on B_R and rapidly decays outside of B_R; if we write the latter integral as a sum of integrals over cubes Q_R,

$$\int \Big|\sum_\theta |f_\theta|^2 * (R^{-2} w_{B_R})\Big|^2 = \sum_{Q_R} \int_{Q_R} \Big|\sum_\theta |f_\theta|^2 * (R^{-2} w_{B_R})\Big|^2.$$

Since B_R is a square of sidelength R, the convolution is approximately constant on such Q_R. Thus

$$\sum_{Q_R} \int_{Q_R} \Big|\sum_\theta |f_\theta|^2 * (R^{-2} w_{B_R})\Big|^2 \lesssim \sum_{Q_R} |Q_R|^{-1} \left(\int W_{Q_R} \sum_\theta |f_\theta|^2\right)^2$$

for suitable weights W_{Q_R} which are approximate cutoffs to the set Q_R. Thus

$$\int_{|\xi|<R^{-1}} \Big|\widehat{\sum_\theta |f_\theta|^2}(\xi)\Big|^2 \lesssim \sum_{Q_R} |Q_R|^{-1} \left(\int W_{Q_R} \sum_\theta |f_\theta|^2\right)^2,$$

which is one of the summands on the right-hand side of 1.2.

More generally, if we consider integrals of the form

$$\int_{|\xi|\sim r}\Big|\widehat{\sum_\theta |f_\theta|^2}(\xi)\Big|^2, \quad R^{-1} < r \le R^{-1/2},$$

then we may instead make use of the approximate orthogonality of the families $\{\sum_{\theta\subseteq\tau}|f_\theta|^2\}_{\ell(\tau)=\frac{1}{rR}}$; notice that, by finite overlap,

$$\int_{|\xi|\sim r}\Big|\widehat{\sum_\theta |f_\theta|^2}(\xi)\Big|^2 \lesssim \sum_\tau \int_{|\xi|\sim r}\Big|\widehat{\sum_{\theta\subseteq\tau} |f_\theta|^2}(\xi)\Big|^2,$$

and that the functions

$$\sum_{\theta\subseteq\tau}|f_\theta|^2 * \chi_{\sim r}^\vee$$

are approximately constant on sets of the form $U\|U_{\tau,R}$, where $\chi_{\sim r}$ is a smooth cutoff to the annulus $|\xi|\sim r$. Thus, as above,

$$\int_{|\xi|\sim r}\Big|\widehat{\sum_\theta |f_\theta|^2}(\xi)\Big|^2 \lesssim \sum_{U\|U_{\tau,R}} |U|^{-1}\Big(\int W_U \sum_{\theta\subseteq\tau}|f_\theta|^2\Big)^2,$$

which is also of the right shape for our theorem.

We may observe from the preceding calculation that we would have proved Theorem 1.2 if, for each τ and each $U\|U_{\tau,R}$, we had the estimate

$$C(\log R)^4|U|^{-1}\int_U \sum_{\theta\subseteq\tau}|f_\theta|^2 \ge \frac{\alpha^2}{(\#\tau)^2},$$

or else $S_U f$ is negligible, say $O(R^{-1000})$. It is natural therefore to split f into pruned pieces for which the non-negligible $S_U f$ satisfy the “good” estimate above, at various scales. Our prunings, following [10], will therefore be written as follows:

$$\begin{aligned} f &= f_N + f^{\mathcal{B}}\\ f_N &= f_{N-1} + f_N^{\mathcal{B}}\\ f_{N-1} &= f_{N-2} + f_{N-1}^{\mathcal{B}}\\ &\cdots\\ f_2 &= f_1 + f_2^{\mathcal{B}}\end{aligned}$$

where f_m is given by trivializing the contributions $S_U f$, $U\|U_{\tau,R}$, $d(\tau)\lesssim(\log R)^{-m}$, for which 1.2 fails.

To illustrate, the first phase of pruning is as follows. Take the wave packet expansion of f at scale R, say

$$f \approx \sum_\theta \sum_{T\in\mathbb{T}_\theta} \psi_T f_\theta,$$

and define f_N to be

$$f_N = \sum_\theta \sum_{T\in\mathbb{T}'_\theta} \psi_T f_\theta,$$

where $\mathbb{T}'_\theta$ is the set of T for which

$$C_{\mathfrak{p}}(\log R)^4|T|^{-1}\int_T |f_\theta|^2 \gtrsim \frac{\alpha^2}{(\#\theta)^2}$$

for a suitable pruning constant $C_{\mathfrak{p}}$. If we apply the L^4 square function estimate/Plancherel/dyadic pigeonholing argument outlined above to f_N, then the contribution of the integral along $|\xi| \sim r$ of f_N will be acceptable for Theorem 1.2 when $r \gtrapprox R^{-1/2}$.

However, the other annular integrals will involve wave envelopes of other dimensions which have not yet been pruned, and it will be necessary to consider deeper prunings. In particular, if we decompose $f_N = f_{N-1} + f_N^{\mathcal{B}}$ by defining

$$f_{N-1} = \sum_\theta f_{N-1,\theta},$$

with $f_{N-1,\theta}$ equal to the sum of the wave envelopes of scale $\sim (\log R) R^{1/2} \times R$ with appropriately high amplitude square functions, then more of the integrals of f_{N-1} will be acceptable; on the other hand, since $f_N^{\mathcal{B}}$ is high-amplitude on small wave packets and low-amplitude on larger wave packets, it must be that $f_N^{\mathcal{B}}$ is dominated by high-frequency contribution (as otherwise low-dominance would imply sufficient local constancy to reach a contradiction).

Proceeding inductively, we replace f by a sum of N functions $f_1 + \sum_{m=2}^N f_m^{\mathcal{B}}$, where the "bad" functions $f_m^{\mathcal{B}}$ have acceptable high-frequency contributions and are also dominated by those contributions, and where the lowest function f_1 satisfies the wave envelope estimate by construction.

We remark on the difference between this work and the work in [8], where logarithmic bounds were derived from canonical-scale decoupling. There, the critical problem was to provide good bounds on integrals like $\int \left| \sum_{\theta \subseteq \tau_k} |f_\theta|^2 * |\eta^\vee_{\sim R_k^{-1}}| \right|^2$, and the pruning was set up to assist this. In this paper, the pruning supplies a decomposition $f = f_1 + \sum_{m=2}^N f_m^{\mathcal{B}}$ such that each $f_m^{\mathcal{B}}$ satisfies good bounds on those integrals by definition, and the problem instead becomes to reducing more general integrals to these.

One additional technical advantage in the current work is the use of wave packets with near-exponential decay, which permits one to improve Schwartz-type decay to decay of the form $e^{-|x|^{1-\varepsilon}}$, while preserving compact support on the Fourier side. Such decay on the spatial side is sufficient to prevent super-logarithmic losses in our setting, particularly when estimating the interference of parallel wave packets via Cauchy-Schwarz. The authors of [8] handled this issue by appealing to wave packets defined by Gaussian weights, which possess the technical difficulty of having noncompact Fourier support. Note too that, by analyticity, the decay $e^{-c|x|^{1/2}}$ could not be improved to $e^{-c|x|}$.

Lastly, we refine the argument of [10] by applying a modified broad/narrow argument and a modified pigeonholing, which are chosen to avoid superlogarithmic losses. Each of these have appeared elsewhere in the literature before; for example, the broad/narrow argument is adapted from [8].

2.2 Initial notation-setting

We begin by reproducing some of the language of [10], with minor modifications. Fix arbitrary $\alpha > 0$, and $R \in 2^{2^{\mathbb{N}}}$ sufficiently large; we will occasionally assume that R is large enough that $\log \log R$ exceeds a universal constant. Throughout the paper, we will use B_R to denote the ball of radius R centered at 0. Let $U_\alpha = \{x \in B_R : |f(x)| > \alpha\}$.

We will need a sequence of scales. Let N be the least integer greater than or equal to $\frac{1}{2} \frac{\log R}{\log \log R}$. Let $R_k := (\log R)^k$ for $k = 0, \ldots, N-1$, and define $R_N := R^{1/2}$. Assuming R is large, we may take $CN \le \log R$ for suitable constants C. We note here that, as long as $R > e^2$, then $R_k > 2^k$, and so it will suffice to prove Theorem 1.2 with "s dyadic" replaced by "$s \in (\log R)^{\mathbb{Z}}$."

Next, let $\{\theta\}$ be a partition of $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ by approximate $R^{-1/2} \times R^{-1}$ rectangles, and similarly let $\{\tau_k\}$ be a partition of $\mathcal{N}_{R_k^{-1}}(\mathbb{P}^1)$ by approximate $R_k^{-1} \times R_k^{-2}$ rectangles; here and throughout the paper, the notations τ_N and θ are interchangeable. We assume that for $k < k'$ and each choice of $\tau_k, \tau_{k'}$ we either have $\tau_{k'} \subseteq \tau_k$ or $\tau_{k'} \cap \tau_{k'} = \varnothing$. We also write τ_0 for the full $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$.

By scaling, it will suffice to consider the case when $\max_\theta \|f_\theta\|_\infty = 1$; since we are bounding $|U_\alpha|$, we will assume also that $\alpha \le R^{1/2}$. By considering the summand on the right-hand side of the inequality in Theorem 1.2 corresponding to $s = 1$, it suffices to consider the case $\alpha \geq 1$.

For each point $p \in \mathbb{P}^1$, let $\mathbf{t}_p$ be the tangent vector to $\mathbb{P}^1$ at p pointing in the positive-x direction. Similarly, write $\mathbf{n}_p$ for the normal vector to $\mathbb{P}^1$ at p pointing in the positive-y direction.

For each fixed τ_k, we will let $U_{\tau_k,R}$ be a rectangle of dimensions $(R/R_k) \times R$ with long side parallel to $\mathbf{n}_{c_{\tau_k}}$. Fix also a tiling of $\mathbb{R}^2$ by translates U of $U_{\tau_k,R}$; we will denote the relationship between U and $U_{\tau_k,R}$ by $U \| U_{\tau_k,R}$, so that the tiling just described is the set $\{U \| U_{\tau_k,R}\}$.

Our next definition requires more ink, so we enclose it in a formal definition.

Definition 2.1 (Sufficiently rapid cutoffs)**.** Fix a small constant $\epsilon_0 > 0$. Define ρ_0 to be a smooth function satisfying the following properties:

- $\operatorname{supp}(\rho_0) \subseteq [-1, 1]^2$.
- $\rho_0 \equiv 1$ on $[-1 + \epsilon_0, 1 - \epsilon_0]^2$.
- $\|\rho^\vee\|_\infty \lesssim 1$.
- There is a constant c such that $|\rho_0^\vee(x)| \leq e^{-c|x|^{1/2}}$ for all $x \in \mathbb{R}^2$.

For each box T, let $\rho_T = \rho_0 \circ R_T$, where R_T is an affine transformation that scales and rotates T to $[-\frac{1}{2}, \frac{1}{2}]^2$. Observe that $\rho_T = 1$ on $(2 - 2\epsilon_0)T$ and $\rho_T = 0$ outside of $2T$. Observe from the outset that $\|\rho_T^\vee\|_1 = \|\rho_0^\vee\|_1 = O(1)$ by change-of-variable.

Additionally, we may note that, if $\{U \| T^*\}$ is the fundamental tiling of $\mathbb{R}^2$ by translates U of the dual T^* of T, say with U having center c_U, then the set of centers $\{c_u : U \| T^*\}$ form a lattice, and by the Poisson summation formula

$$\sum_{U \| T^*} \rho_T^\vee(x + c_U) = \frac{1}{m(T)} \sum_{V \| T} e^{2\pi i x \cdot c_V} \rho_T(c_V) = \frac{1}{m(T)}.$$

This is the reason for the extra factor of 2 in the definition of ρ_T. We will interpret this calculation as the statement that $\{m(T)\rho_U\}_{U \| T^*}$ form a partition of unity.

We will also write $\psi_U(x) = \rho_{U^*}^\vee(x - c_U)$ for any parallelogram U.

Remark 2.2. The function ρ_0 may be constructed as the infinite convolution over $\frac{1}{10n^2} 1_{[-10^{-1}n^{-2}, 10^{-1}n^{-2}]}$.

We will relate different square functions by means of analyzing their high- and low-frequency components. To this end, set φ to be a smooth nonnegative radial bump function on $\mathbb{R}^2$ such that $\varphi(\xi) = 1$ on $|\xi| \leq 1$ and $\varphi(\xi) = 0$ on $|\xi| \geq 2$. For each $r > 0$, we define the cutoff functions

$$\eta_{\leq r}(\xi) = \varphi(r^{-1}\xi), \quad \eta_{>r}(\xi) = \varphi(\xi) - \varphi(r^{-1}\xi), \quad \eta_{\sim r}(\xi) = \varphi(r^{-1}\xi) - \varphi(2r^{-1}\xi).$$

Note in particular that $\eta_{\leq r}(\xi) = 1$ on $|\xi| \leq r$ and $\eta_{\leq r}(\xi) = 0$ on $|\xi| > 2r$, and $\eta_{>r}(\xi) = 1$ on $2r < |\xi| \leq 1$ and $\eta_{>r}(\xi) = 0$ on $|\xi| \in (0, r) \cup (2, \infty)$.

Next, let W_U denote the composition $(W \circ T_{\tau_k})(x - c_U)$, where

$$W(x, y) := \frac{1}{(1 + |x|^2)^{100}(1 + |y|^2)^{100}},$$

and T_{τ_k} is the linear transformation which rotates $2U_{\tau_k,R}$ to $[-R/R_k, R/R_k] \times [-R, R]$ and then rescales to $[-1, 1]^2$. We define $⨍_U g := |U|^{-1} \int g W_U$ for arbitrary g. Since W decays polynomially, we may assume $\psi_U \lesssim W_U$ for every choice of U.

Next, for each k, let w_k be the weight

$$w_k(x) = \frac{c}{(1 + |x|^2 R_k^{-1})^{10}}$$

with c chosen so that $\|w_k\|_1 = 1$.

2.3 Pruning

For suitable constant[1] $C_{\mathfrak{p}} > 0$, we define the pruned set $\mathcal{G}_\theta$ associated to θ as follows.

Definition 2.3. Set[2]

$$\mathcal{G}_\theta := \left\{ U \| U_{\theta,R} : C_{\mathfrak{p}} (\log R)^4 ⨍_U |f_\theta|^2 \geq \frac{\alpha^2}{(\#\theta)^2} \right\}.$$

Define the pruned functions as

$$f_{N,\theta} := \sum_{U \in \mathcal{G}_\theta} \psi_U f_\theta, \quad f_N := \sum_\theta f_{N,\theta}.$$

For $k < N$ and each τ_k, define

$$\mathcal{G}_{\tau_k} := \left\{ U \| U_{\tau_k,R} : C_{\mathfrak{p}} (\log R)^4 ⨍_U \sum_{\theta \subseteq \tau_k} |f_{k+1,\theta}|^2 \geq \frac{\alpha^2}{(\#\tau_k)^2} \right\}$$

and

$$f_{k,\theta} := \sum_{U \in \mathcal{G}_{\tau_k}} \psi_U f_{k+1,\theta} \text{ (where } \tau_k \supseteq \theta) \quad \text{and} \quad f_k = \sum_\theta f_{k,\theta}.$$

We set also $f_k - f_{k-1} =: f_k^{\mathcal{B}}$, and $f_{k,\theta}^{\mathcal{B}} = \sum_{U \notin \mathcal{G}_{\tau_{k-1}}} \psi_U f_{k,\theta}$, where $\theta \subseteq \tau_{k-1}$. If $k' \leq k$, then set $f_{k,\tau_{k'}}^{\mathcal{B}} = \sum_{\theta \subseteq \tau_{k'}} f_{k,\theta}^{\mathcal{B}}$.

The following estimates will be needed:

Lemma 2.4 (Pruning lemmas)**.** *The pruned functions satisfy the following:*

(a) $f_N = f_1 + \sum_{m=2}^N f_m^{\mathcal{B}}$.

(b) $|f_{k,\theta}| \leq |f_{k+1,\theta}| \leq |f_\theta|$.

(c) $\operatorname{supp}(\widehat{f_{k,\theta}}) \subseteq 2(N-k)\theta$ *for all* θ.

Proof. (a): This is just the calculation

$$f_1 + \sum_{m=2}^N f_m^{\mathcal{B}} = f_1 + \sum_{m=2}^N (f_m - f_{m-1}) = f_N.$$

(b): Since $\sum_{U \in \mathcal{G}_{\tau_k}} \psi_U \leq 1$, it follows that

$$|f_{k,\theta}| = |f_{k+1,\theta}| \Big| \sum_{U \in \mathcal{G}_{\tau_k}} \psi_U \Big| \leq |f_{k+1,\theta}|,$$

and similarly

$$|f_{N,\theta}| = |f_\theta| \sum_{U \in \mathcal{G}_{\tau_N}} \psi_U \leq |f_\theta|.$$

(c): We first consider the case $k = N$. For each θ and $U \| U_{\theta,R}$,

$$\widehat{\psi_U f_\theta}(\xi) = \int \widehat{\psi_U}(\eta) \hat{f}_\theta(\xi - \eta) d\eta,$$

[1]The size of $C_{\mathfrak{p}}$ is only constrained by the proof of Lemma 2.13.

[2]Recall from above that we have repurposed the symbol $⨍_U \cdot$ to mean $|U|^{-1} \int W_U \cdot$

which vanishes when there does not exist $\eta \in 2U^* \subseteq B(0, 2R^{-1})$ such that $\xi - \eta \in \theta$, i.e. when $\xi \notin \mathcal{N}_{2R^{-1}}\theta$. Thus $f_{N,\theta}$ has Fourier support in $\theta + B(0, 2R^{-1})$.

More generally, the same calculation gives

$$\operatorname{supp}(\hat{f}_{k,\theta}) \subseteq \theta + 2 \sum_{j=0}^{N-k} U^*_{\tau_{N-j}},$$

where τ_{N-j} is the cap of size $R_{N-j}^{-1} \times R_{N-j}^{-2}$ containing θ. In particular,

$$\operatorname{supp}(\hat{f}_{k,\theta}) \subseteq 2(N-k)\theta \subseteq (\log R)\theta,$$

as claimed. □

2.4 Square functions

In this section, we record a series of lemmas that control the contribution of square functions at various scales. The proofs of these are standard, and have been delayed to the appendix.

Our first lemma encodes that our frequency-localized functions f_θ and $f^{\mathcal{B}}_{m,\theta}$ are approximately constant on small scales.

Lemma 2.5 (Pointwise local constancy lemmas)**.**

(a) For any θ, $|f_\theta|^2 \lesssim |f_\theta|^2 * |\rho_\theta^\vee|$.

(b) For any k, m *and any* x,

$$|f_{m,\tau_k}|^2(x) \lesssim |f_{m,\tau_k}|^2 * w_{R_k}(x).$$

Our second lemma serves as a shorthand for passing between several integrals that are essentially equivalent to the wave-envelope expansion.

Lemma 2.6 (Integrated local constancy lemmas)**.** *Let* $r > 0$ *be dyadic.*

(a) If $r \leq 2R_{k+3}/R$, *then*

$$\int \Big| \sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^\vee_{\sim r} \Big|^2 \lesssim \log R \int \Big| \sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}| \Big|^2.$$

(b) If $k \geq m$, *then*

$$\int \Big| \sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}| \Big|^2 \lesssim \sum_{U \in \mathcal{G}_{\tau_k}} |U| \left(\fint_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \right)^2.$$

Next, we note that, on the superlevel set $\{|f| > \alpha\}$, it is possible to replace f by f_N, so we may appeal to the decomposition $f_N = f_1 + \sum_{m=2}^N f^{\mathcal{B}}_m$.

Lemma 2.7 (Replacement lemma)**.** $|f(x) - f_N(x)| \lesssim \frac{\alpha}{C_{\mathfrak{p}}^{1/2} (\log R)}$.

As a consequence, we will be able to control the size of the superlevel set U_α by the size of the auxiliary set $V_\alpha := \{x : |f_N(x)| > \frac{1}{2}\alpha\}$.

For the next lemma, we will need to define an adjacency relation.

Definition 2.8. For caps τ_k, τ_k' of the same size, we say "τ_k near $\tau_{k'}$" if $\operatorname{dist}(\tau_k, \tau_{k'}) \lesssim (\log R)\operatorname{diam}(\tau_k)$ for a suitably chosen implicit constant. If τ_k, τ_k' do not satisfy this, we write "τ_k not near τ_k'".

Remark 2.9. As defined, we have that for each τ_k, $\#\{\tau_k' : \tau_k \text{ near } \tau_k'\} \lesssim \log R$.

Remark 2.10. If τ_k near τ_k', then $\tau_k \subseteq C \log R(\tau_k' + (c_{\tau_k} - c_{\tau_k'}))$ and symmetrically.

We now mention the two key lemmas that facilitate an efficient wave-envelope estimate. These are standard in high/low calculations, e.g. [4] (in the proof of Theorem 5.4), [9] (in the proof of Lemma 1.4), [5] (Lemmas 11, 12, 13), [6] (in the proof of Theorem 5), and [10] (Lemmas 4, 5, 6).

Lemma 2.11 (Low lemma)**.** *For any $2 \le m \le k \le N$, $0 \le s \le k$, and $r \le R_k^{-1}$,*

$$|f^{\mathcal{B}}_{m,\tau_s}|^2 * \eta^\vee_{\le r}(x) = \sum_{\tau_k \subseteq \tau_s} \sum_{\tau_k' \text{ near } \tau_k} \big(f^{\mathcal{B}}_{m,\tau_k} \overline{f^{\mathcal{B}}_{m,\tau_k'}}\big) * \eta^\vee_{\le r}(x)$$

for any x and any τ_s.

Lemma 2.12 (High Lemmas)**.** *For any $m, k,$ and l such that $2 \le m \le N$ and $k + l \le N$,*

(a)

$$\int \Big| \sum_\theta |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^\vee_{\ge R_k/R} \Big|^2 \lesssim \log R \sum_{\tau_k} \int \Big| \sum_{\theta \subseteq \tau_k} |f_{m,\theta}|^2 * \eta^\vee_{\ge R_k/R} \Big|^2,$$

(b)

$$\int \Big| \sum_{\tau_k} |f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^\vee_{\ge R_k^{-1}} \Big|^2 \lesssim (\log R) \sum_{\tau_k} \int |f^{\mathcal{B}}_{m,\tau_k}|^4,$$

(c)

$$\int \Big| \sum_{\tau_k} \sum_{\tau_k' \text{ near } \tau_k} (f^{\mathcal{B}}_{m,\tau_k} \overline{f^{\mathcal{B}}_{m,\tau_k'}}) * \eta^\vee_{\ge R_{k+l}^{-1}} \Big|^2 \lesssim (\log R)^{l+3} \sum_{\tau_k} \int |f^{\mathcal{B}}_{m,\tau_k}|^4.$$

Next, we will need a tool to ensure that, when taking wave envelope contributions of the bad parts $f^{\mathcal{B}}_m$, we are allowed to disregard the low-frequency envelopes which have not yet been pruned.

Lemma 2.13 (Weak high-domination of bad parts)**.** *Let $2 \le m \le N$ and $0 \le k < m$.*

(a) We have the estimate

$$\Big| \sum_{\tau_{m-1} \subseteq \tau_k} |f^{\mathcal{B}}_{m,\tau_{m-1}}|^2 * \eta^\vee_{\le R_{m-1}/R}(x) \Big| \lesssim \frac{\alpha^2 (\#\tau_{m-1} \subseteq \tau_k)}{C_{\mathfrak{p}} (\log R)^2 (\#\tau_{m-1})^2}.$$

(b) Suppose $\alpha \lesssim (\log R) |f^{\mathcal{B}}_{m,\tau_k}(x)|$. Then

$$\sum_{\tau_{m-1} \subseteq \tau_k} |f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x) \lesssim \Big| \sum_{\tau_{m-1} \subseteq \tau_k} |f^{\mathcal{B}}_{m,\tau_{m-1}}|^2 * \eta^\vee_{\ge R_{m-1}/R}(x) \Big|.$$

3 Proof of Theorem 1.2

3.1 Bounding the broad sets

This portion of the argument follows closely the approach of [10], Section 3. Recall that U_α is defined as the set

$$U_\alpha = \big\{ x \in B_R : |f(x)| > \alpha \big\}.$$

We consider also the auxiliary set

$$V_\alpha = \Big\{ x \in B_R : |f_N(x)| > \frac{1}{2}\alpha \Big\}.$$

To avoid trivialities, we assume $|U_\alpha| > 0$ for the remainder of this section. By the replacement lemma 2.7,

$$U_\alpha \subseteq V_\alpha$$

for large enough R. By the pruning lemmas 2.4,

$$\begin{aligned} V_\alpha &\subseteq \left\{x \in V_\alpha : |f_1|(x) \geq N^{-1}|f_N(x)|\right\} \cup \bigcup_{m=2}^{N} \left\{x \in V_\alpha : |f_m^{\mathcal{B}}|(x) \geq N^{-1}|f_N(x)|\right\} \\ &=: U_\alpha^1 \cup \bigcup_{m=2}^{N} U_\alpha^m \end{aligned}$$

so that

$$|U_\alpha| \leq |U_\alpha^1| + \sum_{m=2}^{N} |U_\alpha^m|.$$

We bound each of these sets in turn.

Proposition 3.1 (Case $m = 1$)**.**

$$\alpha^4 |U_\alpha^1| \lesssim C_{\mathfrak{p}}^2 (\log R)^8 \sum_{U \in \mathcal{G}_{\tau_1}} |U| \left(\fint_U \sum_\theta |f_\theta|^2 \right)^2.$$

Proof. Clearly it suffices to assume $|U_\alpha^1| > 0$. Then there is some $x \in B_R$ such that $|f_1(x)| \geq \frac{1}{2N}\alpha$. Since

$$\begin{aligned} \frac{1}{2N}\alpha \leq |f_1(x)| &= |\sum_{\tau_1} \sum_{\theta \subseteq \tau_1} \sum_{U \in \mathcal{G}_{\tau_1}} \psi_U(x) f_{2,\theta}(x)| \\ &\leq |\sum_{\tau_1} \sum_{\theta \subseteq \tau_1} \sum_{U \in \mathcal{G}_{\tau_1}; x \in (\log R)^3 U} \psi_U(x) f_{2,\theta}(x)| + |\sum_{\tau_1} \sum_{\theta \subseteq \tau_1} \sum_{U \in \mathcal{G}_{\tau_1}; x \notin (\log R)^3 U} \psi_U(x) f_{2,\theta}(x)|, \end{aligned}$$

and, if $x \notin (\log R)^3 U$, the near-exponential decay of ψ_U implies

$$|\psi_U(x) f_{2,\theta}(x)| \lesssim R^{-1000},$$

whereby

$$|\sum_{\tau_1} \sum_{\theta \subseteq \tau_1} \sum_{U \in \mathcal{G}_{\tau_1}; x \notin (\log R)^3 U} \psi_U(x) f_{2,\theta}(x)| \leq R^{-100},$$

we conclude that there is some τ_1 and $U \in \mathcal{G}_{\tau_1}$ with $x \in (\log R)^3 U$.

Since $U \| U_{\tau_1,R}$, U is a rectangle of dimensions $\frac{R}{\log R} \times R$, and that by definition of $\mathcal{G}_{\tau_1}$ we have

$$|U|^{-1} \int W_U \sum_{\theta \subseteq \tau_1} |f_{2,\theta}|^2 \geq \frac{\alpha^2}{(\#\tau_1)^2} \frac{1}{C_{\mathfrak{p}} (\log R)^4}.$$

In particular,

$$\alpha^4 \leq C_{\mathfrak{p}}^2 (\log R)^8 \left(\fint_U \sum_\theta |f_\theta|^2 \right)^2,$$

where we have used the pruning lemmas 2.4.

The above calculation demonstrates that, for each $x \in U_\alpha$ satisfying $|f(x)| \leq 4N|f_1(x)|$, there is some τ_1 and $U \in \mathcal{G}_{\tau_1}$ such that $x \in (\log R)^3 U$. Thus

$$1_{\{x \in U_\alpha : |f(x)| \leq 2N|f_1(x)|\}} \leq \sum_{\tau_1} \sum_{U \in \mathcal{G}_{\tau_1}} 1_{(\log R)^3 U},$$

and upon integrating we achieve

$$\left|\left\{x \in U_\alpha : |f(x)| \le 2N|f_1(x)|\right\}\right| \le \sum_{\tau_1} \sum_{U \in \mathcal{G}_{\tau_1}} (\log R)^6 |U|$$

$$\le 4\alpha^{-4} C_{\mathfrak{p}}^2 (\log R)^{14} \sum_{\tau_1} \sum_{U \in \mathcal{G}_{\tau_1}} \left(\fint_U \sum_{\theta \subseteq \tau_1} |f_\theta|^2 \right)^2,$$

which rearranges to the desired

$$\alpha^4 |U_\alpha^1| \lesssim C_{\mathfrak{p}}^2 (\log R)^{14} \sum_{\tau_1} \sum_{U \in \mathcal{G}_{\tau_1}} \left(\fint_U \sum_{\theta \subseteq \tau_1} |f_\theta|^2 \right)^2.$$

□

We will use the following local bilinear restriction result, demonstrated in [5]:

Theorem 3.2 (Bilinear restriction; Theorem 15 of [5])**.** *Let $S \ge 4$, $\frac{1}{2} \ge E \ge S^{-1/2}$, and $X \subseteq \mathbb{R}^2$ be Lebesgue measurable. Suppose that τ, τ' are E-separated subsets of $\mathcal{N}_{S^{-1}}(\mathbb{P}^1)$. Then, for a partition $\Omega = \{\omega_S\}$ of $\mathcal{N}_{S^{-1}}(\mathbb{P}^1)$ into $\sim S^{-1/2} \times S^{-1}$-caps, we have*

$$\int_X |f_\tau|^2 |f_{\tau'}|^2(x)dx \lesssim E^{-2} \int_{\mathcal{N}_{S^{1/2}}(X)} \Big| \sum_{\omega_S} |f_{\omega_S}|^2 * w_{S^{1/2}}(x) \Big|^2 dx.$$

This will be our initial estimate when we try to estimate f in the broad case. We now define the broad sets on which bilinear methods are appropriate.

Define the mth ($2 \le m \le N$) broad set in U_α to be as follows. Fix any $\tau_l, \tau_l' \subseteq \tau_{l-1}$ non-adjacent caps with $l \le m$, and define

$$\mathrm{Br}_\alpha^m(\tau_l, \tau_l') = \left\{ x \in U_\alpha^m : |f_{m,\tau_{l-1}}^{\mathcal{B}}(x)| \le (\log R)^3 |f_{m,\tau_l}^{\mathcal{B}} f_{m,\tau_l'}^{\mathcal{B}}|^{1/2} \right\}.$$

Proposition 3.3 (Case $2 \le m \le N$)**.**

$$\int_{\mathrm{Br}_\alpha^m(\tau_l,\tau_l')} |f_{m,\tau_{l-1}}^{\mathcal{B}}|^4 \lesssim (\log R)^{30} \sum_{m<k\le N} \sum_{\tau_k \subseteq \tau_{l-1}} \sum_{U \in \mathcal{G}_\tau} |U| \left(\fint_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \right)^2.$$

Proof. Let (c, c^2) denote the center of τ_{l-1}. Write B for the affine transformation

$$B(\xi_1, \xi_2) = (R_{l-1}(\xi_1 - c), R_{l-1}^2(\xi_2 - 2\xi_1 c + c^2)).$$

Then $(\widehat{f_{m,\tau_{l-1}}^{\mathcal{B}}} \circ B^{-1})^\vee$ has Fourier support contained in the set $\mathcal{N}_{NR_{l-1}^2 R^{-1}}$. Additionally, $B(\tau_l)$ and $B(\tau_l')$ have diameters $\sim (\log R)^{-1}$, and

$$\mathcal{P}_{B(\tau_r)}[(\widehat{f_{m,\tau_{l-1}}^{\mathcal{B}}} \circ B^{-1})^\vee] = (\widehat{f_{m,\tau_r}^{\mathcal{B}}} \circ B^{-1})^\vee$$

for each $\tau_r \subseteq \tau_{l-1}$.

Write $s = \min(m + l - 2, N)$. Changing variables by B and applying bilinear restriction, after some routine calculation we obtain

$$\int_{\mathrm{Br}_\alpha^m(\tau_l,\tau_l')} |f_{m,\tau_{l-1}}^{\mathcal{B}}|^4 \lesssim (\log R)^{14} \int_{\mathrm{Br}_\alpha^m(\tau_l,\tau_l') + T(B_{R_{m-1}})} \Big| \sum_{\tau_s \subseteq \tau_{l-1}} |f_{m,\tau_s}^{\mathcal{B}}|^2 * \tilde{w}_{R_{m-1}} \Big|^2,$$

where we have written T for the linear transformation $T(\xi_1,\xi_2) = (R_{l-1}\xi_1, R_{l-1}^2\xi_2)$ and $\tilde{w}_{R_{m-1}}$ for the function $\tilde{w}_{R_{m-1}}(x_1,x_2) = cw_{R_{m-1}}(R_{l-1}^{-1}x_1, R_{l-1}^{-2}x_2)$, with c chosen so that $\|\tilde{w}_{R_{m-1}}\|_1 = 1$. Observe that each $|f^{\mathcal{B}}_{m,\tau_s}|^2$ is approximately constant on balls of radius $\sim (\log R)^{-1}R_s > R_{m-1}$. By the rapid decay of $\tilde{w}_{R_{m-1}}$ and the lower bound on $\sum_{\tau_{l+m-2}\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_{m+l-2}}|^2$ over $\mathrm{Br}^m_\alpha \subseteq U^m_\alpha$, we may apply weak high-domination 2.13 to estimate

$$\int_{\mathrm{Br}^m_\alpha(\tau_l,\tau_l')+T(B_{R_{m-1}})} \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \tilde{w}_{R_{m-1}}\Big|^2 \lesssim \int_{\mathrm{Br}^m_\alpha(\tau_l,\tau_l')+T(B_{R_{m-1}})} \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \tilde{w}_{R_{m-1}} * \eta^\vee_{\geq R_{m-1}/R}\Big|^2,$$

which we may now bound by

$$\int \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \tilde{w}_{R_{m-1}} * \eta^\vee_{\geq R_{m-1}/R}\Big|^2.$$

We now pigeonhole to a dyadic scale. Let $R_{m-1}/R \leq r \leq 2NR_{m-1}^{-1}$ be dyadic such that

$$\int_{\mathbb{R}^2} \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \tilde{w}_{R_{m-1}} * \eta^\vee_{\geq R_{m-1}/R}\Big|^2 \lesssim \log R \int_{\mathbb{R}^2} \Big|\sum_{\tau_s} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \tilde{w}_{R_{m-1}} * \eta^\vee_{\geq R_{m-1}/R} * \eta^\vee_{\sim r}\Big|^2.$$

By Young,

$$\int_{\mathbb{R}^2} \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \tilde{w}_{R_{m-1}} * \eta^\vee_{\geq R_{m-1}/R} * \eta^\vee_{\sim r}\Big|^2 \lesssim \int_{\mathbb{R}^2} \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \eta^\vee_{\sim r}\Big|^2.$$

The remainder of the analysis will be split into cases, depending on the size of r.

Case 1: $r \leq R^{-1/2}$. By the low lemma 2.11,

$$\int \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \eta^\vee_{\sim r}\Big|^2 = \int \Big|\sum_{\theta\subseteq\tau_{l-1}} \sum_{\theta' \text{ near } \theta} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^\vee_{\sim r}\Big|^2.$$

Let k be s.t. $(\log R)r \leq R_k/R \leq (\log R)^2 r$. Since we have assumed $r \geq R_{m-1}/R$, we must have $k \geq m$. By the integrated local constancy lemma 2.6, part (a), and Cauchy-Schwarz, we have

$$\int \Big|\sum_{\theta\subseteq\tau_{l-1}} \sum_{\theta' \text{ near } \theta} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^\vee_{\sim r}\Big|^2 \lesssim (\log R)^3 \sum_{\tau_k\subseteq\tau_{l-1}} \int \Big|\sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^\vee_{\tau_k}|\Big|^2.$$

By part (b) of the integrated local constancy lemma,

$$\sum_{\tau_k\subseteq\tau_{l-1}} \int \Big|\sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^\vee_{\tau_k}|\Big|^2 \lesssim \sum_{\tau_k\subseteq\tau_{l-1}} \sum_{U\in\mathcal{G}_{\tau_k}} |U| \left(\fint_U \sum_{\theta\subseteq\tau_k} |f_\theta|^2 \right)^2.$$

We conclude that

$$\int_{\mathrm{Br}^m_\alpha(\tau_l,\tau_l')} |f^{\mathcal{B}}_{m,\tau_{l-1}}|^4 \lesssim (\log R)^{18} \sum_{\tau_k\subseteq\tau_{l-1}} \sum_{U\in\mathcal{G}_{\tau_k}} |U|^{-1} \left(\int \sum_{\theta\subseteq\tau_k} |f_\theta|^2 W_U \right)^2$$

as claimed.

Case 2: $r > R^{-1/2}$ Let k be such that $R_{k-1}^{-1} \leq r \leq (\log R)R_{k-1}^{-1}$; this time, from the assumption $r \leq 2NR_{m-1}^{-1}$, we conclude that $R_{m-1} \leq 2NR_{k-1}$, i.e. $k \geq m$. By the low lemma 2.11,

$$\int_{\mathbb{R}^2} \Big|\sum_{\tau_s\subseteq\tau_{l-1}} |f^{\mathcal{B}}_{m,\tau_s}|^2 * \eta^\vee_{\sim r}\Big|^2 = \int \Big|\sum_{\tau_{k-2}\subseteq\tau_{l-1}} \sum_{\tau_{k-2}' \text{ near } \tau_{k-2}} (f^{\mathcal{B}}_{m,\tau_k}\overline{f^{\mathcal{B}}_{m,\tau_k'}}) * \eta^\vee_{\sim r}\Big|^2.$$

By part (c) of the high lemma 2.12,

$$\int \Big| \sum_{\tau_{k-2}\subseteq\tau_{l-1}} \sum_{\tau'_{k-2}\text{ near }\tau_{k-2}} (f^{\mathcal{B}}_{m,\tau_k}\overline{f^{\mathcal{B}}_{m,\tau'_k}}) * \eta^{\vee}_{\sim r}\Big|^2 \lesssim (\log R)^4 \sum_{\tau_{k-2}\subseteq\tau_{l-1}} \int |f^{\mathcal{B}}_{m,\tau_{k-2}}|^4$$

$$\lesssim (\log R)^{10} \sum_{\tau_k\subseteq\tau_{l-1}} \int |f^{\mathcal{B}}_{m,\tau_k}|^4.$$

By the reverse square function estimate for $\mathbb{P}^1$ and by splitting $f^{\mathcal{B}}_{m,\tau_k}$ into $O(\log R)$ pieces with disjoint Fourier support,

$$\int |f^{\mathcal{B}}_{m,\tau_k}|^4 \lesssim (\log R)^4 \int \Big|\sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2\Big|^2.$$

So far, in case 2, we have reached the estimate

$$\int_{\mathrm{Br}^m_\alpha(\tau_l,\tau'_l)} |f^{\mathcal{B}}_{m,\tau_{l-1}}|^4 \lesssim (\log R)^{29} \sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2\Big|^2$$

for some $k \geq m$. We consider two sub-cases, depending on if the latter is high- or low-dominated.

Case 2a: Suppose that

$$\sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2\Big|^2 \lesssim \sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\leq R_m/R}\Big|^2.$$

Since $m \leq k$, we have $R_m/R \leq R_k/R$, so by part (a) of the integrated local constancy lemma 2.6, we have

$$\sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\leq R_m/R}\Big|^2 \lesssim \sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{\tau_k}|\Big|^2.$$

Then trivially (using $k \geq m$)

$$\sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{\tau_k}|\Big|^2 \lesssim \sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{\tau_m}|\Big|^2$$

$$\leq \sum_{\tau_m\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_m} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{\tau_m}|\Big|^2$$

and by the integrated local constancy lemma (b) we have

$$\sum_{\tau_m\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_m} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{\tau_m}|\Big|^2 \lessapprox \sum_{\tau_m\subseteq\tau_{l-1}} \sum_{U\in\mathcal{G}_{\tau_m}} |U| \left(\fint_U \sum_{\theta\subseteq\tau_m} |f_\theta|^2 \right)^2.$$

Thus we have the desired

$$\sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2\Big|^2 \lessapprox \sum_{\tau_m\subseteq\tau_{l-1}} \sum_{U\in\mathcal{G}_{\tau_m}} |U| \left(\fint_U \sum_{\theta\subseteq\tau_m} |f_\theta|^2 \right)^2.$$

Case 2b: If we are not in case 2a, then

$$\sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2\Big|^2 \lessapprox \sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\geq R_m/R}\Big|^2.$$

Now let μ be dyadic between R_m/R and $R^{-1/2}$ such that

$$\sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\geq R_m/R}\Big|^2 \lesssim \log R \sum_{\tau_k\subseteq\tau_{l-1}} \int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\sim\mu}\Big|^2.$$

If $\mu \le R_k/R$, then by part (a) of the integrated local constancy 2.6 we have

$$\sum_{\tau_k\subseteq\tau_{l-1}}\int\Big|\sum_{\theta\subseteq\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2*\eta^{\vee}_{\sim\mu}\Big|^2\lesssim\sum_{\tau_k\subseteq\tau_{l-1}}\int\Big|\sum_{\theta\subseteq\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2*|\rho^{\vee}_{\tau_k}|\Big|^2,$$

and by integrated local constancy (b) we have

$$\sum_{\tau_k\subseteq\tau_{l-1}}\int\Big|\sum_{\theta\subseteq\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2*|\rho^{\vee}_{\tau_k}|\Big|^2\lesssim\sum_{\tau_k\subseteq\tau_{l-1}}\sum_{U\in\mathcal{G}_{\tau_k}}|U|\left(\fint_U\sum_{\theta\subseteq\tau_k}|f_\theta|^2\right)^2,$$

and we conclude that

$$\int_{\mathrm{Br}^m_\alpha(\tau_l,\tau_l')}|f^{\mathcal{B}}_{m,\tau_{l-1}}|^4\lesssim(\log R)^{30}\sum_{\tau_k\subseteq\tau_{l-1}}\sum_{U\in\mathcal{G}_{\tau_k}}|U|\left(\fint_U\sum_{\theta\subseteq\tau_k}|f_\theta|^2\right)^2.$$

On the other hand, if $\mu > R_k/R$, then pick $p \ge k$ such that R_p/R is nearest to μ. Then by the high lemma 2.12

$$\sum_{\tau_k\subseteq\tau_{l-1}}\int\Big|\sum_{\theta\subseteq\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2*\eta^{\vee}_{\sim\mu}\Big|^2\lesssim(\log R)\sum_{\tau_p\subseteq\tau_{l-1}}\int\Big|\sum_{\theta\subseteq\tau_p}|f^{\mathcal{B}}_{m,\theta}|^2*\eta^{\vee}_{\sim\mu}\Big|^2,$$

and as above

$$\begin{aligned}\sum_{\tau_p\subseteq\tau_{l-1}}\int\Big|\sum_{\theta\subseteq\tau_p}|f^{\mathcal{B}}_{m,\theta}|^2*\eta^{\vee}_{\sim\mu}\Big|^2&\lesssim\sum_{\tau_p\subseteq\tau_{l-1}}\int\Big|\sum_{\theta\subseteq\tau_p}|f^{\mathcal{B}}_{m,\theta}|^2*|\rho^{\vee}_{\tau_p}|\Big|^2\\&\lesssim\sum_{\tau_p\subseteq\tau_{l-1}}\sum_{U\in\mathcal{G}_{\tau_p}}|U|\left(\fint_U\sum_{\theta\subseteq\tau_p}|f_\theta|^2\right)^2,\end{aligned}$$

from which we have the estimate

$$\int_{\mathrm{Br}^m_\alpha(\tau_l,\tau_l')}|f^{\mathcal{B}}_{m,\tau_{l-1}}|^4\lesssim(\log R)^{30}\sum_{k\le\nu\le N}\sum_{\tau_\nu\subseteq\tau_{l-1}}\sum_{U\in\mathcal{G}_{\tau_k}}|U|\left(\fint_U\sum_{\theta\subseteq\tau_k}|f_\theta|^2\right)^2$$

and we are done.

□

3.2 Broad/narrow analysis

In Propositions 3.1 and 3.3, we produced the desired bounds on the subset of the superlevel set for which f is sufficiently broad at some scale. In this subsection, we perform a broad/narrow analysis to produced the desired wave envelope estimate in each cube of sidelength R.

As a note: for the remainder of the article, we suppress the constant $C_{\mathfrak{p}}$ from the pruning definition as an implicit constant.

Proposition 3.4 (Local wave envelope estimate)**.** For each cube B_R of sidelength R and each $\alpha > 0$,

$$\alpha^4|\{x\in B_R:|f(x)|>\alpha\}|\lesssim(\log R)^{27}\sum_{\substack{R^{-1/2}\le s\le 1\\ s\text{ dyadic}}}\sum_{\tau:\ell(\tau)=s}\sum_{U\in\mathcal{G}_\tau}|U|^{-1}\|S_Uf\|_2^4.$$

We first note a technical obstruction. The common strategy in decoupling theory for performing broad/narrow analysis can be summarized as follows. Fix some scale s and $x \in B_R$, and fix τ_* to be the box of size $d(\tau_*) = s$ which maximizes $|f_{\tau_*}(x)|$. Then since $f(x) = \sum_\tau f_\tau(x)$, it follows (Lemma 7.2 of [3]) that either

$$|f(x)| \le 4|f_{\tau_*}(x)| \quad \text{or} \quad |f(x)| \le s^{-\frac{3}{2}} \max_{\tau \text{ not near } \tau'} |f_\tau(x) f_{\tau'}(x)|^{1/2},$$

where the maximum is taken over those boxes τ, τ' of diameter $d(\tau) = d(\tau') = s$. If we simplify the above as

$$|f(x)| \le 4|f_{\tau_*}(x)| + s^{-\frac{3}{2}} \max_{\tau \text{ not near } \tau'} |f_\tau(x) f_{\tau'}(x)|^{1/2}$$

and iterate by first choosing $s = R_1^{-1}$, then breaking up the first summand by choosing $s = R_2^{-1}$ and rescaling, etc., we achieve the estimate

$$|f(x)| \le 4^N \max_\theta |f_\theta(x)| + P(x),$$

for a suitable nonnegative quantity $P(x)$. Note however that $4^N = R^{\frac{\log 2}{\log\log R}}$, which is larger than our desired error $(\log R)^{O(1)}$ (while nevertheless being $O_\varepsilon(R^\varepsilon)$).

As a consequence, the broad/narrow analysis will need to be carried out more efficiently. We follow an approach demonstrated in Section 4 of [8], where a $(\log R)^{O(1)}$ error was obtained for canonical-scale ($\beta = \frac{1}{2}$) decoupling. Namely, the domain of integration for $|f|^4$ will be successively divided into broad and narrow sets, ranging over many scales. If a point x is broad at some scale, we will be able to productively use Propositions 3.1 and 3.3. If instead x is narrow at all scales, then a trivial estimate will suffice. As suggested by the above analysis, we will need to reduce 4 to a quantity that does not grow too quickly under the iteration.

We proceed to the proof. We will express the various estimates as "decoupling" bounds, though it is worth emphasizing that they are arranged *pointwise* (so this decomposition scheme is really a decomposition of *constants*, not functions with special Fourier support); we do so because of the convenient inductive structure of decoupling-style bounds.

Fix $2 \le m \le N$. We first present a modification of Lemma 8 of [8], which serves to replace the constant 4 in the prior calculation with a much smaller quantity.

Lemma 3.5 (Narrow lemma)**.** *For all sufficiently large R, the following holds. Suppose $1 \le k \le N$ and τ_k is an box of diameter R_k^{-1}. Let $\{\tau_{k+1}\}$ be the boxes of diameter R_{k+1}^{-1} with $\tau_{k+1} \subseteq \tau_k$. Then, for each x, either*

$$|f^{\mathcal{B}}_{m,\tau_k}(x)| \le (\log R)^3 \max_{\tau_{k+1} \text{ not near } \tau'_{k+1}} |f^{\mathcal{B}}_{m,\tau_{k+1}}(x) f^{\mathcal{B}}_{m,\tau'_{k+1}}(x)|^{1/2} \tag{3.1}$$

or

$$|f^{\mathcal{B}}_{m,\tau_k}(x)| \le \left(1 + \frac{1}{\log R}\right) \max_{\tau_{k+1} \subseteq \tau_k} \Big| \sum_{\tau'_{k+1} \text{ near } \tau_{k+1}} f^{\mathcal{B}}_{m,\tau'_{k+1}}(x)\Big|. \tag{3.2}$$

Proof. Fix $\tau^*_{k+1} \subseteq \tau_k$ which realizes the maximum

$$|f^{\mathcal{B}}_{m,\tau^*_{k+1}}(x)| = \max_{\tau_{k+1} \subseteq \tau_k} |f^{\mathcal{B}}_{m,\tau_{k+1}}(x)|.$$

Suppose 3.2 fails. Then, since $f^{\mathcal{B}}_{m,\tau_k}(x) = \sum_{\tau_{k+1} \subseteq \tau_k} f^{\mathcal{B}}_{m,\tau_{k+1}}(x)$, we have the inequality

$$|f^{\mathcal{B}}_{m,\tau_k}(x) - \sum_{\tau_{k+1} \not\sim \tau^*_{k+1}} f^{\mathcal{B}}_{m,\tau_{k+1}}(x)| < \left(1 + \frac{1}{\log R}\right)^{-1} |f^{\mathcal{B}}_{m,\tau_k}(x)|.$$

On the other hand,

$$|f^{\mathcal{B}}_{m,\tau_k}(x) - \sum_{\tau_{k+1}\not\sim\tau^*_{k+1}} f^{\mathcal{B}}_{m,\tau_{k+1}}(x)| \geq |f^{\mathcal{B}}_{m,\tau_k}(x)| - (\#\tau_{k+1}) \max_{\tau_{k+1} \text{ not near } \tau^*_{k+1}} |f^{\mathcal{B}}_{m,\tau_{k+1}}(x)|,$$

where we have used $\#\tau_{k+1}$ to denote $\#\{\tau_{k+1} \subseteq \tau_k\}$; the above implies

$$(\#\tau_{k+1}) \max_{\tau_{k+1} \text{ not near } \tau^*_{k+1}} |f^{\mathcal{B}}_{m,\tau_{k+1}}(x)| > \left(1 - \left(1 + \frac{1}{\log R}\right)^{-1}\right) |f^{\mathcal{B}}_{m,\tau_k}(x)|.$$

Relating the above to 3.1, for each $\tau_{k+1} \subseteq \tau_k$,

$$|f^{\mathcal{B}}_{m,\tau_{k+1}}(x)| \leq |f^{\mathcal{B}}_{m,\tau_{k+1}}(x) f^{\mathcal{B}}_{m,\tau^*_{k+1}}(x)|^{1/2},$$

and thus

$$|f^{\mathcal{B}}_{m,\tau_k}(x)| < (\#\tau_{k+1}) \left(1 - \left(1 + \frac{1}{\log R}\right)^{-1}\right)^{-1} \max_{\tau_{k+1} \text{ not near } \tau^*_{k+1}} |f^{\mathcal{B}}_{m,\tau_{k+1}}(x) f^{\mathcal{B}}_{m,\tau^*_{k+1}}(x)|^{1/2}.$$

The conclusion follows from the estimates

$$\left(1 - \left(1 + \frac{1}{\log R}\right)^{-1}\right)^{-1} \lesssim \log R$$

and

$$\#\tau_{k+1} \sim \frac{R_k^{-1}}{R_{k+1}^{-1}} = \log R,$$

and taking R sufficiently large so that an extra factor of $\log R$ exceeds the above implicit constants. □

We wish to use this to divide the integral of $|f^{\mathcal{B}}_m|^4$ into broad and narrow parts, with a small constant on narrow parts. For the narrow component, we wish to relate $\int |f^{\mathcal{B}}_m|^4$ to $\sum_\tau \int |f^{\mathcal{B}}_{m,\tau}|^4$, so that we may further decompose each $f^{\mathcal{B}}_{m,\tau}$ into broad and narrow components and proceed inductively.

Definition 3.6. We define $\text{Broad}_{1,m}$ to be the set

$$\text{Broad}_{1,m} = \left\{ x \in U^m_\alpha : |f^{\mathcal{B}}_m(x)| \leq (\log R)^3 \max_{\tau_1 \text{ not near } \tau'_1} |f^{\mathcal{B}}_{m,\tau_1}(x) f^{\mathcal{B}}_{m,\tau'_1}(x)|^{1/2} \right\}.$$

The complementary set $\text{Narrow}_{1,m}$ is defined as $U^m_\alpha \setminus \text{Broad}_{1,m}$.

Remark 3.7. It follows that $\text{Broad}_{1,m}$ may be covered by $O(N^2)$-many $\text{Br}^m_\alpha(\tau, \tau')$.

Lemma 3.8 (Decoupling the narrow part ($k = 1$))**.** *There are a collection of boxes τ^*_1 of diameter $\sim R_1^{-1}$ such that*

$$\int_{\text{Narrow}_{1,m}} |f^{\mathcal{B}}_m|^4 \leq \left(1 + \frac{2}{\log R}\right)^4 \sum_{\tau^*_1} \int_{\text{Narrow}_{1,m}} |f^{\mathcal{B}}_{m,\tau^*_1}|^4.$$

Proof. If $x \in \text{Narrow}_{1,m}$, then by the narrow lemma 3.5

$$|f^{\mathcal{B}}_m(x)| \leq \left(1 + \frac{1}{\log R}\right) |f^{\mathcal{B}}_{m,\tau^*_1}(x)|$$

for a suitable rectangle τ_1^* which is a union of three consecutive rectangles τ_1, so certainly for the decomposition $\{\tau_1^*\}$ of $\mathcal{N}_{3R_1^{-1}}\mathbb{P}^1$ into rectangles of diameter $\sim 3R_1^{-1}$, we have

$$|f_m^{\mathcal{B}}(x)| \leq \left(1+\frac{1}{\log R}\right)(\sum_{\tau_1^*}|f_{m,\tau_1^*}^{\mathcal{B}}(x)|^4)^{1/4},$$

and hence

$$\int_{\mathrm{Narrow}_{1,m}}|f_m^{\mathcal{B}}|^4 \leq \left(1+\frac{2}{\log R}\right)^4\sum_{\tau_1^*}\int_{\mathrm{Narrow}_{1,m}}|f_{m,\tau_1^*}^{\mathcal{B}}|^4.$$

□

Definition 3.9. Write, for each τ_1^*,

$$\mathrm{Broad}_{2,m}(\tau_1^*) := \left\{x \in \mathrm{Narrow}_{1,m} : |f_{\tau_1^*,m}^{\mathcal{B}}(x)| \leq (\log R)^3 \max_{\substack{\tau_2 \text{ not near } \tau_2' \\ \tau_2,\tau_2'\subseteq\tau_1^*}} |f_{\tau_2,m}^{\mathcal{B}}(x)f_{\tau_2',m}^{\mathcal{B}}(x)|^{1/2}\right\}$$

where as usual each τ_2 has diameter $\sim R_2^{-1}$. Write also $\mathrm{Narrow}_{2,m}(\tau_1^*) := \mathrm{Narrow}_{1,m} \setminus \mathrm{Broad}_{2,m}(\tau_1^*)$.

Definition 3.10. Let $2 \leq k < N$. Suppose $\tau_k^* \subseteq \tau_{k-1}^*$ have diameter $\sim R_k^{-1}, \sim R_{k-1}^{-1}$, respectively. We inductively write

$$\mathrm{Broad}_{k+1,m}(\tau_k^*) := \left\{x \in \mathrm{Narrow}_{k,m}(\tau_{k-1}^*) : |f_{\tau_k^*,m}^{\mathcal{B}}(x)| \leq (\log R)^3 \max_{\substack{\tau_{k+1} \text{ not near } \tau_{k+1}' \\ \tau_{k+1},\tau_{k+1}'\subseteq\tau_k^*}} |f_{\tau_{k+1},m}^{\mathcal{B}}(x)f_{\tau_{k+1}',m}^{\mathcal{B}}(x)|^{1/2}\right\}$$

and $\mathrm{Narrow}_{k+1,m}(\tau_k^*) := \mathrm{Narrow}_{k,m}(\tau_{k-1}^*) \setminus \mathrm{Broad}_{k+1,m}(\tau_k^*)$.

Lemma 3.11 (Decoupling the narrow part ($k \geq 2$))**.** *Fix any $2 \leq k \leq N$. Then, for each τ_{k-1}^*,*

$$\int_{\mathrm{Narrow}_{k,m}(\tau_{k-1}^*)}|f_{\tau_{k-1}^*,m}^{\mathcal{B}}|^4 \leq \left(1+\frac{2}{\log R}\right)^4\sum_{\tau_k^*\subseteq\tau_{k-1}^*}\int_{\mathrm{Narrow}_{k,m}(\tau_{k-1}^*)}|f_{\tau_k^*,m}^{\mathcal{B}}|^4.$$

Proof. The argument is identical to 3.8. □

Combining Lemmas 3.8 and 3.11, we conclude

$$\int_{U_\alpha^m}|f_m^{\mathcal{B}}|^4 \leq \left(1+\frac{2}{\log R}\right)^{4N}\sum_{\tau_{N-1}^*}\int_{\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)}\sum_{\tau_N^*\subseteq\tau_{N-1}^*}|f_{\tau_N^*,m}^{\mathcal{B}}|^4 + \int_{\mathrm{Broad}_{1,m}}|f_m^{\mathcal{B}}|^4$$
$$+\sum_{k=2}^{N}\left(1+\frac{2}{\log R}\right)^{4k}\sum_{\tau_{k-1}^*}\int_{\mathrm{Broad}_{k,m}(\tau_{k-1}^*)}|f_{\tau_{k-1}^*,m}^{\mathcal{B}}|^4.$$

Remark 3.12. For $1 \leq k \leq N$,

$$\left(1+\frac{2}{\log R}\right)^{4k} \leq \left(1+\frac{2}{\log R}\right)^{\frac{\log R}{2}\frac{4}{\log\log R}} \lesssim 1.$$

Our next steps are bounding each of the summands in turn.

Lemma 3.13 (Narrow bound)**.** *We have*

$$\sum_{\tau_{N-1}^*}\int_{\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)}\sum_{\tau_N^*\subseteq\tau_{N-1}^*}|f^{\mathcal{B}}_{\tau_N^*,m}|^4\lesssim\sum_{U\in\mathcal{G}_\theta}|U|^{-1}\left(\int|f_\theta|^2(y)W_U(y)dy\right)^2.$$

Proof. Note that each τ_N^* is either equal to some θ or a union of three adjacent θ. In particular, for each $x\in\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)$,

$$\sum_{\tau_N^*\subseteq\tau_{N-1}^*}|f^{\mathcal{B}}_{\tau_N^*,m}(x)|^4\le 3^3\sum_{\theta\subseteq\tau_{N-1}^*}|f^{\mathcal{B}}_{m,\theta}(x)|^4,$$

hence

$$\begin{aligned}\sum_{\tau_{N-1}^*}\int_{\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)}\sum_{\tau_N^*\subseteq\tau_{N-1}^*}|f^{\mathcal{B}}_{\tau_N^*,m}|^4&\lesssim\sum_{\tau_{N-1}^*}\int_{\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)}\sum_{\theta\subseteq\tau_{N-1}^*}|f^{\mathcal{B}}_{m,\theta}|^4\\&\le\sum_{\tau_{N-1}^*}\int_{B_R}\sum_{\theta\subseteq\tau_{N-1}^*}|f_{N,\theta}|^4.\end{aligned}$$

By the definition of the pruning, for each θ,

$$\int_{B_R}|f_{N,\theta}|^4=\int_{B_R}|\sum_{U\in\mathcal{G}_\theta}\psi_U f_\theta|^4=\sum_{U\in\mathcal{G}_\theta}\int_{B_R}|\psi_U|^4|f_\theta|^4.$$

Since each $|\psi_U|\le 1$, we have the trivial bound

$$\sum_{U\in\mathcal{G}_\theta}\int_{B_R}|\psi_U|^4|f_\theta|^4\le\sum_{U\in\mathcal{G}_\theta}\int_{B_R}|\psi_U|^2|f_\theta|^4.$$

By the local constancy lemma (a),

$$\begin{aligned}\sum_{U\in\mathcal{G}_\theta}\int_{B_R}|\psi_U|^2|f_\theta|^4&\lesssim\sum_{U\in\mathcal{G}_\theta}\int_{B_R}|\psi_U|^2\left(|f_\theta|^2*|\rho_\theta^\vee|\right)^2\\&=\sum_{U\in\mathcal{G}_\theta}\int_{B_R}|\psi_U|^2(x)\left(\int|f_\theta|^2(y)|\rho_\theta^\vee|(x-y)dy\right)^2dx.\end{aligned}$$

By Minkowski,

$$\begin{aligned}&\sum_{U\in\mathcal{G}_\theta}\int_{B_R}|\psi_U|^2(x)\left(\int|f_\theta|^2(y)|\rho_\theta^\vee|(x-y)dy\right)^2dx\\&\qquad\le\sum_{U\in\mathcal{G}_\theta}\left(\int|f_\theta|^2(y)\left(\int|\psi_U|^2(x)|\rho_\theta^\vee|^2(x-y)dx\right)^{1/2}dy\right)^2\\&\qquad\le\sum_{U\in\mathcal{G}_\theta}\left(\int|f_\theta|^2(y)\left(\int|\psi_U|^2(x)|\rho_\theta^\vee|^2(x-y)dx\right)^{1/2}dy\right)^2.\end{aligned}$$

By the rapid decay of $\rho_\theta^\vee$ outside of θ^*,

$$\int|\psi_U|^2(x)|\rho_\theta^\vee|^2(x-y)dx\lesssim\sup_{x\in y+\theta^*}|\psi_U|^2(x)\int|\rho_\theta^\vee|^2(x-y)dx\lesssim W_U^2(y)|U|^{-1},$$

and so

$$\sum_{U\in\mathcal{G}_\theta}\left(\int|f_\theta|^2(y)\left(\int|\psi_U|^2(x)|\rho_\theta^\vee|^2(x-y)dx\right)^{1/2}\right)^2dy\lesssim\sum_{U\in\mathcal{G}_\theta}|U|^{-1}\left(\int|f_\theta|^2(y)W_U(y)dy\right)^2,$$

and hence

$$\sum_{\tau_{N-1}^*}\int_{\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)}\sum_{\tau_N^*\subseteq\tau_{N-1}^*}|f^{\mathcal{B}}_{\tau_N^*,m}|^4\lesssim\sum_{U\in\mathcal{G}_\theta}|U|^{-1}\left(\int|f_\theta|^2(y)W_U(y)dy\right)^2$$

as claimed. ☐

Lemma 3.14 (Broad bound, $k=1$). *We have*

$$\alpha^4|\mathrm{Broad}_{1,m}|\lesssim(\log R)^{10}\sum_{m<k<N}\sum_{\tau:\ell(\tau)=R_k^{-1}}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau}|f_\theta|^2\right)^2.$$

Proof. Suppose first $m=1$. Then $\mathrm{Broad}_{1,1}\subseteq U_\alpha^1$, so by 3.1 we have

$$\alpha^4|\mathrm{Broad}_{1,1}|\lesssim(\log R)^8\sum_{U\in\mathcal{G}_{\tau_1}}|U|\left(⨍_U\sum_{\theta}|f_\theta|^2\right)^2.$$

Suppose next $2\le m\le N$. By the definition of the first broad set,

$$\mathrm{Broad}_{1,m}=\bigcup_{\tau_1\text{ not near }\tau_1'}\mathrm{Br}_\alpha^m(\tau_1,\tau_1'),$$

and so, since there are $\lesssim\log R$ -many τ_1,

$$\alpha^4|\mathrm{Broad}_{1,m}|\lesssim(\log R)^2\alpha^4\max_{\tau_1\text{ not near }\tau_1'}|\mathrm{Br}_\alpha^m(\tau_1,\tau_1')|.$$

By Prop. 3.3, we conclude that

$$\alpha^4|\mathrm{Broad}_{1,m}|\lesssim(\log R)^{10}\sum_{m<k<N}\sum_{\tau:\ell(\tau)=R_k^{-1}}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau}|f_\theta|^2\right)^2.$$

☐

Lemma 3.15 (Broad bound, $2\le k\le N$). *We have*

$$\sum_{\tau_{k-1}^*}\int_{\mathrm{Broad}_{k,m}(\tau_{k-1}^*)}|f^{\mathcal{B}}_{\tau_{k-1}^*,m}|^4\lesssim(\log R)^{19}\sum_{m<s\le N}\sum_{\tau_s\subseteq\tau_{k-1}^*}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau_s}|f^{\mathcal{B}}_{m,\theta}|^2\right)^2.$$

Proof. By the definition of the broad set, for each τ_{k-1}^* and each $x\in\mathrm{Broad}_{k,m}(\tau_{k-1}^*)$ there is some pair $\tau_k^*,\tau_k^{*\prime}\subseteq\tau_{k-1}^*$ not near such $|f^{\mathcal{B}}_{m,\tau_{k-1}^*}(x)|\le(\log R)^3|f^{\mathcal{B}}_{m,\tau_k^*}(x)f^{\mathcal{B}}_{m,\tau_k^{*\prime}}(x)|^{1/2}$, i.e.,

$$\mathrm{Broad}_{k,m}(\tau_{k-1}^*)\subseteq\bigcup_{\substack{\tau_k^*,\tau_k^{*\prime}\subseteq\tau_{k-1}^*\\ \tau_k^*\text{ not near }\tau_k^{*\prime}}}\mathrm{Br}_\alpha^m(\tau_k^*,\tau_k^{*\prime}).$$

Thus

$$\sum_{\tau_{k-1}^*}\int_{\mathrm{Broad}_{k,m}(\tau_{k-1}^*)}|f^{\mathcal{B}}_{m,\tau_{k-1}^*}|^4\lesssim(\log R)^{12}\sum_{\tau_{k-1}^*}\sum_{\substack{\tau_k^*,\tau_k^{*\prime}\subseteq\tau_{k-1}^*\\ \tau_k^*\text{ not near }\tau_k^{*\prime}}}\int_{\mathrm{Br}_\alpha^m(\tau_k^*,\tau_k^{*\prime})}|f^{\mathcal{B}}_{m,\tau_{k-1}^*}|^4.$$

By Prop. 3.3,

$$\sum_{\tau_{k-1}^*}\sum_{\substack{\tau_k^*,\tau_k^{*\prime}\subseteq\tau_{k-1}^*\\ \tau_k^*\text{ not near }\tau_k^{*\prime}}}\int_{\mathrm{Br}_\alpha^m(\tau_k^*,\tau_k^{*\prime})}|f_{m,\tau_{k-1}^*}^{\mathcal{B}}|^4\lesssim(\log R)^8\sum_{m<s\le N}\sum_{\tau_s}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau}|f_\theta|^2\right)^2,$$

where we have used that there are $O((\log R)^2)$-many pairs $\tau_k^*,\tau_k^{*\prime}\subseteq\tau_{k-1}^*$.

□

Proof of Prop. 3.4. By the replacement lemma 2.7,

$$\alpha^4|U_\alpha|\le\alpha^4|V_\alpha|.$$

Write

$$\alpha^4|V_\alpha|\le\sum_{m=1}^N\alpha^4|U_\alpha^m|$$

and, by the definition of the broad/narrow sets above, we may write

$$\alpha^4|U_\alpha^m|=\alpha^4|\mathrm{Broad}_{1,m}|+\alpha^4|\mathrm{Narrow}_{1,m}|.$$

Since $\mathrm{Narrow}_{1,m}\subseteq U_\alpha^m$, we have the bound

$$\alpha^4|\mathrm{Narrow}_{1,m}|\lesssim\int_{\mathrm{Narrow}_{1,m}}|f_m^{\mathcal{B}}|^4.$$

By the definition of the broad/narrow sets,

$$\begin{aligned}\int_{\mathrm{Narrow}_{1,m}}|f_m^{\mathcal{B}}|^4\le&\left(1+\frac{2}{\log R}\right)^{4N}\sum_{\tau_{N-1}^*}\int_{\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)}\sum_{\tau_N^*\subseteq\tau_{N-1}^*}|f_{\tau_N^*,m}^{\mathcal{B}}|^4\\&+\sum_{k=2}^N\left(1+\frac{2}{\log R}\right)^{4k}\sum_{\tau_{k-1}^*}\int_{\mathrm{Broad}_{k,m}(\tau_{k-1}^*)}|f_{\tau_{k-1}^*,m}^{\mathcal{B}}|^4.\end{aligned}$$

By the narrow bound 3.13,

$$\sum_{\tau_{N-1}^*}\int_{\mathrm{Narrow}_{N,m}(\tau_{N-1}^*)}\sum_{\tau_N^*\subseteq\tau_{N-1}^*}|f_{\tau_N^*,m}^{\mathcal{B}}|^4\lesssim\sum_{U\in\mathcal{G}_\theta}|U|^{-1}\left(\int|f_\theta|^2(y)W_U(y)dy\right)^2.$$

By the broad bounds 3.14 and 3.15,

$$\alpha^4|\mathrm{Broad}_{1,m}|\lesssim(\log R)^{10}\sum_{m<k<N}\sum_{\tau:\ell(\tau)=R_k^{-1}}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau}|f_\theta|^2\right)^2$$

and

$$\sum_{\tau_{k-1}^*}\int_{\mathrm{Broad}_{k,m}(\tau_{k-1}^*)}|f_{\tau_{k-1}^*,m}^{\mathcal{B}}|^4\lesssim(\log R)^{30}\sum_{m<s\le N}\sum_{\tau_s}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau_s}|f_{m,\theta}^{\mathcal{B}}|^2\right)^2.$$

Thus

$$\alpha^4|U_\alpha^m|\lesssim(\log R)^{30}\sum_{m<s\le N}\sum_{\tau_s}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau_s}|f_{m,\theta}^{\mathcal{B}}|^2\right)^2,$$

and hence, since $N\le\log R$,

$$\alpha^4|U_\alpha|\lesssim(\log R)^{31}\sum_{1\le s\le N}\sum_{\tau_s}\sum_{U\in\mathcal{G}_\tau}|U|\left(⨍_U\sum_{\theta\subseteq\tau_s}|f_{m,\theta}^{\mathcal{B}}|^2\right)^2$$

as claimed.

□

3.3 Reduction to local estimates

In the above subsections we produced bounds on the measure of the set $U_\alpha = \{x \in B_R : |f(x)| > \alpha\}$. In this subsection we note that, if we can prove Theorem 1.2 in the special case that $\{x \in \mathbb{R}^2 : |f(x)| > \alpha\} \subseteq Q_R$ for a suitable cube Q_R of radius R, then we can conclude that Theorem 1.2 is true in the general case.

Proof that Prop. 3.4 implies Theorem 1.2. Fix a $O(1)$-overlapping cover of $\mathbb{R}^2$ by cubes Q_R of radius R. Write ρ_{B_R} for a Schwartz function satisfying the following criteria:

- ρ_{B_R} radially symmetric, real, and nonnegative.
- $\rho_{B_R} \gtrsim 1_{B_R}$.
- $\mathrm{supp}(\hat{\rho}_{B_R}) \subseteq B_{2/R}$.
- $\sum_{Q_R} \rho_{B_R}(c_{Q_R} - \cdot) \lesssim 1$.
- ρ_{B_R} decays rapidly outside of B_R.

For each Q_R, write $\rho_{Q_R} = \rho_{B_R}(c_{Q_R} - \cdot)$. By the triangle inequality, there is a subcollection Θ of the θ such that, writing $f' = \sum_{\theta \in \Theta} f_\theta$, we have

$$\alpha^4 |\{x \in \mathbb{R}^2 : |f(x)| > 10\alpha\}| \lesssim \alpha^4 |\{x \in \mathbb{R}^2 : |f'(x)| > \alpha\}|,$$

and such that the $2R^{-1}$-neighborhoods of the $\theta \in \Theta$ are pairwise disjoint. Then $f'\rho_{Q_R}$ has Fourier support in the $\sim R^{-1}$-neighborhood of $\mathbb{P}^1$. By Prop. 3.4, for each Q'_R,

$$\alpha^4 |\{x \in Q_R : |f'\rho_{Q'_R}(x)| > \alpha\}| \lesssim \sum_{\substack{R^{-1/2} \le s \le 1 \\ s \text{ dyadic}}} \sum_{\tau : \ell(\tau) = s} \sum_{U \in \mathcal{G}_\tau} |U|^{-1} \|S_U[f'\rho_{Q'_R}]\|_2^4.$$

By trivial bounds on f and the rapid decay of ρ_{B_R},

$$\{x \in \mathbb{R}^2 : |f'\rho_{Q'_R}(x)| > \alpha\} \subseteq 2Q'_R,$$

and so

$$\sum_{Q_R} \alpha^4 |\{x \in Q_R : |f'\rho_{Q'_R}(x)| > \alpha\}| \lesssim \max_{Q_R} \alpha^4 |\{x \in Q_R : |f'\rho_{Q'_R}(x)| > \alpha\}|.$$

By Proposition 3.4, for each Q_R,

$$\alpha^4 |\{x \in Q_R : |f'\rho_{Q'_R}(x)| > \alpha\}| \lesssim (\log R)^{31} \sum_{\substack{R^{-1/2} \le s \le 1 \\ s \text{ dyadic}}} \sum_{\tau : \ell(\tau) = s} \sum_{U \in \mathcal{G}_\tau} |U|^{-1} \|S_U[f'\rho_{Q'_R}]\|_2^4.$$

Adding over all Q'_R, we get

$$\sum_{Q_R, Q'_R} \alpha^4 |\{x \in Q_R : |f'\rho_{Q'_R}(x)| > \alpha\}| \lesssim (\log R)^{31} \sum_{Q'_R} \sum_{\substack{R^{-1/2} \le s \le 1 \\ s \text{ dyadic}}} \sum_{\tau : \ell(\tau) = s} \sum_{U \in \mathcal{G}_\tau} |U|^{-1} \|S_U[f'\rho_{Q'_R}]\|_2^4.$$

If we commute the sum over Q'_R to the inside and use a trivial estimate we conclude

$$\sum_{Q_R, Q'_R} \alpha^4 |\{x \in Q_R : |f'\rho_{Q'_R}(x)| > \alpha\}| \lesssim (\log R)^{31} \sum_{\substack{R^{-1/2} \le s \le 1 \\ s \text{ dyadic}}} \sum_{\tau : \ell(\tau) = s} \sum_{U \in \mathcal{G}_\tau} |U|^{-1} \|S_U[\sum_{Q'_R} f'\rho_{Q'_R}]\|_2^4,$$

i.e.

$$\sum_{Q'_R}\alpha^4|\{x\in\mathbb{R}^2:|f'\rho_{Q'_R}(x)|>\alpha\}|\lesssim(\log R)^{31}\sum_{\substack{R^{-1/2}\le s\le 1\\ s\text{ dyadic}}}\sum_{\tau:\ell(\tau)=s}\sum_{U\in\mathcal{G}_\tau}|U|^{-1}\|S_Uf'\|_2^4.$$

Finally, by rapid decay,

$$\sum_{Q'_R}\alpha^4|\{x\in\mathbb{R}^2:|f'\rho_{Q'_R}(x)|>\alpha\}|\gtrsim\alpha^4|\{x\in\mathbb{R}^2:|f'(x)|\gtrsim\alpha\}|,$$

whereas trivially $S_Uf\ge S_Uf'$ pointwise, so we conclude

$$\alpha^4|\{x\in\mathbb{R}^2:|f(x)|\gtrsim\alpha\}|\lesssim(\log R)^{31}\sum_{\substack{R^{-1/2}\le s\le 1\\ s\text{ dyadic}}}\sum_{\tau:\ell(\tau)=s}\sum_{U\in\mathcal{G}_\tau}|U|^{-1}\|S_Uf\|_2^4.$$

Since this is true for all choices of α, we may change variables to conclude

$$\alpha^4|\{x\in\mathbb{R}^2:|f(x)|>\alpha\}|\lesssim(\log R)^{31}\sum_{\substack{R^{-1/2}\le s\le 1\\ s\text{ dyadic}}}\sum_{\tau:\ell(\tau)=s}\sum_{U\in\mathcal{G}_\tau}|U|^{-1}\|S_Uf\|_2^4,$$

as claimed.

□

4 Proof of Theorem 1.1

In this section we verify that the wave envelope estimate 1.2 is strong enough to imply Theorem 1.1. This is essentially proven in section 4 of [10], but the latter included $O_\varepsilon(R^\varepsilon)$-lossy pigeonholing steps. Here we perform a more restricted pigeonholing which suffices for the result, and then quote the corresponding incidence geometry calculation in [10].

We will focus on the case $p\ge 4$, where we will have an upper bound for Theorem 1.1 with power law $R^{\beta(p-\frac{p}{q}-1)-1}$. Under the assumption $\frac{3}{p}+\frac{1}{q}\le 1$, the remaining case is $3\le p\le 4$, where an upper bound $\max\left(1,R^{\beta(\frac{p}{2}-\frac{p}{q})}\right)$ is needed; this is handled in section 4 of [10], and the proof there requires no modification for our purposes.

We begin with the partial decoupling statement in the case $p\ge 4$.

Proposition 4.1. Suppose $p\ge 4$. Let $1\le k\le N$ be arbitrary, and fix a canonical scale cap τ_k. Suppose as before that $\Gamma_\beta(R^{-1})$ is a partition of $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ into approximate $R^{-\beta}\times R^{-1}$ boxes γ. Assume $f=\sum_\gamma f_\gamma$ satisfies the following regularity properties:

(a) $\|f_\gamma\|_\infty\sim 1$ or $f_\gamma=0$ for each γ.

(b) $\|f_\gamma\|_p^p\sim_p\|f_\gamma\|_2^2$ for each γ and each $p\ge 1$.

Write γ_k for approximate boxes of dimensions $\sim\max(R^{-\beta},R_k/R)\times R^{-1}$. Then

$$\begin{aligned}\sum_{U\in\mathcal{G}_{\tau_k}}|U|\left(\fint_U\sum_{\theta\subseteq\tau_k}|f_\theta|^2\right)^2\lesssim&(\log R)^{p-4}(\#\tau_k)^{p-4}\alpha^{4-p}\\ &\times\left(\max_{\gamma_k\subseteq\tau_k}\#(\gamma\subseteq\gamma_k)\times\#(\gamma\subseteq\tau_k)\right)^{\frac{p}{2}-1}\sum_{\gamma\subseteq\tau_k}\|f_\gamma\|_p^p,\end{aligned}\tag{4.1}$$

where E_2 is as in the statement of Theorem 1.2.

Proof. For each $\theta \subseteq \tau_k$, the small caps $\gamma_k \subseteq \theta$ are either adjacent or are $\sim \max(R^{-\beta}, R_k/R) \geq R_k/R$-separated. Fix any $U \in \mathcal{G}_{\tau_k}$. Since $U \| U_{\tau_k,R}$ has dimensions $R/R_k \times R$, we conclude that the f_{γ_k} are locally orthogonal on U. Thus

$$\int W_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \lesssim \int W_U \sum_{\gamma_k \subseteq \tau_k} |f_{\gamma_k}|^2,$$

and so, appealing to the definition of $\mathcal{G}_{\tau_k}$,

$$\frac{\alpha^2}{(\#\tau_k)^2} \lesssim (\log R)^2 |U|^{-1} \int W_U \sum_{\gamma_k \subseteq \tau_k} |f_{\gamma_k}|^2,$$

where we have suppressed the dependence on $C_{\mathfrak{p}}$. Multiplying the left-hand side of 4.1 by the $(\frac{p}{2}-2)$-power of the latter display, we obtain the estimate

$$\sum_{U \in \mathcal{G}_{\tau_k}} |U| \left(⨍_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \right)^2 \lesssim (\#\tau_k)^{p-4} \alpha^{4-p} (\log R)^{p-4} \sum_{U \in \mathcal{G}_{\tau_k}} |U| \left(⨍_U \sum_{\gamma_k \subseteq \tau_k} |f_{\gamma_k}|^2 \right)^{\frac{p}{2}}. \tag{4.2}$$

Uniformity assumption (a) implies

$$\| \sum_{\gamma_k \subseteq \tau_k} |f_{\gamma_k}|^2 \|_\infty \lesssim [\max_{\gamma_k \subseteq \tau_k} \#(\gamma \subseteq \gamma_k)] \times \#(\gamma \subseteq \tau_k).$$

By removing factors of $\| \sum_{\gamma_k \subseteq \tau_k} |f_{\gamma_k}|^2 \|_\infty$ from 4.2, we obtain

$$\begin{aligned} \sum_{U \in \mathcal{G}_{\tau_k}} |U| \left(⨍_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \right)^2 \lesssim & (\#\tau_k)^{p-4} \alpha^{4-p} (\log R)^{p-4} \\ & \times \left(\max_{\gamma_k \subseteq \tau_k} \#(\gamma \subseteq \gamma_k) \times \#(\gamma \subseteq \tau_k) \right)^{\frac{p}{2}-1} \\ & \times \sum_{U \in \mathcal{G}_{\tau_k}} \int W_U \sum_{\gamma_k \subseteq \tau_k} |f_{\gamma_k}|^2, \end{aligned}$$

and by local orthogonality and uniformity assumption (b)

$$\sum_{U \in \mathcal{G}_{\tau_k}} \int W_U \sum_{\gamma_k \subseteq \tau_k} |f_{\gamma_k}|^2 \lesssim \int \sum_{\gamma \subseteq \tau_k} |f_\gamma|^2 \sim_p \sum_{\gamma \subseteq \tau_k} \|f_\gamma\|_p^p.$$

Together we get the estimate

$$\begin{aligned} \sum_{U \in \mathcal{G}_{\tau_k}} |U| \left(⨍_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \right)^2 \lesssim & (\log R)^{p-4} (\#\tau_k)^{p-4} \alpha^{4-p} \\ & \times \left(\max_{\gamma_k \subseteq \tau_k} \#(\gamma \subseteq \gamma_k) \times \#(\gamma \subseteq \tau_k) \right)^{\frac{p}{2}-1} \sum_{\gamma \subseteq \tau_k} \|f_\gamma\|_p^p, \end{aligned}$$

as claimed. □

Remark 4.2. Suppose that $p = 2 + 2/\beta$ and $q = p$. Plugging in the bounds $\#\tau_k \leq R_k$, $\max_{\gamma_k \subseteq \tau_k} \#(\gamma \subseteq \gamma_k) \leq \max(1, R^{\beta-1} R_k)$, and $\#(\gamma \subseteq \tau_k) \leq R_k^{-1} R^\beta$, and $R_k \leq R^{1/2}$, this immediately implies the estimate

$$\sum_{U \in \mathcal{G}_{\tau_k}} |U| \left(⨍_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \right)^2 \lesssim (\log R)^{p-4} \alpha^{4-p} R^{\beta(p-2)-1} \sum_{\gamma \subseteq \tau_k} \|f_\gamma\|_p^p,$$

and hence, by Theorem 1.2,

$$\alpha^p |U_\alpha| \lesssim (\log R)^{E_2+p-3} R^{\beta(p-2)-1} \sum_\gamma \|f_\gamma\|_p^p$$

as claimed. It essentially remains to remove assumptions (a) and (b) above, and to track the dependence on q.

Proof of Theorem 1.1 for $p \geq 4$. Consider the decomposition

$$f = \sum_{\gamma \in \Gamma_\beta(R^{-1})} f_\gamma \ .$$

By scaling we may assume that $\max_\theta \|f_\theta\|_\infty = 1$. Then we may write

$$f = \sum_{R^{-O(1)} \leq \lambda \leq R^{O(1)}} \sum_{\substack{\gamma \in \Gamma_\beta(R^{-1}) \\ \|f_\gamma\|_\infty \sim \lambda}} f_\gamma + R^{-1000}\eta,$$

where the λ range over dyadic numbers, and η is rapidly decaying outside of B_R. We abbreviate

$$\Gamma_\beta^\lambda(R^{-1}) = \{\gamma \in \Gamma_\beta(R^{-1}) : \|f_\gamma\|_\infty \sim \lambda\}.$$

Then, for each λ, consider the wave envelope expansion

$$\sum_{\gamma \in \Gamma_\beta^\lambda(R^{-1})} f_\gamma = \sum_{\gamma \in \Gamma_\beta^\lambda(R^{-1})} \sum_U \psi_U f_\gamma,$$

where each U has dimensions $\sim R^\beta \times R$ and has long edge parallel to $\mathbf{n}_{c_\gamma}$. Since $\gamma \in \Gamma_\beta^\lambda(R^{-1})$, there is some U such that $\|\psi_U f\|_\infty \sim \lambda$. If we write $\mathcal{U}_\lambda = \mathcal{U}_\lambda^\gamma$ for the set of U for which $\|\psi_U f_\gamma\|_\infty \sim \lambda$, then for all $\gamma \in \Gamma_\beta^\lambda(R^{-1})$

$$\|\sum_{U \in \mathcal{U}_\lambda} \psi_U f_\gamma\|_p^p \sim_p (\#\mathcal{U}_\lambda)|U|\lambda^p,$$

and so

$$\|\frac{1}{\lambda}\sum_{U \in \mathcal{U}_\lambda} \psi_U f_\gamma\|_p^p \sim_p (\#\mathcal{U}_\lambda)|U| \sim \|\frac{1}{\lambda}\sum_{U \in \mathcal{U}_\lambda} \psi_U f_\gamma\|_2^2.$$

For each $1 \leq \mathfrak{r} \leq R$ dyadic and each λ, write $\Gamma_\beta^{\lambda;\mathfrak{r}}(R^{-1})$ to be the collection of $\gamma \in \Gamma_\beta^\lambda(R^{-1})$ such that $\#\mathcal{U}_\lambda^\gamma \sim \mathfrak{r}$. Define for $\gamma \in \Gamma_\beta^\lambda(R^{-1})$

$$g_\gamma^{(\lambda)} = \frac{1}{\lambda}\sum_{U \in \mathcal{U}_\lambda} \psi_U f_\gamma$$

and

$$g^{(\lambda,\mathfrak{r})} = \sum_{\gamma \in \Gamma_\beta^{\lambda;\mathfrak{r}}(R^{-1})} g_\gamma^{(\lambda)}.$$

Then for each $\lambda, \mathfrak{r}$, and $\mathfrak{a} > 0$ we have

$$\mathfrak{a}^4 |\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}| \lesssim (\log R)^{E_2} \sum_{1 \leq k \leq N} \sum_{\tau_k} \sum_{U \in \mathcal{G}_{\tau_k}[\mathfrak{a}]} |U| \left(\fint_U \sum_{\theta \subseteq \tau_k} |\lambda g_\theta^{(\lambda,\mathfrak{r})}|^2 \right)^2,$$

where we have written $\mathcal{G}_{\tau_k}[\mathfrak{a}]$ to record that the pruning is with respect to the parameter $\mathfrak{a}$. For each $1 \le k \le N$ and each $1 \le \mathfrak{s} \le R^{1/2}$, let $\mathcal{T}_k(\mathfrak{s})$ denote the collection of τ_k such that $\#\{\gamma \subseteq \tau_k : g_\gamma^{(\lambda,\mathfrak{r})} \neq 0\} \sim \mathfrak{s}$. By pigeonholing, for each k we may find $\mathfrak{s}_* = \mathfrak{s}_*^k$ such that

$$\mathfrak{a}^4 |\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}| \lesssim (\log R)^{E_2+1} \sum_{1\le k\le N} \sum_{\tau_k \in \mathcal{T}_k(\mathfrak{s}_*^k)} \sum_{U \in \mathcal{G}_{\tau_k}[\mathfrak{a}]} |U| \left(\fint_U \sum_{\theta \subseteq \tau_k} |\lambda g_\theta^{(\lambda,\mathfrak{r})}|^2 \right)^2 .$$

By Prop. 4.1, we have

$$\mathfrak{a}^p |\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}| \lesssim (\log R)^{E_2+p-3} \sum_{1\le k\le N} (\#\mathcal{T}_k(\mathfrak{s}_*^k))^{p-4} \sum_{\tau_k \in \mathcal{T}_k(\mathfrak{s}_*^k)} \left(\mathfrak{s}_*^k \max_{\gamma_k \subseteq \tau_k} \#(\gamma \subseteq \gamma_k) \right)^{\frac{p}{2}-1} \sum_{\gamma \subseteq \tau_k} \|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^p,$$

and by pigeonholing to a single dyadic $1 \le k_* \le N < \log R$ we have

$$\begin{aligned} &\mathfrak{a}^p |\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}| \\ &\quad \lesssim (\log R)^{E_2+p-2} (\#\mathcal{T}_{k_*}(\mathfrak{s}_*^{k_*}))^{p-4} \sum_{\tau_{k_*} \in \mathcal{T}_{k_*}(\mathfrak{s}_*^{k_*})} \left(\mathfrak{s}_*^{k_*} \max_{\gamma_{k_*} \subseteq \tau_{k_*}} \#(\gamma \subseteq \gamma_{k_*}) \right)^{\frac{p}{2}-1} \sum_{\gamma \subseteq \tau_{k_*}} \|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^p \\ &\quad \sim_p (\log R)^{E_2+p-2} (\#\mathcal{T}_{k_*}(\mathfrak{s}_*^{k_*}))^{p-3} \left(\mathfrak{s}_*^{k_*} \max_{\gamma_{k_*} \subseteq \tau_{k_*}} \#(\gamma \subseteq \gamma_{k_*}) \right)^{\frac{p}{2}-1} \mathfrak{s}_*^{k_*} \lambda^p \mathfrak{r} |U|. \end{aligned}$$

We now appeal to the following lemma, which is essentially proved in [10]:

Lemma 4.3 (Case 2 of the proof of Theorem 5 in [10])**.**

$$(\#\mathcal{T}_{k_*}(\mathfrak{s}_*^{k_*}))^{p-4} \left(\mathfrak{s}_*^{k_*} \max_{\substack{\gamma_{k_*} \subseteq \tau_{k_*} \\ \tau_{k_*} \in \mathcal{T}_{k_*}(\mathfrak{s}_*^{k_*})}} \#(\gamma \subseteq \gamma_{k_*}) \right)^{\frac{p}{2}-1} \lesssim R^{\beta(p-\frac{p}{q}-1)-1} \left(\mathfrak{s}_*^{k_*} \times \#\mathcal{T}_{k_*}(\mathfrak{s}_*^{k_*}) \right)^{\frac{p}{q}-1}.$$

. □

Consequently,

$$\begin{aligned} &\mathfrak{a}^p \left| \left\{ x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a} \right\} \right| \\ &\quad \lesssim_p (\log R)^{E_2+p-2} R^{\beta(p-\frac{p}{q}-1)-1} \left(\#\mathcal{T}_{k_*}(\mathfrak{s}_*^{k_*}) \right)^{\frac{p}{q}} (\mathfrak{s}_*^{k_*})^{\frac{p}{q}} \lambda^p \mathfrak{r} |U| \\ &\quad \sim_p (\log R)^{E_2+p-2} R^{\beta(p-\frac{p}{q}-1)-1} \left(\sum_\gamma \|g_\gamma^{(\lambda,\mathfrak{r})}\|_p^q \right)^{\frac{p}{q}}. \end{aligned} \tag{4.3}$$

On the other hand,

$$f = \sum_{R^{-O(1)} \le \lambda \le R^{O(1)}} \lambda g^{(\lambda)} + R^{-1000} \eta,$$

and consequently, for a suitable λ_*,

$$\begin{aligned} \alpha^p |\{x : |f(x)| > \alpha\}| &\le \alpha^p \sum_{R^{-O(1)} \le \lambda \le R^{O(1)}} \left| \left\{ x : |\lambda g^{(\lambda)}(x)| \gtrsim \frac{\alpha}{\log R} \right\} \right| \\ &\lesssim (\log R)^{p+1} \left(\frac{\alpha}{\log R} \right)^p \left| \left\{ x : |\lambda_* g^{(\lambda_*)}(x)| \gtrsim \frac{\alpha}{\log R} \right\} \right|, \end{aligned}$$

and hence, for a suitable $\mathfrak{r}$,

$$\alpha^p |\{x : |f(x)| > \alpha\}| \lesssim (\log R)^{2p+1} \left(\frac{\alpha}{(\log R)^2} \right)^p \left| \left\{ x : |\lambda_* g^{(\lambda_*,\mathfrak{r})}(x)| \gtrsim \frac{\alpha}{(\log R)^2} \right\} \right|,$$

which by 4.3, applied to $\mathfrak{a} = \frac{\alpha}{(\log R)^2}$, implies

$$\alpha^p|\{x : |f(x)| > \alpha\}| \lesssim (\log R)^{E_2+3p-1} R^{\beta(p-\frac{p}{q}-1)-1} \Big(\sum_\gamma \|\lambda_* g_\gamma^{(\lambda_*,\mathfrak{r})}\|_p^q \Big)^{\frac{p}{q}}.$$

Finally, we note that each $\lambda_* g_\gamma^{(\lambda_*,\mathfrak{r})}$ is obtained by taking a subsum of a partition of unity applied to f_γ, so we conclude that

$$\alpha^p|\{x : |f(x)| > \alpha\}| \lesssim (\log R)^{E_2+3p-1} R^{\beta(p-\frac{p}{q}-1)-1} \Big(\sum_\gamma \|f_\gamma\|_p^q \Big)^{\frac{p}{q}}$$

as claimed. □

5 Appendix: Proofs of square function lemmas

In this appendix, we record the proofs of the critical lemmas for the high/low method in Fourier analysis that are appropriate for our sequence of scales. The proofs are essentially identical to those in [10], but we record them for completeness, in addition to verifying that the losses are as claimed.

Lemma 5.1 (Pointwise local constancy lemmas). *(a) For any θ, $|f_\theta|^2 \lesssim |f_\theta|^2 * |\rho_\theta^\vee|$.*

(b) For any k, m and any x,

$$|f_{m,\tau_k}|^2(x) \lesssim |f_{m,\tau_k}|^2 * w_{R_k}(x).$$

Proof. (a): Note first that

$$|f_\theta|^2(y) = |f_\theta * \rho_\theta^\vee|^2(y) \le \left| \int_{\mathbb{R}} |f_\theta|(z)|\rho_\theta|^{1/2}(y-z)|\rho_\theta|^{1/2}(y-z)dz \right|^2,$$

by considering the Fourier support. By Hölder,

$$\left| \int_{\mathbb{R}} |f_\theta|(z)|\rho_\theta|^{1/2}(y-z)|\rho_\theta|^{1/2}(y-z)dz \right|^2 \le \|\rho_\theta^\vee\|_1 \left(|f_\theta|^2 * |\rho_\theta^\vee| \right)(y).$$

Note that, by change-of-variable, $\|\rho_\theta^\vee\|_1 = \|\rho_0^\vee\|_1 = O(1)$ independent of R. Thus

$$|f_\theta|^2 \lesssim |f_\theta|^2 * |\rho_\theta^\vee|$$

as claimed.

(b): By the pruning lemma, $|f_{m,\tau_k}|^2$ has Fourier support contained in $\bigcup_{\theta,\theta' \subseteq \tau_k} (N-m+2)(\theta-\theta')$, which is in turn contained in the set $B_{\frac{1}{2}(\log R)R_k^{-1}}$. Let ρ_k be a real smooth radially symmetric cutoff function that is equal to 1 on $B_{\frac{1}{2}(\log R)R_k^{-1}}$ and is supported in $B_{(\log R)R_k^{-1}}$. By the same calculation as in (a),

$$|f_{m,\tau_k}|^2 = |f_{m,\tau_k}|^2 * \rho_k^\vee \lesssim |f_{m,\tau_k}|^2 * |\rho_k^\vee|.$$

On the other hand, we clearly have $|\rho_k^\vee| \lesssim w_{R_k}$, and we are done. □

Lemma 5.2 (Integrated local constancy lemmas). *(a) If $r \le \frac{1}{2} R_{k+3}/R$, then*

$$\int \Big| \sum_{\theta \subseteq \tau_k} |f_{m,\theta}^{\mathcal{B}}|^2 * \eta_{\sim r}^\vee \Big|^2 \lesssim \int \Big| \sum_{\theta \subseteq \tau_k} |f_{m,\theta}^{\mathcal{B}}|^2 * |\rho_{(\log R)^3 U_{\tau_k,R}^*}^\vee| \Big|^2.$$

(b) If $k \geq m$, then

$$\int \Big| \sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{(\log R) U^*_{\tau_k,R}}| \Big|^2 \lesssim \sum_{U \in \mathcal{G}_{\tau_k}} |U| \Bigg(\fint_U \sum_{\theta \subseteq \tau_k} |f_\theta|^2 \Bigg)^2 .$$

Proof. (a): The Fourier support of $\sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\sim r}$ is contained in the set

$$(N - m + 2) \bigcup_{\theta \subseteq \tau_k} (\theta - \theta) \cap B_{2r} \subseteq (\log R)^3 U^*_{\tau_k,R},$$

where $U^*_{\tau_k,R}$ is a rectangle of dimensions $R_k/R \times R^{-1}$ with long edge parallel to $\mathbf{t}_{c_{\tau_k}}$. Thus

$$\begin{aligned}
\int \Big| \sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\sim r} \Big|^2 &= \int \Big| \sum_{\theta \subseteq \tau_k} \widehat{|f^{\mathcal{B}}_{m,\theta}|^2} \eta_{\sim r} \Big|^2 \\
&\lesssim \int \Big| \sum_{\theta \subseteq \tau_k} \widehat{|f^{\mathcal{B}}_{m,\theta}|^2} \rho_{(\log R)^3 U^*_{\tau_k,R}} \Big|^2 \\
&\leq \int \Big| \sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}| \Big|^2
\end{aligned}$$

as claimed.

(b): Since $k \geq m$, $|f^{\mathcal{B}}_{m,\theta}| \leq |f_{k,\theta}| \leq |f_{k+1,\theta}| \leq |f_\theta|$ by the pruning lemmas, so

$$\int \Big| \sum_{\theta \subseteq \tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}| \Big|^2 \leq \int \Big| \sum_{\theta \subseteq \tau_k} |f_{k,\theta}|^2 * |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}| \Big|^2 .$$

By the definition of the pruning,

$$\int \Big[\sum_{\theta \subseteq \tau_k} |f_{k,\theta}|^2 * |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}| \Big]^2 = \int \Big[\sum_{\theta \subseteq \tau_k} \int \Big| \sum_{U \in \mathcal{G}_{\tau_k}} \psi_U f_{k+1,\theta}(y) \Big|^2 |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}|(x-y) dy \Big]^2 dx,$$

which, since $\sum_{U \in \mathcal{G}_{\tau_k}} \psi_U \leq 1$, may be bounded from above by

$$\int \Big[\sum_{\theta \subseteq \tau_k} \int \sum_{U \in \mathcal{G}_{\tau_k}} \psi_U(y) |f_\theta(y)|^2 |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}|(x-y) dy \Big]^2 dx.$$

We may remove the ψ_U from the dy integral by replacing it with $\tilde{\psi_U}(x) = \max_{z \in x + U_{\tau_k,R}} |\psi_U(z)|$; note that for each y and $x \in y + U_{\tau_k,R}$ we have $\psi_U(y) \leq \tilde{\psi_U}(x)$. Thus

$$\begin{aligned}
&\int \Big[\sum_{\theta \subseteq \tau_k} \int \sum_{U \in \mathcal{G}_{\tau_k}} \psi_U(y) |f_\theta(y)|^2 |\rho^{\vee}_{(\log R) U^*_{\tau_k,R}}|(x-y) dy \Big]^2 dx \\
&\quad \leq \int \Big[\sum_{U \in \mathcal{G}_{\tau_k}} \tilde{\psi_U}(x) \sum_{\theta \subseteq \tau_k} \int_{x + U_{\tau_k,R}} |f_\theta(y)|^2 |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}|(x-y) dy \Big]^2 dx \\
&\quad + \int \Big[\sum_{\theta \subseteq \tau_k} \int_{\mathbb{R}^2 \setminus (x + U_{\tau_k,R})} \sum_{U \in \mathcal{G}_{\tau_k}} \psi_U(y) |f_\theta(y)|^2 |\rho^{\vee}_{(\log R)^3 U^*_{\tau_k,R}}|(x-y) dy \Big]^2 dx \\
&\quad =: (I) + (II).
\end{aligned}$$

Note that $|\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}|$ decays almost-exponentially outside of $(\log R)^{-3}U_{\tau_k,R}$, so when $y \notin x + U_{\tau_k,R}$ we have $|\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}(x-y)| \lesssim R^{-100}$. By Minkowski, (II) may be controlled via

$$\int \left[\sum_{\theta \subseteq \tau_k} \int_{\mathbb{R}^2 \setminus (x+U_{\tau_k,R})} \sum_{U \in \mathcal{G}_{\tau_k}} \psi_U(y)|f_\theta(y)|^2 |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}|(x-y)dy \right]^2 dx$$
$$\le \left(\int \sum_{\theta\subseteq\tau_k} \sum_{U\in\mathcal{G}_{\tau_k}} \psi_U(y)|f_\theta(y)|^2 \left[\int_{\mathbb{R}^2 \setminus (y+U_{\tau_k,R})} |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}|^2 (x-y)dx \right]^{1/2} dy \right)^2$$
$$\lesssim R^{-200} \left(\sum_{U\in\mathcal{G}_{\tau_k}} \int \psi_U(y) \sum_{\theta\subseteq\tau_k} |f_\theta(y)|^2 dy \right)^2 \le R^{-100}.$$

On the first integral (I), we may estimate

$$\int \left[\sum_{U\in\mathcal{G}_{\tau_k}} \tilde{\psi_U}(x) \sum_{\theta\subseteq\tau_k} \int_{x+U_{\tau_k,R}} |f_\theta(y)|^2 |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}|(x-y)dy \right]^2 dx$$
$$\lesssim \int \sum_{U\in\mathcal{G}_{\tau_k}} \tilde{\psi_U}(x) \left[\sum_{\theta\subseteq\tau_k} \int_{x+U_{\tau_k,R}} |f_\theta(y)|^2 |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}|(x-y)dy \right]^2 dx.$$

By Minkowski,

$$\sum_{U\in\mathcal{G}_{\tau_k}} \int \tilde{\psi_U}(x) \left[\sum_{\theta\subseteq\tau_k} \int |f_\theta(y)|^2 |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}|(x-y)dy \right]^2 dx$$
$$\le \sum_{U\in\mathcal{G}_{\tau_k}} \left(\int \sum_{\theta\subseteq\tau_k} |f_\theta|^2(y) \left[\int \tilde{\psi_U}(x) |\rho^\vee_{(\log R)^3 U^*_{\tau_k,R}}|^2 (x-y)dx \right]^{1/2} dy \right)^2$$
$$\lesssim (\log R) \sum_{U\in\mathcal{G}_{\tau_k}} |U|^{-1} \left(\int W_U \sum_{\theta\subseteq\tau_k} |f_\theta|^2(y)dy \right)^2.$$

We conclude that

$$\int \Big| \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^\vee_{\tau_k}| \Big|^2 \lesssim (\log R) \sum_{U\in\mathcal{G}_{\tau_k}} |U|^{-1} \left(\int W_U \sum_{\theta\subseteq\tau_k} |f_\theta|^2(y)dy \right)^2$$

as claimed. □

Lemma 5.3 (Replacement lemma)**.** $|f(x) - f_N(x)| \lesssim \frac{\alpha}{C_{\mathfrak{p}}^{1/2}(\log R)}$.

Proof. Consider the difference

$$|f(x) - f_N(x)| \le \sum_\theta \sum_{U \notin \mathcal{G}_\theta} \psi_U(x)|f_\theta(x)|.$$

By an analogue of local constancy (a),

$$|\psi_U f_\theta| \lesssim \left(|\psi_U f_\theta|^2 * |\rho^\vee_{2\theta}| \right)^{1/2},$$

so

$$\begin{aligned}|f(x)-f_N(x)| &\le \sum_\theta \sum_{U\notin\mathcal{G}_\theta} \left(|\psi_U f_\theta|^2 * |\rho_{2\theta}^\vee|\right)^{1/2} \\ &= \sum_\theta \sum_{U\notin\mathcal{G}_\theta} \left(\int |\psi_U f_\theta|^2(y)|\rho_{2\theta}^\vee|(x-y)dy\right)^{1/2}.\end{aligned}$$

Next, since $\psi_U \lesssim W_U$,

$$\begin{aligned}|f(x)-f_N(x)| &\le \sum_\theta \sum_{U\notin\mathcal{G}_\theta} \left(\int W_U(y)|f_\theta|^2(y)\psi_U(y)|\rho_{2\theta}^\vee|(x-y)dy\right)^{1/2} \\ &\le \sum_\theta \max_{U\notin\mathcal{G}_\theta}\left(\int W_U(y)|f_\theta|^2(y)dy\right)^{1/2} \\ &\quad\times \sum_{U\notin\mathcal{G}_\theta}\left(\sup_y \psi_U(y)|\rho_{2\theta}^\vee|(x-y)\right)^{1/2}\end{aligned}$$

by Cauchy-Schwarz. By the rapid decay of ψ_U outside of U and local constancy of $\rho_{2\theta}^\vee$,

$$\begin{aligned}\sum_{U\notin\mathcal{G}_\theta} \|\psi_U(\cdot)\rho_{2\theta}^\vee(x-\cdot)\|_{L^\infty(\mathbb{R}^2)} &\lesssim \sum_U \|\rho_{2\theta}^\vee(x-\cdot)\|_{L^\infty(U)} \\ &\lesssim |U|^{-1}\sum_U \|\rho_{2\theta}^\vee(x-\cdot)\|_{L^1(U)} \\ &= |U|^{-1}\|\rho_{2\theta}^\vee\|_{L^1(\mathbb{R}^2)} \\ &\lesssim |U|^{-1},\end{aligned}$$

so that

$$|f(x)-f_N(x)| \lesssim |U|^{-1/2}\sum_\theta \max_{U\notin\mathcal{G}_\theta}\left(\int W_U|f_\theta|^2(y)dy\right)^{1/2}.$$

Finally, by the definition of $\mathcal{G}_\theta$,

$$|f(x)-f_N(x)| \lesssim \sum_\theta \max_{U\notin\mathcal{G}_\theta} \frac{\alpha}{(\#\theta)C_{\mathfrak{p}}^{1/2}(\log R)} = \frac{\alpha}{C_{\mathfrak{p}}^{1/2}(\log R)}$$

as claimed. □

Lemma 5.4 (Low lemma)**.** *For any $2 \le m \le k \le N$, $0 \le s \le k$, and $r \le R_k^{-1}$,*

$$|f_{m,\tau_s}^{\mathcal{B}}|^2 * \eta_{\le r}^\vee(x) = \sum_{\tau_k\subseteq\tau_s} \sum_{\tau_k':\tau_k \text{ near } \tau_k'} \left(f_{m,\tau_k}^{\mathcal{B}}\overline{f_{m,\tau_k'}^{\mathcal{B}}}\right) * \eta_{\le r}^\vee(x)$$

for any x and any τ_s.

Proof. Indeed,

$$\begin{aligned}|f_{m,\tau_s}^{\mathcal{B}}|^2 * \eta_{\le r}^\vee(x) &= \int_{\mathbb{R}^2} |f_{m,\tau_s}^{\mathcal{B}}|^2(x-y)\eta_{\le r}^\vee(y)dy \\ &= \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\left[\widehat{f_{m,\tau_s}^{\mathcal{B}}} * \widehat{\overline{f_{m,\tau_s}^{\mathcal{B}}}}(\xi)\right]\eta_{\le r}(\xi)d\xi \\ &= \sum_{\tau_k,\tau_k'\subseteq\tau_s}\int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\left[\widehat{f_{m,\tau_k}^{\mathcal{B}}} * \widehat{\overline{f_{m,\tau_k'}^{\mathcal{B}}}}(\xi)\right]\eta_{\le r}(\xi)d\xi.\end{aligned}$$

Note that each $\widehat{f^{\mathcal{B}}_{m,\tau_k}}$ has support in the set $\bigcup_{\theta\subseteq\tau_k}(N-m+2)\theta \subseteq (N-m+2)\tau_k$; thus the convolution in the latter integrand is supported in the set $(N-m+2)(\tau_k-\tau_k') \subseteq (\log R)(\tau_k-\tau_k')$, which is contained in the ball $B_{C(\log R)R_k^{-1}}(c_{\tau_k}-c_{\tau_k'})$ for suitable universal constant C. Since $\eta_{\le r}$ has support in the ball of radius $2R_k^{-1}$, and the diameter of each τ_k is $\sim R_k^{-1}$, we conclude that for each τ_k there are $\lesssim \log R$-many neighboring τ_k' such that the support of $\widehat{f^{\mathcal{B}}_{m,\tau_k}} * \widehat{\overline{f^{\mathcal{B}}_{m,\tau_k'}}}$ has nontrivial intersection with the support of $\eta_{\le r}$. Thus

$$\sum_{\tau_k,\tau_k'\subseteq\tau_s}\int_{\mathbb{R}^2} e^{-2\pi i x\cdot\xi}\Big[\widehat{f^{\mathcal{B}}_{m,\tau_k}} * \widehat{\overline{f^{\mathcal{B}}_{m,\tau_k'}}}(\xi)\Big]\eta_{\le r}(\xi)d\xi$$
$$= \sum_{\substack{\tau_k,\tau_k'\subseteq\tau_s\\ \tau_k \text{ near } \tau_k'}}\int_{\mathbb{R}^2} e^{-2\pi i x\cdot\xi}\Big[\widehat{f^{\mathcal{B}}_{m,\tau_k}} * \widehat{\overline{f^{\mathcal{B}}_{m,\tau_k}}}(\xi)\Big]\eta_{\le r}(\xi)d\xi.$$

By Plancherel again, we conclude. □

Lemma 5.5 (High Lemmas)**.** *For any $m,k,$ and l such that $2\le m\le N$ and $k+l\le N$,*

(a)
$$\int\Big|\sum_{\theta}|f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\ge R_k/R}\Big|^2 \lesssim \log R\sum_{\tau_k}\int\Big|\sum_{\theta\subseteq\tau_k}|f_{m,\theta}|^2 * \eta^{\vee}_{\ge R_k/R}\Big|^2,$$

(b)
$$\int\Big|\sum_{\tau_k}|f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^{\vee}_{\ge R_k^{-1}}\Big|^2 \lesssim (\log R)\sum_{\tau_k}\int|f^{\mathcal{B}}_{m,\tau_k}|^4,$$

(c)
$$\int\Big|\sum_{\tau_k}\sum_{\tau_k' \text{ near } \tau_k}(f^{\mathcal{B}}_{m,\tau_k}\overline{f^{\mathcal{B}}_{m,\tau_k'}}) * \eta^{\vee}_{\ge R_{k+l}^{-1}}\Big|^2 \lesssim (\log R)^{l+3}\sum_{\tau_k}\int|f^{\mathcal{B}}_{m,\tau_k}|^4.$$

Proof. (a): By Plancherel,

$$\int\Big|\sum_{\theta}|f^{\mathcal{B}}_{m,\theta}|^2 * \eta^{\vee}_{\ge R_k/R}\Big|^2 = \int_{|\xi|\ge R_k/R}\Big|\sum_{\tau_k}\sum_{\theta\subseteq\tau_k}\widehat{|f^{\mathcal{B}}_{m,\theta}|^2}(\xi)\eta_{\ge R_k/R}(\xi)\Big|^2.$$

The supports of the summands $\sum_{\theta\subseteq\tau_k}\widehat{|f^{\mathcal{B}}_{m,\theta}|^2}(\xi)\eta_{\ge R_k/R}(\xi)$, ranging over distinct τ_k, have greatest overlap on the circle of radius R_k/R, where the overlap is $O(N)$. By Cauchy-Schwarz,

$$\int_{|\xi|\ge R_k/R}\Big|\sum_{\tau_k}\sum_{\theta\subseteq\tau_k}\widehat{|f^{\mathcal{B}}_{m,\theta}|^2}(\xi)\eta_{\ge R_k/R}(\xi)\Big|^2 \lesssim (\log R)\sum_{\tau_k}\int_{|\xi|\ge R_k/R}\Big|\sum_{\theta\subseteq\tau_k}\widehat{|f^{\mathcal{B}}_{m,\theta}|^2}(\xi)\eta_{\ge R_k/R}(\xi)\Big|^2.$$

We conclude by enlarging the domain of integration on the right-hand side and using Plancherel.

(b): By Plancherel,

$$\int\Big|\sum_{\tau_k}|f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^{\vee}_{\ge R_k^{-1}}\Big|^2 = \int_{|\xi|\ge R_k^{-1}}\Big|\sum_{\tau_k}\widehat{|f^{\mathcal{B}}_{m,\tau_k}|^2}(\xi)\eta_{\ge R_k^{-1}}(\xi)\Big|^2.$$

Each $\widehat{|f^{\mathcal{B}}_{m,\tau_k}|^2}$ is supported in the set $(N-m+2)(\tau_k-\tau_k)\subseteq(\log R)(\tau_k-\tau_k)$, and the maximal overlap between these for distinct τ_k in the region $|\xi|\ge R_k^{-1}$ occurs when $|\xi| = R_k^{-1}$, where the overlap is $\lesssim \log R$. By Cauchy-Schwarz and Plancherel,

$$\int_{|\xi|\ge R_k^{-1}}\Big|\sum_{\tau_k}\widehat{|f^{\mathcal{B}}_{m,\tau_k}|^2}(\xi)\eta_{\ge R_k^{-1}}(\xi)\Big|^2 \lesssim \log R\sum_{\tau_k}\int\Big||f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^{\vee}_{\ge R_k^{-1}}\Big|^2.$$

Lastly, $\|\eta^\vee_{\geq R_k^{-1}}\|_1 = O(1)$ by a change-of-variable, thus

$$\int \Big|\sum_{\tau_k} |f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^\vee_{\geq R_k^{-1}}\Big|^2 \lesssim (\log R) \sum_{\tau_k} \int |f^{\mathcal{B}}_{m,\tau_k}|^4$$

as claimed.

(c): Reasoning as in (b), note that $\Big[f^{\mathcal{B}}_{m,\tau_k}\overline{f^{\mathcal{B}}_{m,\tau_k'}}\Big] * \eta^\vee_{\sim R_{k+l}^{-1}}$ has Fourier support in the set $(N - m+2)(\tau_k - \tau_k')$. Note that $\tau_k - \tau_k'$ is contained in a set of the form $(c_{\tau_k} - c_{\tau_k'}) + C(\log R)(\tau_k - \tau_k) \subseteq C'(\log R)^2(\tau_k - \tau_k)$ (c.f. Remark 2.10). As this is the case for each τ_k' for which τ_k' near τ_k, we conclude that $\sum_{\tau_k':\tau_k' \text{ near } \tau_k}\Big[f^{\mathcal{B}}_{m,\tau_k}\overline{f^{\mathcal{B}}_{m,\tau_k'}}\Big] * \eta^\vee_{\sim R_{k+l}^{-1}}$ has Fourier support in the set $C'(\log R)^2(\tau_k - \tau_k)$. On the circle of radius R_{k+l}^{-1}, the overlap between these sets is $O((\log R)^{l+2})$, so

$$\int \Big|\sum_{\tau_k}\sum_{\tau_k' \text{ near } \tau_k} (f^{\mathcal{B}}_{m,\tau_k}\overline{f^{\mathcal{B}}_{m,\tau_k'}}) * \eta^\vee_{\geq R_{k+l}^{-1}}\Big|^2 \lesssim (\log R)^{l+2} \sum_{\tau_k} \int \Big|\sum_{\tau_k' \text{ near } \tau_k} |f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^\vee_{\geq R_{k+l}^{-1}}\Big|^2.$$

By Cauchy-Schwarz,

$$\sum_{\tau_k}\int \Big|\sum_{\tau_k' \text{ near } \tau_k} |f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^\vee_{\geq R_{k+l}^{-1}}\Big|^2 \lesssim (\log R) \sum_{\tau_k} \int \Big||f^{\mathcal{B}}_{m,\tau_k}|^2 * \eta^\vee_{\geq R_{k+l}^{-1}}\Big|^2,$$

and since $\|\eta_{\geq R_{k+l}^{-1}}\|_1 = O(1)$ we conclude that

$$\int \Big|\sum_{\tau_k}\sum_{\tau_k' \text{ near } \tau_k} (f^{\mathcal{B}}_{m,\tau_k}\overline{f^{\mathcal{B}}_{m,\tau_k'}}) * \eta^\vee_{\geq R_{k+l}^{-1}}\Big|^2 \lesssim (\log R)^{l+3} \sum_{\tau_k} \int |f^{\mathcal{B}}_{m,\tau_k}|^4$$

as claimed. $\square$

Lemma 5.6 (Weak high-domination of bad parts)**.** *Let $2 \leq m \leq N$ and $0 \leq k < m$.*

(a) We have the estimate

$$\Big|\sum_{\tau_{m-1}\subseteq\tau_k} |f^{\mathcal{B}}_{m,\tau_r}|^2 * \eta^\vee_{\leq R_{m-1}/R}(x)\Big| \lesssim \frac{\alpha^2(\#\tau_{m-1}\subseteq\tau_k)}{C_{\mathfrak{p}}(\log R)^2(\#\tau_{m-1})^2}.$$

(b) Suppose $\alpha \lesssim (\log R)|f^{\mathcal{B}}_{m,\tau_k}(x)|$. Then

$$\sum_{\tau_{m-1}\subseteq\tau_k} |f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x) \lesssim \Big|\sum_{\tau_{m-1}\subseteq\tau_k} |f^{\mathcal{B}}_{m,\tau_{m-1}}|^2 * \eta^\vee_{\geq R_{m-1}/R}(x)\Big|.$$

Proof. (a): By the low lemma,

$$\sum_{\tau_r\subseteq\tau_k} |f^{\mathcal{B}}_{m,\tau_r}|^2 * \eta^\vee_{\leq R_{m-1}/R}(x) = \sum_{\theta\subseteq\tau_k}\sum_{\theta' \text{ near } \theta} (f^{\mathcal{B}}_{m,\theta}\overline{f^{\mathcal{B}}_{m,\theta}}) * \eta^\vee_{\leq R_{m-1}/R}(x).$$

By the definition of "near,"

$$\Big|\sum_{\theta\subseteq\tau_k}\sum_{\theta' \text{ near } \theta} (f^{\mathcal{B}}_{m,\theta}\overline{f^{\mathcal{B}}_{m,\theta}}) * \eta^\vee_{\leq R_{m-1}/R}(x)\Big| \lesssim (\log R) \sum_{\theta\subseteq\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * |\eta^\vee_{\leq R_{m-1}/R}|(x).$$

By local constancy,

$$\sum_{\theta\subseteq\tau_k}|f_{m,\theta}^{\mathcal{B}}|^2*|\eta^\vee_{\le R_{m-1}/R}|(x)\lesssim\sum_{\theta\subseteq\tau_k}|f_{m,\theta}^{\mathcal{B}}|^2*|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|(x).$$

If $\theta\subseteq\tau_{m-1}$,

$$|f_{m,\theta}^{\mathcal{B}}|^2*|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|(x)=\int\Big|\sum_{U\notin\mathcal{G}_{\tau_{m-1}}}\psi_U f_{m,\theta}\Big|^2(y)\Big(|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|\Big)(x-y)dy.$$

Since ψ_U are all real and nonnegative,

$$\begin{aligned}\int\Big|\sum_{U\notin\mathcal{G}_{\tau_{m-1}}}\psi_U f_{m,\theta}\Big|^2(y)\Big(|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|\Big)(x-y)dy\\ =\int\sum_{U\notin\mathcal{G}_{\tau_{m-1}}}\psi_U(y)|f_{m,\theta}|^2(y)\sum_{U'\notin\mathcal{G}_{\tau_{m-1}}}\psi_{U'}(y)\Big(|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|\Big)(x-y)dy.\end{aligned}$$

Since $\{\psi_{U'}\}_{U'}$ form a partition of unity, $\sum_{U'\notin\mathcal{G}_{\tau_{m-1}}}\psi_{U'}(y)\le 1$, and so

$$\begin{aligned}\int\sum_{U\notin\mathcal{G}_{\tau_{m-1}}}\psi_U(y)|f_{m,\theta}|^2(y)\sum_{U'\notin\mathcal{G}_{\tau_{m-1}}}\psi_{U'}(y)\Big(|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|\Big)(x-y)dy\\ \le\int\sum_{U\notin\mathcal{G}_{\tau_{m-1}}}\psi_U(y)|f_{m,\theta}|^2(y)\Big(|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|\Big)(x-y)dy,\end{aligned}$$

which by Hölder is bounded from above by

$$\sum_{U\notin\mathcal{G}_{\tau_{m-1}}}\Big\|\psi_U^{1/2}(y)\Big(|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_m/R}|\Big)(x-y)\Big\|_{L^\infty_y}\int\psi_U^{1/2}|f_{m,\theta}|^2(y)dy.$$

Note that, for each x, the function $y\mapsto|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_m/R}|(x-y)$ is approximately constant on rectangles of dimensions $\sim(\log R)^{-1}R\times R/R_m$, with long edge parallel to $\mathbf{n}_{c_\theta}$. By rapid decay of ψ_U outside of U,

$$\begin{aligned}\sum_{U\|U_{\tau_{m-1},R}}\Big\|\psi_U^{1/2}(y)\Big(|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_m/R}|\Big)(x-y)\Big\|_{L^\infty_y}&\lesssim\sum_{U\|U_{\tau_{m-1},R}}\Big\||\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_m/R}|(x-y)\Big\|_{L^\infty_y(U)}\\ &\le\sum_{U\|U_{\tau_{m-1},R}}\sum_{\substack{V\sim(\log R)^{-1}R\times R/R_m\\ V\subseteq U}}\Big\||\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_m/R}|(x-y)\Big\|_{L^\infty_y(V)}\\ &\lesssim\sum_{U\|U_{\tau_{m-1},R}}\sum_{\substack{V\sim(\log R)^{-1}R\times R/R_m\\ V\subseteq U}}|V|^{-1}\Big\||\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_m/R}|(x-y)\Big\|_{L^1_y(V)}\\ &=(\log R)|U|^{-1}\Big\||\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_m/R}|(x-y)\Big\|_{L^1_y(\mathbb{R}^2)}\\ &\lesssim(\log R)|U|^{-1}.\end{aligned}$$

Additionally, the polynomial decay of W_U allows us to take $\psi_U^{1/2}\lesssim W_U$, so in total we get

$$|f_{m,\theta}^{\mathcal{B}}|^2*|\rho^\vee_{(\log R)\theta}|*|\eta^\vee_{\le R_{m-1}/R}|(x)\lesssim(\log R)\max_{U\notin\mathcal{G}_{\tau_{m-1}}}⨍_U|f_{m,\theta}|^2(y)dy.$$

If we sum over all $\theta \subseteq \tau_{m-1}$, and use the hypothesis $U \notin \mathcal{G}_{\tau_{m-1}}$, we see that

$$\begin{aligned}\sum_{\tau_{m-1}\subseteq\tau_k}\sum_{\theta\subseteq\tau_{m-1}} |f^{\mathcal{B}}_{m,\theta}|^2 * |\rho^{\vee}_{(\log R)\theta}| * |\eta^{\vee}_{\leq R_{m-1}/R}|(x) \\ \lesssim (\log R)^2 \sum_{\tau_{m-1}\subseteq\tau_k} \sup_{U\notin\mathcal{G}_{\tau_{m-1}}} \fint_U \sum_{\theta\subseteq\tau_{m-1}} |f_{m,\theta}|^2(y)dy \\ \leq \frac{\alpha^2(\#\tau_{m-1}\subseteq\tau_k)}{(\#\tau_{m-1})^2}\frac{1}{C_{\mathfrak{p}}(\log R)^2}.\end{aligned}$$

(b): Write $f^{\mathcal{B}}_{m,\tau_k} = \sum_{\tau_{m-1}\subseteq\tau_k} f^{\mathcal{B}}_{m,\tau_{m-1}}$, where $f^{\mathcal{B}}_{m,\tau_{m-1}} = \sum_{U\notin\mathcal{G}_{\tau_{m-1}}}\psi_U\sum_{\theta\subseteq\tau_{m-1}} f_{m,\theta}$. By Cauchy-Schwarz,

$$\alpha \lesssim (\#\tau_{m-1}\subseteq\tau_k)^{1/2}(\log R)\left(\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x)\right)^{1/2}.$$

We assume for the sake of contradiction that

$$\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x) \leq C_{\mathfrak{p}}^{1/2}\Big|\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2 * \eta^{\vee}_{\leq R_{m-1}/R}(x)\Big|.$$

By (a),

$$\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x) \lesssim C_{\mathfrak{p}}^{1/2}\frac{\alpha^2(\#\tau_{m-1}\subseteq\tau_k)}{(\#\tau_{m-1})^2 C_{\mathfrak{p}}(\log R)^2}.$$

On the other hand, we assumed the estimate

$$\alpha^2 \lesssim (\#\tau_{m-1}\subseteq\tau_k)(\log R)^2\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x),$$

so that

$$\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x) \lesssim C_{\mathfrak{p}}^{1/2}\frac{(\log R)^2}{C_{\mathfrak{p}}(\log R)^2}\frac{(\#\tau_{m-1}\subseteq\tau_k)^2}{(\#\tau_{m-1})^2}\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x),$$

i.e.

$$C_{\mathfrak{p}}^{1/2} \lesssim 1.$$

If $C_{\mathfrak{p}}$ is chosen as a sufficiently large universal constant (i.e. independently of f, R), then we conclude by contradiction that

$$\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x) > C_{\mathfrak{p}}^{1/2}\Big|\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2 * \eta^{\vee}_{\geq R_{m-1}/R}(x)\Big|,$$

i.e.

$$\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2(x) \leq \frac{C_{\mathfrak{p}}^{1/2}}{C_{\mathfrak{p}}^{1/2}-1}\Big|\sum_{\tau_{m-1}\subseteq\tau_k}|f^{\mathcal{B}}_{m,\tau_{m-1}}|^2 * \eta^{\vee}_{\leq R_{m-1}/R}(x)\Big|.$$

Since $C_{\mathfrak{p}}$ is chosen to be a large constant, we conclude that the prefactor $\frac{C_{\mathfrak{p}}^{1/2}}{C_{\mathfrak{p}}^{1/2}-1}$ is $O(1)$, so we are done. □

6 Acknowledgements

The author would like to thank Terence Tao and Hong Wang for many helpful comments and suggestions. The author would also like to thank Jaume de Dios Pont for suggesting the use of Gevrey-class functions in place of Gaussians for the purpose of defining rapidly-decaying wave packets.

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