UCLA Analysis qualifying exam solutions

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These are solutions to old analysis qualifying exams for UCLA, accessible on the math UCLA website. Varying levels of detail are presented, and not every problem after the first solved problem is solved. Please send any corrections to johnsrude (at) math.ucla.edu; if you did not obtain this document directly from the source, please first check that I have not already made that correction.

Note that Adam Lott has compiled solutions back to 2009, accessible here. The solutions presented here are my own; the solutions presented there are the compiled work of many individuals and should probably be treated as rather more reliable.

I lastly make several remarks contrasting features of the solutions presented here and those of the solutions that would be expected, or desirable, in the qualifying exam itself. Firstly, these have been prepared outside of a testing environment, slowly over a long period of time, in part as preparation for teaching (though I almost always avoid using outside sources, except to recall this-or-that technical condition for a theorem). Secondly, they are much longer than a typical submission on the exam: I attempt to clearly spell out as many details as are needed for someone to understand the solution, *assuming that they have no idea how to solve it themselves*. As such, these look much more like homework submissions than exam submissions; in the latter, one is writing a sketch directed at a seasoned examiner, who knows the question and knows how people are likely to solve the question.

Next, I do not take too much care to restrict the methods used in the solutions to those which are covered in the 245A/B/C, 246A/B classes, which are expected to be known to those students. In part, this is logistically necessary: the exams are in part written by the faculty who teach those courses (who vary from year-to-year), and so each year's exam is slightly biased towards the particular topics that professor focused on. As such, it would be too much work to make sure that all my methods are geared towards solving problems from the perspective of someone who has just taken those classes.

Status of solutions:

F19P1	✓	S20P1	✓	F20P1	✓	S21P1	✓		
F19P2	✓	S20P2	✓	F20P2	✓	S21P2	✓		
F19P3	✓	S20P3	✓	F20P3	✓	S21P3	✓		
F19P4	✓	S20P4	✓	F20P4	✓	S21P4	✓		
F19P5	✓	S20P5	✓	F20P5	✓	S21P5	✓		
F19P6	✓	S20P6	✓	F20P6	✓	S21P6	✓		
F19P7	✓	S20P7	✓	F20P7	✓	S21P7	✓		
F19P8	✓	S20P8	✓	F20P8	✓	S21P8	✓		
F19P9	✓	S20P9	✓	F20P9	✓	S21P9	✓		
F19P10	✓	S20P10	✓	F20P10	✓	S21P10	✓		
F19P11	✓	S20P11	✓	F20P11	✓	S21P11	✓		
F19P12	✓	S20P12	✓	F20P12	✓	S21P12	✓		
F21P1	✓	S22P1	✓	F22P1	✓	S23P1	✓	F23P1	✓
F21P2	✓	S22P2	✓	F22P2	✓	S23P2	✓	F23P2	✓
F21P3	✓	S22P3	✓	F22P3	✓	S23P3	✓	F23P3	✓
F21P4	✓	S22P4	✓	F22P4	✓	S23P4	✓	F23P4	✓
F21P5	✓	S22P5	✓	F22P5	✓	S23P5	✓	F23P5	✓
F21P6	✓	S22P6	✓	F22P6	✓	S23P6	✗	F23P6	✓
F21P7	✓	S22P7	✓	F22P7	✓	S23P7	✓	F23P7	✓
F21P8	✓	S22P8	✓	F22P8	✓	S23P8	✓	F23P8	✓
F21P9	✓	S22P9	✓	F22P9	✓	S23P9	✓	F23P9	✓
F21P10	✓	S22P10	✓	F22P10	✓	S23P10	✓	F23P10	✓
F21P11	✓	S22P11	✓	F22P11	✓	S23P11	✓	F23P11	✓
F21P12	✓	S22P12	✓	F22P12	✓	S23P12	✓	F23P12	✓

1 Spring 2019

Spring 2019 Problem 1. Let $f \in C^2(\mathbb{R})$ be a real-valued function that is uniformly bounded on $\mathbb{R}$. Prove that there exists a point $c \in \mathbb{R}$ such that $f''(c) = 0$.

Proof. We will show the contrapositive: if f'' is nowhere vanishing, then f is unbounded. Replacing f by $-f$ if necessary, we may assume that $f'' > 0$ on all of $\mathbb{R}$. We divide into cases, depending on the sign of f':

Case 1: Suppose there is some $x_0 \in \mathbb{R}$ such that $\varepsilon := f'(x_0) > 0$. Then, for each $y \geq x_0$,

$$f'(y) = f'(x_0) + \int_{x_0}^{y} f''(t)dt \geq f'(x_0) = \varepsilon$$

so that, for each $z \geq x_0$,

$$f(z) = f(x_0) + \int_{x_0}^{z} f'(y)dy \geq f(x_0) + \varepsilon(z - x_0)$$

Taking z to be large, we conclude that f is not bounded above, as was to be shown.

Case 2: Suppose $f'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then, if $y > x_0$,

$$f'(y) = f'(x_0) + \int_{x_0}^{y} f''(t)dt > f'(x_0) = 0$$

so that $f'(y) > 0$, and we may apply Case 1.

Case 3: If we are not in one of the previous cases, then $f' < 0$ on all of $\mathbb{R}$. Since $f'' > 0$, we have that f' is increasing, so writing $f'(0) =: -\varepsilon < 0$ we have

$$y < 0 \implies f'(y) \leq f'(0) = -\varepsilon$$

Then, for $z < 0$,

$$f(z) = f(0) - \int_{z}^{0} f'(y)dy \geq f(0) + \varepsilon|z|$$

so that f is unbounded as $z \to -\infty$, as was to be shown.

$\square$

Spring 2019 Problem 2. Let μ be a Borel probability measure on $[0, 1]$ that has no atoms (this means that $\mu(\{t\}) = 0$ for any $t \in [0, 1]$). Let also $\mu_1, \mu_2, \ldots$ be Borel probability measures on $[0, 1]$. Assume that μ_n converges to μ in the weak* topology. Denote $F(t) := \mu([0, t])$ and $F_n(t) := \mu_n([0, t])$ for each $n \geq 1$ and $t \in [0, 1]$. Prove that F_n converges uniformly to F.

Proof. We recall that $\mu_n \rightharpoonup \mu$ weak-$*$'ly on $[0, 1]$ if, for each $f \in C([0, 1]; \mathbb{R})$, we have

$$\int f d\mu_n \to \int f d\mu, \quad (n \to \infty)$$

Since the functions F, F_n are monotone and F is continuous, it will suffice to show pointwise convergence; we will establish the implication to uniform convergence at the end. Note that $F_n(1) = 1 = F(1)$

and for each n, so we need to consider $t < 1$. Let $0 < t < 1$, and for $k \in \mathbb{N}$ sufficiently large so that $\frac{2}{k} < 1 - t$ we write

$$h_k(x) = \begin{cases} 1 & 0 \leq x \leq t \\ 1 - k|x - t| & t \leq x \leq t + \frac{1}{k} \\ 0 & t + \frac{1}{k} \leq x \leq 1 \end{cases}$$

Then $\{h_k\}_k$ is a sequence of functions uniformly bounded by the integrable function $1_{[0,1]}$, and limit pointwise to the indicator $1_{[0,t]}$. By the dominated convergence theorem,

$$F(t) = \int 1_{[0,t]} d\mu = \lim_{k\to\infty} \int h_k d\mu, \quad F_n(t) = \int 1_{[0,t]} d\mu_n = \lim_{k\to\infty} \int h_k d\mu_n$$

We also write, for each k,

$$u_k(x) = \begin{cases} 0 & x \leq t - \frac{1}{k} \\ k(x - t + \frac{1}{k}) & t - \frac{1}{k} \leq x \leq t \\ 1 & t \leq x \leq t + \frac{1}{k} \\ 1 - k(x - t - \frac{1}{k}) & t + \frac{1}{k} \leq x \leq t + \frac{2}{k} \\ 0 & x \geq t + \frac{2}{k} \end{cases}$$

Note that each u_k is continuous, $u_k \leq 1_{[t-\frac{1}{k}, t+\frac{2}{k}]}$, and $h_k - 1_{[0,t]} \leq u_k$.

Let now $\varepsilon > 0$. By the atomless condition and finiteness of μ, continuity from above implies that there exists $k \in \mathbb{N}$ such that

$$\mu([t - \frac{1}{k}, t + \frac{2}{k}]) < \frac{\varepsilon}{2}$$

It follows then that

$$\begin{aligned} \limsup_{n\to\infty} |F_n(t) - F(t)| &\leq \limsup_{n\to\infty} \int |1_{[0,t]} - h_k| d\mu_n + \limsup_{n\to\infty} \left| \int h_k d\mu_n - \int h_k d\mu \right| \\ &\quad + \limsup_{n\to\infty} \int |h_k - 1_{[0,t]}| d\mu \\ &\leq \limsup_{n\to\infty} \int u_k d\mu_n + 0 + \limsup_{n\to\infty} \int u_k d\mu \\ &= 2 \int u_k d\mu < \varepsilon \end{aligned}$$

Since ε was arbitrary, we conclude that $F_n(t) \to F(t)$ as $n \to \infty$ for each $t \in (0, 1]$.

We'll omit the $t = 0$ case, as it is similar but technically somewhat simpler than what is already done above. So now we accept that $F_n \to F$ pointwise on $[0, 1]$.

We next demonstrate uniform convergence. Let $\varepsilon > 0$ be arbitrary. Let δ be an ε-modulus of continuity for F, i.e. $|x - y| < \delta$ implies $|F(x) - F(y)| < \varepsilon$; since F is continuous on a compact domain, this exists. Let $0 = t_1 < \ldots < t_N = 1$ be a $\delta/2$-net of $[0, 1]$, i.e. a finite subset such that any $t \in [0, 1]$ has some t_j such that $|t - t_j| < \frac{\delta}{2}$. Let $n_1, \ldots, n_N \in \mathbb{N}$ be such that

$$k_j \geq n_j \implies |F_{k_j}(t_j) - F(t_j)| < \varepsilon$$

Then, for each $t \in [0,1]$, if $t_j \leq t \leq t_{j+1}$, then $|t - t_j| < \delta$ and $|t - t_{j+1}| < \delta$, so for $n \geq \max(n_1, \dots, n_N)$,

$$|F_n(t) - F(t)| \leq |F_n(t) - F(t_{j+1})| + |F(t_{j+1}) - F(t)| < |F_n(t) - F(t_{j+1})| + \varepsilon$$

Note that, by monotonicity,

$$F_n(t) \leq F_n(t_{j+1}) \leq F(t_{j+1}) + \varepsilon$$

and

$$F_n(t) \geq F_n(t_j) \geq F(t_j) - \varepsilon \geq F(t_{j+1}) - 2\varepsilon$$

so that

$$|F_n(t) - F(t_{j+1})| < 2\varepsilon$$

Thus we have justified that, for each $\varepsilon > 0$, we may find N_ε large enough so that $t \in [0,1]$ and $n \geq N_\varepsilon$ implies $|F_n(t) - F(t)| < 3\varepsilon$. In particular, $F_n \to F$ uniformly, as was to be established.

$\square$

Spring 2019 Problem 3. (a) Let f be a positive continuous function on $\mathbb{R}$ such that $\lim_{|t|\to\infty} f(t) = 0$. Show that the set $\{hf : h \in L^1(\mathbb{R}, m), \|h\|_1 \leq K\}$ is a closed nowhere dense set in $L^1(\mathbb{R}, m)$, for any $K \geq 1$ (m denotes the Lebesgue measure on $\mathbb{R}$).

(b) Let $\{f_n\}_n$ be a sequence of positive continuous functions on $\mathbb{R}$ such that for each n we have $\lim_{|t|\to\infty} f(t) = 0$. Show that there exists $g \in L^1(\mathbb{R}, m)$ such that $g/f_n \notin L^1(\mathbb{R}, m)\ \forall n$.

Proof. (a): Throughout we will write H for the set in question. We begin by showing that H has no interior. Since H is the image of a linear mapping $h \mapsto hf$ on $L^1(\mathbb{R}, m)$, H is a linear subspace, so it will suffice to show that 0 is not interior. Let $\varepsilon > 0$ be arbitrary. Let $R > 0$ be such that $|t| \geq R$ implies $|f(t)| < 10^{-2}K^{-1}\varepsilon$. Let then $g = \varepsilon 1_{[R,R+1]}$; then $g \in L^1(\mathbb{R}, m)$ and $\|g\|_1 = \|g - 0\|_1 \leq \varepsilon$. On the other hand, $g \notin H$: if $h \in L^1(\mathbb{R}, m)$ is such that $g = hf$, then for any $R \leq t \leq R+1$

$$\varepsilon = g(t) = f(t)h(t), \quad \text{so} \quad f(t) \neq 0 \quad \text{and} \quad h(t) = \frac{\varepsilon}{f(t)}$$

and so

$$\|h\|_1 \geq \int_R^{R+1} \frac{\varepsilon}{f(t)} dt > \int_R^{R+1} 10^2 K dt = 10^2 K$$

So any $h \in L^1(\mathbb{R}, m)$ satisfying $g = hf$ must have $\|h\|_1 \not\leq K$. In particular, $g \notin H$. Since we have found an element of H^c in every open ball about 0, we conclude that 0 is not interior to H. As remarked previously, this implies that H has empty interior.

We next show that H is closed. Suppose $g_n = h_n f$ is a sequence in H and $g \in L^1(\mathbb{R}, m)$ is such that $g_n \to g$ in L^1. It remains to show that g/f has finite L^1 norm.

For $n \in \mathbb{N}$, write $1 > \varepsilon_n > 0$ for a number such that $f > \varepsilon_n$ on $[-n, n]$. Let $K_n \in \mathbb{N}$ be such that $k \geq K_n$ implies $\|g_k - g\|_{L^1} < 2^{-n}\varepsilon_n^2$. Then

$$\int_{-n}^{n} \left|\frac{g_{K_n}}{f} - \frac{g}{f}\right| dm \leq \varepsilon_n^{-1} \int |g_{K_n} - g| dm \leq 2^{-n}\varepsilon_n$$

and hence

$$\int_{-n}^{n} \left|\frac{g}{f}\right| dm \leq \int_{-n}^{n} |h_{K_n}| + \int_{-n}^{n} \left|\frac{g_{K_n}}{f} - \frac{g}{f}\right| dm \leq K + 2^{-n}\varepsilon_n$$

so that
$$\int |\frac{g}{f}|dm = \limsup_{n\to\infty} \int_{-n}^{n} \Big|\frac{g}{f}\Big| dm \le K < \infty$$
as was to be established.

(b): For each $n, K \in \mathbb{N}$, write $U_{n,K}$ for the set
$$U_{n,K} = \{g \in L^1(\mathbb{R}, m) : \int \frac{|g|}{f_n} > K\}$$
By (a), each $U_{n,K}$ is a dense open subset of $L^1(\mathbb{R}, m)$. By Baire category, there exists $g \in \bigcap_{n,K=1}^{\infty} U_{n,K}$. We verify that g satisfies the conditions required. For each n, we have
$$g \in \bigcap_{n,K=1}^{\infty} U_{n,K}$$
so that
$$\forall K \quad \int \frac{|g|}{f_n} > K$$
i.e. $\frac{g}{f_n} \notin L^1(\mathbb{R}, m)$. Since this holds for each n, we are done.

$\square$

Spring 2019 Problem 4. Let $\mathcal{V}$ be the subspace of $L^\infty([0,1], \mu)$ (where μ is the Lebesgue measure on $[0,1]$) defined by
$$\mathcal{V} = \{f \in L^\infty([0,1], \mu) : \lim_{n\to\infty} n \int_{[0,1/n]} f d\mu \text{ exists}\}$$
(a) Prove that there exists $\varphi \in L^\infty([0,1], \mu)^*$ (i.e., a continuous functional on $L^\infty([0,1], \mu)$) such that $\varphi(f) = \lim_{n\to\infty} n \int_{[0,1/n]} f d\mu$ for every $f \in \mathcal{V}$.

(b) Show that, given any $\varphi \in L^\infty([0,1], \mu)^*$ satisfying the condition in (a) above, there exists no $g \in L^1([0,1], \mu)$ such that $\varphi(f) = \int fg d\mu$ for all $f \in L^\infty([0,1], \mu)$.

Proof. (a): Note first that, for each n,
$$|n \int_{[0,1/n]} f dm| \le \|f\|_\infty \times n \int_{[0,1/n]} dm = \|f\|_\infty$$
so the linear map $\varphi_0 : \mathcal{V} \to \mathbb{R}$, $\varphi_0(f) = \lim_{n\to\infty} n \int_{[0,1/n]} f dm$ satisfies the bound
$$|\varphi_0(f)| \le \sup_n \|f\|_\infty = \|f\|_\infty$$
By Hahn-Banach, there exists $\varphi : L^\infty([0,1], m) \to \mathbb{R}$ linear with norm bounded by 1 such that $\varphi|_{\mathcal{V}} = \varphi_0$, as was to be shown.

(b): Suppose to the contrary that $g \in L^1([0,1], m)$ is such that, for any $f \in L^\infty([0,1], m)$,
$$\varphi(f) = \int fg dm$$

where φ is as in (a). In particular, testing against $f = 1 \in \mathcal{V}$,

$$1 = \varphi(1) = \int gdm$$

so certainly $\|g\|_1 \geq 1$. On the other hand, for any $\varepsilon > 0$ and any $f \in L^\infty([0,1], m)$ such that $f|_{[0,\varepsilon]} \equiv 0$, we have

$$0 = \varphi(f) = \int fgdm$$

In particular, $g|_{(\varepsilon,1]} \equiv 0$ a.e. Since $\varepsilon > 0$ was arbitrary, we conclude that g has essential support contained in $\{0\}$, i.e. $g \equiv 0$ as an element of $L^1([0,1], m)$. But this contradicts the estimate $\|g\|_1 \geq 1$ from earlier, and we're done.

$\square$

Spring 2019 Problem 5. (a) Prove that $L^p([0,1], \mu)$ are separable Banach spaces for $1 \leq p < \infty$ but $L^\infty([0,1], \mu)$ is not (where μ is Lebesgue measure on $[0,1]$).

(b) Prove that there exists no linear bounded surjective map $T : L^p([0,1], \mu) \to L^1([0,1], \mu)$.

Proof. (a): For each $1 \geq s > r > 0$, we have

$$\|1_{[0,s]} - 1_{[0,r]}\|_\infty = \|1_{(r,s]}\|_\infty = 1$$

so the family $\{B(1_{[0,r]}, \frac{1}{2})\}_{0<r\leq 1}$ of nonempty open sets is pairwise disjoint and uncountable, hence $L^\infty([0,1])$ is not separable.

Fix now $1 \leq p < \infty$. It is unclear what is expected to be assumed for the purposes of this problem; we choose to take for granted that continuous functions are dense in L^p, and the Stone-Weierstrass theorem.

We claim that polynomials with rational coefficients are dense in $L^p([0,1], \mu)$. To see this, fix $f \in L^p([0,1], \mu)$ and $\varepsilon > 0$. By the density of continuous functions, there is some $g \in C([0,1])$ such that $\|f - g\|_p < \varepsilon/2$. By the Stone-Weierstrass theorem, we may find some polynomial P such that

$$\|P - g\|_\infty < \frac{\varepsilon}{4}$$

Let the form of P be

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$

with $a_0, \ldots, a_n \in \mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we may find $b_0, \ldots, b_n \in \mathbb{Q}$ satisfying the estimates

$$|b_k - a_k| \leq \frac{\varepsilon}{4(n+1)}, \quad (0 \leq k \leq n)$$

If we then write $Q(x) = b_n x^n + \ldots + b_1 x + b_0$, we compute

$$\|Q - P\|_{L^\infty([0,1])} \leq \sum_{k=0}^{n} |b_k - a_k| \leq \frac{\varepsilon}{4}$$

so that, by Hölder,

$$\|g - Q\|_p \leq \|g - P\|_p + \|P - Q\|_p \leq \|g - P\|_\infty + \|P - Q\|_\infty < \frac{\varepsilon}{2}$$

and hence

$$\|f-Q\|_p \leq \|f-g\|_p + \|g-Q\|_p < \varepsilon$$

as was to be established. Since polynomials with rational coefficients form a countable set, we conclude that L^p is separable.

(b): Suppose, for the sake of contradiction, $p > 1$ and $T : L^p([0,1],\mu) \to L^1([0,1],\mu)$ is a continuous linear surjection. Abusing notation slightly, we write T^* for the induced dual map $L^\infty([0,1],\mu) \to L^{p'}([0,1],\mu)$, where p' is the usual dual exponent in $[1,\infty)$. Observe that T^* is linear and bounded; we claim that further there is $c > 0$ such that $\|T^*g\| \geq c\|g\|$ for each $g \in L^\infty([0,1],\mu)$.

Indeed, we may compute by duality

$$\|T^*g\|_{L^{p'}} = \sup_{\|f\|_p<1} \langle T^*g, f\rangle = \sup_{\|f\|_p<1} \langle g, Tf\rangle = \sup\{\langle g,h\rangle : h \in T[B(0,1)]\}$$

(writing $B(0,1)$ for the open unit ball in L^p). If $g \neq 0$, then there is $h \in L^1([0,1],\mu)$ with $\|h\| \leq \frac{1}{2}$ and $\langle g,h\rangle \geq \frac{1}{4}\|g\|_\infty$. Since T is surjective, the open mapping theorem implies that $T[B(0,1)]$ is open, so there is some $\lambda > 0$ such that $\lambda h \in T[B(0,1)]$. Hence

$$\|T^*g\|_{L^{p'}} \geq \langle g, \lambda h\rangle = \lambda\langle g,h\rangle \geq \frac{\lambda}{4}\|g\|_\infty$$

so the desired conclusion holds with $c = \frac{\lambda}{4}$.

Finally, we use this estimate to reach a contradiction. Let $\{g_\alpha\}_\alpha$ be an uncountable family of elements of $L^\infty([0,1];\mu)$ such that any pair $\alpha \neq \alpha'$ have $\|g_\alpha - g_{\alpha'}\|_\infty \geq 1$. Then, by the preceding, $\{T^*g_\alpha\}_\alpha$ is an uncountable family of elements of $L^{p'}$ which are $\geq c$-separated, so the metric balls of radius $\frac{c}{2}$ centered at the T^*g_α are disjoint. However, this contradicts the fact from (a) that $L^{p'}([0,1],\mu)$ is separable. As such, no such T exists.

□

Spring 2019 Problem 6. Let $\mathcal{H}$ be a Hilbert space and $\{\xi_n\}_n$ a sequence of vectors in $\mathcal{H}$ such that $\|\xi_n\| = 1$ for all n.

(a) Show that if $\{\xi_n\}_n$ converges weakly to a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, then $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$.

(b) Show that if $\lim_{n,m\to\infty} \|\xi_n + \xi_m\| = 2$, then there exists a vector $\xi \in \mathcal{H}$ such that $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$.

Proof. (a): Observe that

$$\begin{aligned}
\|\xi_n - \xi\|^2 &= \langle \xi_n - \xi, \xi_n - \xi\rangle \\
&= \|\xi_n\|^2 + \|\xi\|^2 - 2\mathrm{Re}\langle \xi_n, \xi\rangle \\
&= 2 - 2\mathrm{Re}\langle \xi_n, \xi\rangle \\
&\to 2 - 2\mathrm{Re}\langle \xi, \xi\rangle = 0
\end{aligned}$$

as was to be shown.

(b): Write $\mathcal{H}' = \overline{\mathrm{span}}\{\xi_n : n \in \mathbb{N}\}$; thus $\mathcal{H}' \subseteq \mathcal{H}$ is the closed subspace spanned by the ξ_n. By construction, $\mathcal{H}'$ is separable, so the Banach-Alaoglu theorem implies that the unit ball of $\mathcal{H}'$ is weakly sequentially compact. Thus there is some $\xi \in \mathcal{H}$ that is a weak limit point for $\{\xi_n\}_n$, i.e. there is a subsequence such that $\xi_{n_k} \rightharpoonup \xi$ weakly. By (a), $\|\xi_{n_k} - \xi\| \to 0$.

Fix now $1 > \varepsilon > 0$ and let $N \in \mathbb{N}$ be sufficiently large so that $n, m \geq N$ implies $\|\xi_n + \xi_m\| \geq 2 - \varepsilon$, and if $n_k \geq N$ we have $\|\xi_{n_k} - \xi\| < \varepsilon$. Then, for each $n \geq N$, and any k such that $n_k \geq N$,

$$\begin{aligned} \|\xi_n - \xi\| &\leq \|\xi_n - \xi_{n_k}\| + \|\xi_{n_k} - \xi\| \\ &< \|\xi_n - \xi_{n_k}\| + \varepsilon \end{aligned}$$

whereas

$$(2-\varepsilon)^2 \leq \|\xi_n + \xi_{n_k}\|^2 = 2 + \mathrm{Re}\langle \xi_n, \xi_{n_k}\rangle$$

so that

$$\mathrm{Re}\langle \xi_n, \xi_{n_k}\rangle \geq 2 - 2\varepsilon + \varepsilon^2$$

Consequently,

$$\|\xi_n - \xi_{n_k}\|^2 = 2 - 2\mathrm{Re}\langle \xi_n, \xi_{n_k}\rangle \leq 2\varepsilon - \varepsilon^2$$

so that

$$\|\xi_n - \xi\| < \varepsilon + \sqrt{2\varepsilon - \varepsilon^2}$$

Thus, for each $0 < \varepsilon < 1$, we have found N so that $n \geq N$ implies the preceding inequality. It follows directly that $\|\xi_n - \xi\| \to 0$ as $n \to \infty$, as was to be shown.

□

Spring 2019 Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be entire non-constant, and let us set

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} \log_+ |f(re^{i\varphi})| d\varphi.$$

Here $\log_+ s = \max(\log s, 0)$. Show that $T(r) \to \infty$ as $r \to \infty$.

Proof. We separately handle the cases where f is polynomial, and where f is transcendental. Suppose first that f is a polynomial of degree n. Then, for R sufficiently large, there is $c > 0$ such that for all $|z| \geq R$,

$$|f(z)| \geq c|z|^n$$

Then, for $r \geq R$ large enough that $r^n c > 1$,

$$T(r) \geq \log(r^n c) = \log c + n \log r$$

and we conclude that T is divergent as $r \to \infty$.

Now we assume that f is entire and nonpolynomial. Under the assumption, we may find $|\alpha| \leq 1$ such that $f_\alpha := f - \alpha$ has infinitely many zeroes, and 0 is not one of them. For $R > 1$ such that f_α is nonvanishing on $|z| = R$, let $B_R(z)$ be the (rescaled) Blaschke factor

$$\prod_{j=1}^{n} \frac{(z/R) - (z_j/R)}{1 - \overline{(z_j/R)}(z/R)}$$

where $z_1, \ldots, z_n$ are the zeroes of f_α on $\{|z| < R\}$. Then there exists a zero-free holomorphic function g_R defined on a neighborhood of $\{|z| \leq R\}$ for which

$$f_\alpha(z) = g_R(z) B_R(z)$$

Observe that $|B_R(Re^{i\theta})| = 1$ for all θ. Consequently,

$$\log|f_\alpha(Re^{i\theta})| = \log|g_R(Re^{i\theta})|$$

Notice that $z \mapsto \log|g_R(z)|$ is harmonic. Thus

$$\int_0^{2\pi} \log|g_R(Re^{i\theta})|d\theta = 2\pi\log|g_R(0)| = 2\pi\log|f_\alpha(0)| - 2\pi\sum_{j=1}^{n}\log\frac{|z_j|}{R}$$

Since $|z_j| \leq R$ for each j,

$$\frac{1}{\log R}\int_0^{2\pi} \log|g_R(Re^{i\theta})|d\theta \geq \frac{2\pi}{\log R}|\log|f_\alpha(0)|| + 2\pi\log 2\#\{z \in \mathbb{C} : f(z) = 0, |z| \leq \frac{R}{2}\}$$

The quantity counted in the previous display diverges as $R \to +\infty$. Consequently,

$$\lim_{R\to+\infty}\int_0^{2\pi}\log|f_\alpha(Re^{i\theta})|d\theta = +\infty$$

Finally, observe that, for each z,

$$\max(\log|f(z) - \alpha|, 0) \leq \max(\log|f(z)|, 0) + \max(\log|\alpha|, 0) + \log 2 = \max(\log|f(z)|, 0) + \log 2$$

so that

$$\int_0^{2\pi}\max(\log|f(Re^{i\theta})|, 0) \geq -2\pi\log 2 + \int_0^{2\pi}\log|f(Re^{i\theta}|)d\theta$$

from which the desired conclusion follows.

$\square$

Spring 2019 Problem 8. Show that

$$\sin z - z\cos z = \frac{z^3}{3}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right), \quad z \in \mathbb{C},$$

where $(\lambda_n)_{n\geq1}$ is a sequence in $\mathbb{C}$, $\lambda_n \neq 0$ for all n, such that

$$\sum_{n=1}^{\infty}\frac{1}{|\lambda_n|^2} < \infty.$$

Proof. Let f be the function on the left-hand side. First, taking expansions at 0,

$$\sin z - z\cos z = z - \frac{z^3}{3!} + O(z^5) - z + \frac{z^3}{2!} + O(z^5) = \frac{z^3}{3} + O(z^5)$$

hence f vanishes to order 3 at 0. Next, from the easy estimates

$$|\sin z| \lesssim e^{|z|}, \quad |z\cos z| \lesssim |z|e^{|z|} = e^{|z|+\log|z|} = e^{|z|(1+o(1))}$$

we see that f has order at most 1. It follows that the zeroes $\{\eta_n\}_n \subseteq \mathbb{C} \setminus \{0\}$ of f away from 0 satisfy

$$\sum_n \frac{1}{|\eta_n|^{1+1}} < \infty$$

On the other hand, it is immediate to see that f is even. Thus the zeroes η_n come in plus/minus pairs. Let $\{\lambda_n\}_n$ be a choice of representatives; thus, $\lambda_n \neq \pm\lambda_m$ for $n \neq m$, and each η_n is equal to some $\pm\lambda_m$. Then the preceding bound implies

$$\sum_n \frac{1}{|\lambda_n|^2} < \infty$$

as was to be established.

Next, we note that f has infinitely many zeroes. Indeed, for k an integer,

$$f(k\pi) = (-1)^{k+1}k\pi$$

so by the intermediate value theorem we have a zero between $k\pi$ and $(k+1)\pi$ for each integer k.

Next, since f has order 1, it has the Hadamard factorization

$$f(z) = z^3 e^g \prod_{n=1}^{\infty}(1 - \frac{z}{\eta_n})e^{z/\eta_n}$$

with g a polynomial of degree at most 1. Collecting together the $\pm$ pairs of the zeroes, this factorization rearranges as

$$f(z) = z^3 e^g \prod_{n=1}^{\infty}(1 - \frac{z^2}{\lambda^2})$$

Indeed, the infinite product converges locally uniformly, so we may rearrange freely. It remains to consider the polynomial g.

Finally, observe that

$$\frac{f'}{f} = \frac{3}{z} + g' + \sum_{n=1}^{\infty} \frac{-2z}{\lambda_n^2 - z^2}$$

The infinite series is $O(z)$ near 0; additionally,

$$\frac{f'}{f} = \frac{z \sin z}{\sin z - z \cos z} = \frac{3}{z} + O(z)$$

near 0, so $g' = O(z)$. Thus g is constant. Since we have already identified $f(z)$ to have leading expansion $\frac{z^3}{3} + O(z^5)$, and this agrees with the $\frac{z^3}{3}\prod_n(1 - \frac{z^2}{\lambda_n^2})$ quantity to leading order, we conclude that in fact $g = 0$. Thus the desired factorization holds.

□

Spring 2019 Problem 9. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}(\mathbb{D})$ be the space of functions holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Let

$$\mathcal{U} = \{f \in \mathcal{A}(\mathbb{D}); |f(z)| = 1 \quad \text{for all } z \in \partial\mathbb{D}\}.$$

Show that $f \in \mathcal{U}$ if and only if f is a finite Blaschke product,

$$f(z) = \lambda \prod_{j=1}^{N} \frac{z - a_j}{1 - \bar{a}_j z},$$

for some $a_j \in \mathbb{D}$, $1 \leq j \leq N < \infty$ and $|\lambda| = 1$.

Proof. We only attend to the forward direction, as the reverse direction is trivial. Let $f \in \mathcal{U}$. Suppose first that f has no zeroes in $\mathbb{D}$. If f is constant, then we are done; otherwise, by the maximum principle, $|f(0)| < 1$. Write $c = f(0)$ and

$$\phi : \mathbb{D} \to \mathbb{D}, \quad \phi(z) = \frac{z - c}{1 - \bar{c}z}$$

Then certainly ϕ is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Additionally, note that for $|z| = 1$

$$\begin{aligned} |z - c|^2 &= |z|^2 - z\bar{c} - \bar{z}c + |c|^2 \\ &= 1 - z\bar{c} - \bar{z}c + |\bar{c}z|^2 \\ &= |1 - \bar{c}z|^2 \end{aligned}$$

so that $|\phi(z)| = 1$. Consequently, $f \circ \phi$ is another element of $\mathcal{U}$. But $(f \circ \phi)(c) = 0$, so $f(\phi(c)) = 0$, so $\phi(c)$ is a zero of f. Since ϕ is analytic, nonconstant, and takes magnitude 1 on $\partial\mathbb{D}$, we conclude that $|\phi(c)| < 1$, so f does have a zero. By contradiction, we conclude that any element of $\mathcal{U}$ without zeroes is a constant, and we are done in this case.

We now consider the case that f has N zeroes for some $N \in \mathbb{N}$. Counting multiplicity, write them as $a_1, \ldots, a_N \in \mathbb{D}$ (observe that the boundary condition necessitates that no $a_j \in \partial\mathbb{D}$). Write then

$$g(z) = f(z) \prod_{j=1}^{N} \frac{1 - \bar{a}_j z}{z - a_j}$$

Then g is analytic in $\mathbb{D}$, and (since the linear factors in the numerator vanish at $1/\bar{a}_j \notin \overline{\mathbb{D}}$) have no zeroes in $\overline{\mathbb{D}}$. Since each factor has magnitude 1 on $\partial\mathbb{D}$, we have that $g \in \mathcal{U}$. By the previous case, $g \equiv \lambda$ for some $\lambda \in \partial\mathbb{D}$, i.e.

$$f(z) = \lambda \prod_{j=1}^{N} \frac{z - a_j}{1 - \bar{a}_j z}$$

on $\mathbb{D}$, as was to be shown.

Lastly, suppose $f \in \mathcal{U}$ has infinitely many distinct zeroes $\{a_n\}_n$. Then the latter set accumulates to some element a of $\overline{\mathbb{D}}$. Since f is continuous on $\overline{\mathbb{D}}$ and takes magnitude 1 on $\partial\mathbb{D}$, $a \in \mathbb{D}$. But then by the uniqueness principle, $f \equiv 0$, whereas f must extend continuously to nonzero quantities on the boundary, a contradiction. Thus we are done in all cases.

□

Spring 2019 Problem 10. For $a > 0, b > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(x + a)^2 + b^2} dx.$$

Proof. We will evaluate the integral as

$$\int_0^\infty \frac{\log x}{(x+a)^2+b^2} = \frac{1}{4b}\arctan(b/a)\cdot\log(a^2+b^2)$$

via integrating $\frac{(\log z)}{(z+a)^2+b^2}$ around a keyhole contour.

Write $\log z$ for the branch of the logarithm with cut along $\mathbb{R}_{\geq 0}$, defined to have imaginary part between $[0,2\pi]$; we will freely write $\log x \in \mathbb{R}$ and $\log x \in \mathbb{R}+2\pi i$, as context warrants. For $1>\varepsilon>0$ and $R>1$ two parameters, write $\gamma_j = \gamma_j^{(\varepsilon,R)}$ $(1\leq j\leq 4)$ for the curves

$$\begin{aligned}\gamma_1(t) &= t \quad \big(\varepsilon\leq t\leq R\big),\\ \gamma_2(t) &= Re^{it} \quad (0\leq t\leq 2\pi),\\ \gamma_3(t) &= R-t \quad \big(0\leq t\leq R-\varepsilon\big),\\ \gamma_4(t) &= \varepsilon e^{i(2\pi-t)} \quad (0\leq t\leq 2\pi)\end{aligned}$$

Let also $\Gamma = \Gamma^{(\varepsilon,R)}$ for the closed curve given by traversing $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ in that order. Then the function

$$f(z) = \frac{(\log z)^2}{(z+a)^2+b^2}$$

has singularities at $z=-a+ib$ and $z=-a-ib$, and is otherwise holomorphic in the domain cut out by Γ. We will always assume that R is sufficiently large, and ε sufficiently small, so that $-a+ib$ is contained in that domain.

Considering the factorization

$$(z+a)^2+b^2 = (z+a+ib)(z+a-ib)$$

we see that f has a simple pole at $-a+ib$, so

$$\operatorname{Res}\left[f(z),-a+ib\right] = \operatorname{Res}\left[\frac{(\log z)^2/(z+a+ib)}{(z+a-ib)},-a+ib\right] = \frac{(\log(-a+ib))^2}{2ib}$$

Similarly, there is a simple pole at $-a-ib$ with residue

$$\operatorname{Res}\left[f(z),-a-ib\right] = \operatorname{Res}\left[\frac{(\log z)^2/(z+a-ib)}{(z+a+ib)},-a-ib\right] = -\frac{(\log(-a-ib))^2}{2ib}$$

Thus, by the residue theorem,

$$\int_\Gamma f(z)dz = \frac{\pi}{b}\left((\log(-a+ib))^2-(\log(-a-ib))^2\right) =: c_{a,b}$$

We now analyze the components of $\int_\Gamma = \sum_{j=1}^4 \int_{\gamma_j}$. First,

$$\int_{\gamma_1} f(z)dz = \int_\varepsilon^R \frac{(\log x)^2}{(x+a)^2+b^2}dx$$

Next,

$$\int_{\gamma_3} f(z)dz = \int_R^\varepsilon \frac{(\log x+2\pi i)^2}{(x+a)^2+b^2}dx = -\int_\varepsilon^R \frac{(\log x)^2+4\pi i\log x-4\pi^2}{(x+a)^2+b^2}dx$$

will be related to the preceding integral. Next,

$$\int_{\gamma_2} f(z)dz = \int_0^{2\pi} \frac{(\log R + it)^2}{(Re^{it}+a)^2+b^2} iRe^{it}dt = i\frac{(\log R)^2}{R}\int_0^{2\pi} \frac{1+\frac{it}{\log R}}{(e^{it}+\frac{a}{R})^2+(b/R)^2}e^{it}dt$$

For R sufficiently large, the integrand is pointwise dominated by 4 in magnitude. Thus, by a trivial estimate,

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)dz = 0$$

Finally,

$$\int_{\gamma_4} f(z)dz = \int_0^{2\pi} \frac{(\log\varepsilon + i(\pi - t))^2}{(\varepsilon e^{i(\pi-t)}+a)^2+b^2}(-i\varepsilon e^{i(\pi-t)})dt$$

For ε sufficiently small, the integrand is pointwise bounded by $\frac{4\pi\varepsilon(\log\varepsilon)^2}{a^2+b^2}$ in magnitude. Thus, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0^+}\int_{\gamma_4} f(z)dz = \int_0^{\pi} 0dt = 0$$

where we are of course using the estimate $\varepsilon(\log\varepsilon)^2 = o(1)$.

Now we combine the components. Note that

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = -\int_{\varepsilon}^{R} \frac{4\pi i\log x - 4\pi^2}{(x+a)^2+b^2}dx$$

so that, sending $\varepsilon\to 0$ and $R\to\infty$ in $\int_\Gamma = \sum_{j=1}^4 \int_{\gamma_j}$,

$$\int_0^\infty \frac{\log x}{(x+a)^2+b^2}dx = -\frac{c_{a,b}}{4\pi i} + \frac{\pi}{i}\int_0^\infty \frac{1}{(x+a)^2+b^2}dx$$

Analyzing the second summand using real-variable techniques,

$$\int_0^\infty \frac{1}{(x+a)^2+b^2}dx = \frac{1}{a}\int_1^\infty \frac{1}{x^2+(b/a)^2}dx = \frac{1}{b}\int_{\frac{a}{b}}^\infty \frac{1}{x^2+1}dx$$

which is just

$$\int_0^\infty \frac{1}{(x+a)^2+b^2}dx = \frac{1}{b}\left(\frac{\pi}{2} - \arctan(\frac{a}{b})\right)$$

and hence

$$\int_0^\infty \frac{\log x}{(x+a)^2+b^2}dx = \frac{c_{a,b}}{4\pi i} + \frac{\pi}{bi}\left(\frac{\pi}{2} - \arctan(\frac{a}{b})\right)$$

Finally, recalling

$$c_{a,b} = \frac{\pi}{b}\left((\log(-a+ib))^2 - (\log(-a-ib))^2\right)$$

and letting $\theta = \arctan(b/a)$ and $r = \sqrt{a^2+b^2}$, we have

$$\log(-a+ib) = \log r + i(\pi-\theta), \quad \log(-a-ib) = \log r + i(\pi+\theta)$$

so that

$$(\log(-a+ib))^2 - (\log(-a-ib))^2 = 4\pi\theta - 2i\theta\log r$$

and thus (using $\arctan(a/b) = \frac{\pi}{2} - \arctan(b/a)$)

$$\begin{aligned}\int_0^\infty \frac{\log x}{(x+a)^2+b^2}dx &= -\frac{1}{4bi}(4\pi\theta - 2i\theta\log r) + \frac{\pi}{bi}\theta \\ &= \frac{\theta\log r}{2b} \\ &= \frac{1}{4b}\arctan(b/a)\cdot\log(a^2+b^2)\end{aligned}$$

□

Spring 2019 Problem 11. Let $u \in C^\infty(\mathbb{R})$ be smooth 2π-periodic. Show that there exists a bounded holomorphic function f_+ in the upper half-plane $\mathrm{Im} z > 0$ and a bounded holomorphic function f_- in the lower half-plane $\mathrm{Im} z < 0$, such that

$$u(x) = \lim_{\varepsilon\to 0^+}(f_+(x+i\varepsilon) - f_-(x-i\varepsilon)), \quad x \in \mathbb{R}.$$

Proof. needs finishing

It is clear that we may freely add constants to u without altering the truth of the question; as such, we may assume $\int_0^{2\pi} u(x)dx = 0$. For each $n \in \mathbb{N}$, write

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-inx}dx$$

and

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{inx}dx$$

Note that these integrals all converge. We first remark on the size of the a_n, b_n. Trivially we have

$$|a_n| \leq \frac{1}{2\pi}\int_0^{2\pi}|u(x)|dx \leq \|u\|_\infty$$

and similarly for the b_n; thus these coefficients are uniformly bounded. Moreover, since u is smooth, for each $n \in \mathbb{N}$ we may integrate by parts to obtain

$$\int_0^{2\pi} u(x)e^{-inx}dx = -\frac{1}{n^2}\int_0^{2\pi} u(x)\frac{d^2}{dx^2}\left[e^{-inx}\right]dx = -\frac{1}{n^2}\int_0^{2\pi} u^{(2)}(x)e^{-inx}dx$$

so that

$$|a_n| \leq \frac{1}{2\pi n^2}\int_0^{2\pi}|u^{(2)}(x)|dx \leq \frac{\|u^{(2)}\|_\infty}{2\pi}$$

and hence $\{a_n\}_{n\in\mathbb{N}}$ is summable. Similarly, $\{b_n\}_{n\in\mathbb{N}}$ is summable.

Define the auxiliary functions g_+, g_- by

$$g_+(z) = \sum_{n=1}^{\infty} a_n z^n, \quad g_-(z) = -\sum_{n=1}^{\infty} b_n z^{-n}$$

Since the coefficients are bounded, the series converge on the unit disk $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, respectively. Further, since the coefficients are summable, we obtain

$$|g_+(z)| \leq \sum_{n=1}^{\infty} n^{-2} \|u^{(2)}\|_\infty |z|^n \leq \|u\|_\infty + \sum_{n=1}^{\infty} n^{-2} \|u^{(2)}\|_\infty$$

for all $z \in \mathbb{D}$, and we may find a similar upper bound on g_-; thus the g_+, g_- are bounded on $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, respectively.

We now define

$$f_+(z) = g_+(e^{iz}) \quad (\mathrm{Im}(z) > 0)$$

and

$$f_-(z) = g_-(e^{iz}) \quad (\mathrm{Im}(z) < 0)$$

Note then that

$$\begin{aligned}
f_+(x + i\varepsilon) - f_-(x - i\varepsilon) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} u(t)(e^{in(-t+x+i\varepsilon)} + e^{in(t-x+i\varepsilon)}) dt \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} e^{-n\varepsilon} u(t) \cos(n(x-t)) dt \\
&= \frac{i}{\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} e^{-n\varepsilon} u(t) \cos(n(x-t)) dt
\end{aligned}$$

where in commuting the sum into the integral we are using the exponential decay to guarantee enough summability to use Fubini-Tonelli. As a consequence,

$$\begin{aligned}
f_+(x + i\varepsilon) - f_-(x - i\varepsilon) &= \frac{1}{\pi} \int_0^{2\pi} u(t) \mathrm{Re}\Big[\frac{e^{i(-t+x+i\varepsilon)}}{1 - e^{i(-t+x+i\varepsilon)}}\Big] dt \\
&= \frac{1}{\pi} \mathrm{Re}\left[\int_0^{2\pi} u(t) \frac{1}{e^{i(t-x-i\varepsilon)} - 1} dt\right]
\end{aligned}$$

It remains to estimate the integration kernel. Without loss of generality we will take $x = 0$. Then the kernel takes the form

$$\frac{1}{e^{it+\varepsilon} - 1}$$

which has mean zero on $t \in [0, 2\pi]$. It is also (2π)-periodic, so for each $1 \gg \eta > 0$ we may write

$$
\begin{aligned}
f_+(x+i\varepsilon) - f_-(x-i\varepsilon) &= \frac{1}{\pi}\mathrm{Re}\left[\int_{|t|\le\eta} u(t)\frac{1}{e^{it+\varepsilon}-1}dt + \int_{\eta\le|t|\le\pi} u(t)\frac{1}{e^{it+\varepsilon}-1}dt\right] \\
&= \frac{1}{\pi}\mathrm{Re}\left[\int_{|t|\le\eta} u(t)\frac{1}{e^{it+\varepsilon}-1}dt\right] + \frac{1}{\pi}\int_{\eta\le|t|\le\pi} u(t)\mathrm{Re}\left[\frac{1}{e^{it+\varepsilon}-1}\right]dt \\
&= \frac{1}{\pi}\mathrm{Re}\left[\int_{|t|\le\eta} u(t)\frac{1}{e^{it+\varepsilon}-1}dt\right] - \frac{1}{2\pi}\int_{\eta\le|t|\le\pi} u(t)dt \\
&= \frac{1}{\pi}\mathrm{Re}\left[\int_{|t|\le\eta} u(t)\frac{1}{e^{it+\varepsilon}-1}dt\right] + \frac{1}{2\pi}\int_{|t|\le\eta} u(t)dt \\
&= \frac{1}{2\pi}\int_{|t|\le\eta} u(t)\mathrm{Re}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right]dt
\end{aligned}
$$

Considering the kernel $1_{[-\eta,\eta]}(t)\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}$, we wish to argue that for $\eta = \varepsilon$ we have that the distribution tends to $2\pi\delta_0$ as $\varepsilon \to 0^+$. Since, for each $|t| \le \eta$, we have

$$
\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1} = \frac{e^{2\varepsilon} - e^{it+\varepsilon} + e^{-it+\varepsilon} - 1}{(e^\varepsilon\cos t - 1)^2 + e^{2\varepsilon}\sin^2 t} = \frac{e^{2\varepsilon} - 1 + 2ie^\varepsilon \sin t}{e^{2\varepsilon} - 2e^\varepsilon\cos t + 1}
$$

for which

$$
\left|\frac{\mathrm{Im}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right]}{\mathrm{Re}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right]}\right| = \frac{2e^\varepsilon|\sin t|}{e^{2\varepsilon}-1} \lesssim \varepsilon^{-1}\eta
$$

where we have used the small-angle approximation $\sin t \sim t$ and the first-order expansion $e^{2\varepsilon} \sim 1 + 2\varepsilon$. Thus, if $\eta \ll \varepsilon$,

$$
\begin{aligned}
\left|\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right| &= \sqrt{\left|\mathrm{Re}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right]\right|^2 + \left|\mathrm{Im}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right]\right|^2} \\
&= \left|\mathrm{Re}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right]\right|(1+O(\varepsilon))
\end{aligned}
$$

On the other hand,

$$
\left|\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right| = \frac{|e^{it+\varepsilon}+1|}{|e^{it+\varepsilon}-1|} = \frac{e^\varepsilon + 1 + O(\varepsilon^2)}{\varepsilon + O(\varepsilon^2)} = \varepsilon^{-1}(e^\varepsilon+1)(1+O(\varepsilon))
$$

so that

$$
\mathrm{Re}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right] = \varepsilon^{-1}(e^\varepsilon+1)(1+O(\varepsilon))
$$

Using this, and the first-order Taylor expansion of u, we obtain the estimate

$$
\begin{aligned}
f_+(x+i\varepsilon) - f_-(x-i\varepsilon) &= \frac{1}{2\pi}\int_{|t|\le\varepsilon^2} u(t)\varepsilon^{-1}(e^\varepsilon+1)(1+O(\varepsilon))dt \\
&=
\end{aligned}
$$

Thus
$$\text{Re}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right] \le \left|\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right| \lesssim \text{Re}\left[\frac{e^{it+\varepsilon}+1}{e^{it+\varepsilon}-1}\right]$$

$$\frac{1}{e^{it+\varepsilon}-1} = \dots$$

□

Spring 2019 Problem 12. Let $\mathcal{H}$ be the vector space of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that
$$\int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty.$$
Here $d\mu(z) = e^{-|z|^2} d\lambda(z)$, where $d\lambda(z)$ is the Lebesgue measure on $\mathbb{C}$.

1. Show that $\mathcal{H}$ is a closed subspace of $L^2(\mathbb{C}, d\mu)$.
2. Show that for all $f \in \mathcal{H}$, we have
$$f(z) = \frac{1}{\pi} \int_{\mathbb{C}} f(w) e^{z\bar{w}} d\mu(w), \quad z \in \mathbb{C}.$$

Hint for 2): Show that the normalized monomials
$$e_n(z) = \frac{1}{(\pi n!)^{1/2}} z^n, \quad n = 0, 1, \dots$$
form an orthonormal basis of $\mathcal{H}$.

Proof. Throughout this problem, we will simply write $\|\cdot\|_2$ for the exponentially weighted L^2 norm indicated in the problem.

(1): It will suffice to show that, if $\{f_n\}_n$ is a sequence of elements of $\mathcal{H}$ and $f \in L^2(\mathbb{C}, d\mu)$ with $\lim_{n\to\infty} \|f_n - f\|_2 = 0$, then $f \in \mathcal{H}$.

To this end, fix such $\{f_n\}_n$ and f. We will show a locally uniform Cauchy condition on the f_n. Fix $z \in \mathbb{C}$; then we have for each $n, m \in \mathbb{N}$ and $r > |z|$,
$$f_n(z) - f_m(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f_n(w) - f_m(w)}{w - z} dw$$
so that, for each $R > \rho > 2|z|$,
$$f_n(z) - f_m(z) = \frac{1}{2\pi i I(\rho, R)} \int_\rho^R \int_{|w|=r} \frac{f_n(w) - f_m(w)}{w - z} e^{-r^2} dw r dr$$
where we have written
$$I(\rho, R) = \int_\rho^R e^{-r^2} r dr$$

Thus

$$|f_n(z) - f_m(z)| \le \frac{1}{2\pi I(\rho, R)} \iint_{\rho \le |w| \le R} |f_n(w) - f_m(w)| e^{-|w|^2} \frac{|w|}{|w| - |z|} d\lambda(w)$$

which by Cauchy-Schwartz supplies

$$\begin{aligned}|f_n(z) - f_m(z)| \le & \frac{1}{2\pi I(\rho, R)} \left(\iint_{\mathbb{C}} |f_n(w) - f_m(w)|^2 e^{-|w|^2} d\lambda(w) \right)^{1/2} \\ & \times \left(\iint_{\rho \le |w| \le R} e^{-|w|^2} \left[\frac{|w|}{|w| - |z|} \right]^2 d\lambda(w) \right)^{1/2}\end{aligned}$$

Observe that $\rho \ge 2|z|$, so for any $|w| \ge \rho$ we have

$$\frac{|w|}{|w| - |z|} = \frac{1}{1 - |z/w|} \le \frac{1}{1 - \frac{1}{2}} = 2$$

and hence

$$\left(\iint_{\rho \le |w| \le R} e^{-|w|^2} \left[\frac{|w|}{|w| - |z|} \right]^2 d\lambda(w) \right)^{1/2} \le (8\pi I(\rho, R))^{1/2}$$

Thus we have

$$|f_n(z) - f_m(z)| \le \frac{1}{\sqrt{2\pi I(\rho, R)}} \|f_n - f_m\|_2$$

for any $R > \rho \ge |z|$. On the other hand, we may compute

$$I(\rho, R) = \frac{1}{2} \left(e^{-\rho^{1/2}} - e^{-R^{1/2}} \right) \gtrsim \frac{1}{4} e^{-\rho^{1/2}}$$

for all R sufficiently large. Thus

$$|f_n(z) - f_m(z)| \lesssim e^{\frac{1}{2}\rho^{1/2}} \|f_n - f_m\|_2$$

uniformly over $|z| < \frac{1}{2}\rho$ and $n, m \in \mathbb{N}$. It follows that the $\{f_n\}_n$ are uniformly Cauchy on each compact set, hence $f_n \to f$ locally uniformly, hence f is the locally uniform limit of analytic functions, hence is analytic. We conclude $f \in \mathcal{H}$, as was to be shown.

(2): We follow the hint. Note that, for any $n, m \in \mathbb{Z}_{\ge 0}$,

$$\begin{aligned}\int_{\mathbb{C}} e_n(z) \overline{e_m}(z) d\mu(z) &= \frac{1}{\pi (n!)^{1/2} (m!)^{1/2}} \int_0^\infty r^{n+m} \int_0^{2\pi} e^{i(n-m)} e^{-r^2} r d\theta dr \\ &= \frac{2}{(n!)^{1/2} (m!)^{1/2}} \int_0^\infty r^{n+m+1} e^{-r^2} \delta_{n=m} dr \\ &= \delta_{n=m} \frac{2}{n!} \int_0^\infty r^{2n+1} e^{-r^2} dr \\ &= \delta_{n=m} \frac{1}{n!} \int_0^\infty u^n e^{-u} du \\ &= \delta_{n=m}\end{aligned}$$

where in the last step we use the well-known Gamma integral

$$\int_0^\infty t^n e^{-t} dt = n!$$

provable by induction and integration-by-parts.

Thus $\{e_n\}_n$ form an orthonormal family. We claim that it is complete. Suppose $f \in \mathcal{H}$ is orthogonal to all e_n. Then, writing

$$f(z) = \sum_{k=0}^\infty a_k z^k$$

we have

$$0 = \int f(z)\overline{e_n}(z) d\mu(z) = \frac{1}{(\pi n!)^{1/2}} \int \sum_{k=0}^\infty a_k z^k \bar{z}^n e^{-|z|^2} d\lambda(z)$$

Pick now any $R > 0$. Since the series defining f has infinite radius of convergence, for each $\varepsilon > 0$ we may find $C_\varepsilon > 1$ such that $|a_k| \leq C_\varepsilon \varepsilon^k$ for all k by the Cauchy-Hadamard formula for the radius of convergence. Thus, taking $\varepsilon \ll R^{-1}$,

$$\begin{aligned}\int_{|z|\leq R} \sum_{k=0}^\infty \left|a_k z^k \bar{z}^n\right| e^{-|z|^2} d\lambda(z) &\leq C_\varepsilon \int_{|z|\leq R} \sum_{k=0}^\infty \varepsilon^k |z|^{n+k} e^{-|z|^2} d\lambda(z) \\ &\leq C_\varepsilon \varepsilon^{-n} \int_{|z|\leq R} \sum_{k=0}^\infty \varepsilon^k |z|^k e^{-|z|^2} d\lambda(z) \\ &= C_\varepsilon \varepsilon^{-n} \int_{|z|\leq R} \frac{1}{1-|\varepsilon z|} e^{-|z|^2} d\lambda(z) < \infty\end{aligned}$$

so that, for each R,

$$\int_{|z|\leq R} \sum_{k=0}^\infty a_k z^k \bar{z}^n e^{-|z|^2} d\lambda(z) = \sum_{k=0}^\infty \int_{|z|\leq R} a_k z^k \bar{z}^n e^{-|z|^2} d\lambda(z)$$

and the latter is equal to

$$\sum_{k=0}^\infty \delta_{k=n} \int_0^R a_k r^{k+n+1} e^{-r^2} dr = \int_0^R a_n r^{2n+1} e^{-r^2} dr$$

Taking $R \to \infty$, we conclude that

$$0 = \int_{\mathbb{C}} \sum_{k=0}^\infty a_k z^k \bar{z}^n e^{-|z|^2} d\lambda(z) = a_n \int_0^\infty r^{2n+1} e^{-r^2} dr = a_n n!$$

so that $a_n = 0$. Since this holds for each n, we conclude that $f = \sum_n a_n z^n$ is the zero function. Thus the kernel of the complement of $\mathrm{span}(\{e_n\}_n)$ in $\mathcal{H}$ is trivial, so $\mathrm{span}(\{e_n\}_n) = \mathcal{H}$, as claimed.

We now attack the problem at hand. Since $\{e_n\}_n$ are an orthonormal (topological) basis for $\mathcal{H}$, we have the reproducing formula

$$f = \sum_{n=0}^\infty \langle f, e_n \rangle e_n$$

Expanding this,

$$\sum_{n=0}^{\infty}\langle f, e_n\rangle e_n(z) = \sum_{n=0}^{\infty}\frac{z^n}{\pi n!}\int_{\mathbb{C}} f(w)\bar{w}^n d\mu(w)$$

We wish to commute the sum into the integral. We verify this by demonstrating absolute integrability, namely,

$$\begin{aligned}
\sum_{n=0}^{\infty}\int_{\mathbb{C}}\left|\frac{z^n}{\pi n!}f(w)\bar{w}^n\right|d\mu(w) &= \sum_{n=0}^{\infty}\frac{|z|^n}{\pi n!}\int_0^{\infty}\int_0^{2\pi}|f(re^{it})|r^{n+1}e^{-r^2}dtdr \\
&= \frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi}|f(re^{it})|e^{-r^2}\sum_{n=0}^{\infty}\frac{|rz|^n}{n!}dtrdr \\
&= \frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi}|f(re^{it})|e^{-r^2+r|z|}dtrdr \\
&= \frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi}|f(re^{it})|e^{-\frac{1}{2}r^2}e^{-\frac{1}{2}r^2+r|z|}dtrdr \\
&\le \frac{1}{\pi}\left(\int_0^{\infty}\int_0^{2\pi}|f(re^{it})|^2e^{-r^2}dtrdr\right)^{1/2}\left(\int_0^{\infty}\int_0^{2\pi}e^{-r^2+2r|z|}dtrdr\right)^{1/2} \\
&< \infty
\end{aligned}$$

so that

$$\sum_{n=0}^{\infty}\frac{z^n}{\pi n!}\int_{\mathbb{C}} f(w)\bar{w}^n d\mu(w) = \int_{\mathbb{C}}\sum_{n=0}^{\infty}\frac{z^n}{\pi n!}f(w)\bar{w}^n d\mu(w) = \int_{\mathbb{C}} f(w)e^{z\bar{w}}d\mu(w)$$

as was to be established.

$\square$

2 Fall 2019

Fall 2019 Problem 1. Given σ-finite measures $\mu_1, \mu_2, \nu_1, \nu_2$ on a measurable space $(X, \mathcal{X})$, suppose that $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$. Prove that the product measures $\mu_1 \otimes \mu_2$ and $\nu_1 \otimes \nu_2$ on $(X \times X, \mathcal{X} \otimes \mathcal{X})$ satisfy $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$ and the Radon-Nikodym derivatives obey

$$\frac{\mathrm{d}(\mu_1 \otimes \mu_2)}{\mathrm{d}(\nu_1 \otimes \nu_2)}(x, y) = \frac{\mathrm{d}\mu_1}{\mathrm{d}\nu_1}(x)\frac{\mathrm{d}\mu_2}{\mathrm{d}\nu_2}(y)$$

for $\nu_1 \otimes \nu_2$ almost every $(x, y) \in X \times X$.

Proof. Let $N \in \mathcal{X} \otimes \mathcal{X}$ be $\nu_1 \otimes \nu_2$-null. Observe that $\mu_1 \otimes \mu_2$ and $\nu_1 \otimes \nu_2$ are σ-finite. Then

$$0 = (\nu_1 \otimes \nu_2)(N) = \int_{X \times X} 1_N(x, y) d(\nu_1 \otimes \nu_2)(x, y)$$

Since the integrand is nonnegative and $\nu_1 \otimes \nu_2$ is σ-finite, Fubini-Tonelli implies

$$\int_{X \times X} 1_N(x, y) d(\nu_1 \otimes \nu_2)(x, y) = \int_X \int_X 1_N(x, y) d\nu_1(x) d\nu_2(y)$$

Since the preceding vanishes, it follows that there is a ν_2-nullset N_2 such that for every $y \in X \setminus N_2$ we have $\nu_1(\{x \in X : (x, y) \in N\}) = 0$.

We use this to show that $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$. Similarly as above,

$$(\mu_1 \otimes \mu_2)(N) = \int_{X \times X} 1_N(x, y) d(\mu_1 \otimes \mu_2)(x, y)$$

Since the integrand is nonnegative and $\mu_1 \otimes \mu_2$ is σ-finite, Fubini-Tonelli implies

$$\int_{X \times X} 1_N(x, y) d(\mu_1 \otimes \mu_2)(x, y) = \int_X \int_X 1_N(x, y) d\mu_1(x) d\mu_2(y)$$

Since $\mu_2 \ll \nu_2$, we have $\mu_2(N_2) = 0$. Thus

$$\int_X \int_X 1_N(x, y) d\mu_1(x) d\mu_2(y) = \int_{X \setminus N_2} \int_X 1_N(x, y) d\mu_1(x) d\mu_2(y)$$

For each $y \in X \setminus N_2$, the interior integral is $\mu_1(\{x \in X : (x, y) \in N\})$. By the last paragraph, the latter set is ν_1-null, hence is μ_1-null by absolute continuity. Thus

$$(\mu_1 \otimes \mu_2)(N) = 0$$

and we reach the conclusion that $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$.

Let

$$\mathcal{A} = \{A \in \mathcal{X} \otimes \mathcal{X} : (\mu_1 \otimes \mu_2)(A) = \int_{X \times X} 1_A(x, y) \frac{d\mu_1}{d\nu_1}(x) \frac{d\mu_2}{d\nu_2}(y) d(\nu_1 \otimes \nu_2)(x, y)\}$$

We first claim that $\mathcal{A}$ is a σ-algebra. It is clear that $\mathcal{A}$ is closed under countable disjoint unions. In the event that $A = X \times X$ is the full set, then

$$(\mu_1 \otimes \mu_2)(A) = \mu_1(X)\mu_2(X) = \left(\int_X d\mu_1\right)\left(\int_X d\mu_2\right) = \left(\int_X \frac{d\mu_1}{d\nu_1}(x) d\nu_1(x)\right)\left(\int_X \frac{d\mu_2}{d\nu_2}(y) d\nu_2(y)\right)$$

and Fubini implies that $A \in \mathcal{A}$. Finally, these last two facts and a straightforward demonstration imply that $\mathcal{A}$ is closed under complementation, so it is a σ-algebra.

Next, if $A, B \in \mathcal{X}$, then by Fubini

$$\begin{aligned}
&\int_{X\times X} 1_{A\times B}(x,y)\frac{d\mu_1}{d\nu_1}(x)\frac{d\mu_2}{d\nu_2}(y)d(\nu_1\otimes\nu_2)(x,y)\\
&= \left(\int_X 1_A(x)\frac{d\mu_1}{d\nu_1}(x)d\nu_1(x)\right)\left(\int_X 1_B(y)\frac{d\mu_2}{d\nu_2}(y)d\nu_2(y)\right)\\
&= \mu_1(A)\mu_2(B)
\end{aligned}$$

so that $A \times B \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, it follows that $\mathcal{X}\otimes\mathcal{X} \subseteq \mathcal{A}$. Since the reverse inclusion is true definitionally, it follows that

$$(\mu_1\otimes\mu_2)(A) = \int_{X\times X} 1_A(x,y)\frac{d\mu_1}{d\nu_1}(x)\frac{d\mu_2}{d\nu_2}(y)d(\nu_1\otimes\nu_2)(x,y)$$

for all $A \in \mathcal{X}\otimes\mathcal{X}$. Since Radon-Nikodym derivatives are equal $(\nu_1\otimes\nu_2)$-a.e., it follows that

$$\frac{d(\mu_1\otimes\mu_2)}{d(\nu_1\otimes\nu_2)}(x,y) = \frac{d\mu_1}{d\nu_1}(x)\frac{d\mu_2}{d\nu_2}(y)$$

for $(\nu_1\otimes\nu_2)$-a.e. $(x,y)\in X\times X$.

□

Fall 2019 Problem 2. Let μ be a finite Borel measure on $\mathbb{R}$ with $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$ and let $\varphi(t) = \int_{\mathbb{R}} e^{itx}d\mu(x)$. Prove that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\varphi(t)|^2\mathrm{d}t = 0$$

Proof. Let $\chi : \mathbb{R}\to\mathbb{R}$ be smooth and compactly supported. Then

$$\int_{\mathbb{R}}|\varphi(t)|^2\chi(t)dt = \iiint e^{it(x-y)}\chi(t)d\mu(x)d\mu(y)dt$$

Since μ is finite, the triple integral is over a σ-finite measure space. Note too that

$$\iiint |e^{it(x-y)}\chi(t)|d\mu(x)d\mu(y)dt = \iiint |\chi(t)|d\mu(x)d\mu(y)dt = \mu(\mathbb{R})^2\int|\chi(t)|dt < \infty$$

so we are in a setting to apply Fubini. Consequently we may write

$$\iiint e^{it(x-y)}\chi(t)d\mu(x)d\mu(y)dt = \iint\left(\int e^{it(x-y)}\chi(t)dt\right)d\mu(x)d\mu(y)$$

The inside integral may be evaluated directly as

$$\int e^{it(x-y)}\chi(t)dt = \hat{\chi}(x-y)$$

where $\hat{\cdot}$ represents the (non-unitary) Fourier transform.

Consider now the special case where $\chi(t) = \chi_0(T^{-1}t), \chi_0 \gtrsim 1_{[-1,1]}, |\hat{\chi}_0| \lesssim 1$. Then $\hat{\chi}(t) = T\hat{\chi}_0(Tt)$. Then we see

$$\int_{-T}^{T} |\varphi(t)|^2 dt \lesssim \int_{\mathbb{R}} |\varphi(t)|^2 \chi(t) dt = T \iint \hat{\chi}_0(T(x-y)) d\mu(x) d\mu(y)$$

It remains to establish that the latter double integral is $o_{T\to\infty}(1)$. We first claim that $(\mu \otimes \mu)(\Delta) = 0$, where $\Delta \subseteq \mathbb{R} \times \mathbb{R}$ is the diagonal. To demonstrate this, note by Fubini that

$$(\mu \otimes \mu)(\Delta) = \iint 1_\Delta(x,y) d(\mu \otimes \mu)(x,y) = \int \left(\int 1_\Delta(x,y) d\mu(x) \right) d\mu(y) = 0$$

where we have used that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$.

Next, $\mu \otimes \mu$ is Borel and finite, hence is outer regular. Thus, for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $(\mu \otimes \mu)(\{(x,y) : |x-y| < \delta\}) < \varepsilon$.

We use this to reach our conclusion. Given $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ as in the previous paragraph. Since $\chi_0 \in C_c^\infty(\mathbb{R})$, it in particular follows that there is $T_0 > 0$ such that $|\hat{\chi}_0(t)| \le \varepsilon$ for all $t \ge T_0$. For $T > \delta^{-1}T_0$, we then have

$$\int_{-T}^{T} |\varphi(t)|^2 \lesssim T \iint_{|x-y|<\delta} \hat{\chi}_0(T(x-y)) d\mu(x) d\mu(y) + T \iint_{|x-y|\ge\delta} \hat{\chi}_0(T(x-y)) d\mu(x) d\mu(y)$$

By the choice of δ, the first summand is $O(\varepsilon T)$. Since $T > \delta^{-1}T_0$, it follows that each value $\hat{\chi}_0(T(x-y))$ in the second integral is $\lesssim \varepsilon$. Since $(\mu \otimes \mu)(\mathbb{R}^2) < \infty$, it follows that the second integral is $\lesssim_\mu \varepsilon T$. Consequently,

$$\int_{-T}^{T} |\varphi(t)|^2 dt \lesssim_\mu \varepsilon T$$

for all T sufficiently large depending on $\varepsilon > 0$. Consequently,

$$\frac{1}{T} \limsup_{T\to\infty} |\varphi(t)|^2 \lesssim \varepsilon$$

for all $\varepsilon > 0$; the desired result follows. $\square$

Fall 2019 Problem 3. Consider a measure space $(X, \mathcal{X})$ with σ-finite measure μ and $p \in (1, \infty)$. Let $L^{p,\infty}$ be the set of measurable $f : X \to \mathbb{R}$ with $[f]_p = \sup_{t>0} t\mu(|f| > t)^{1/p}$ finite. Let

$$\|f\|_{p,\infty} = \sup_{\substack{E \in \mathcal{X} \\ \mu(E) \in (0,\infty)}} \frac{1}{\mu(E)^{1-1/p}} \int_E |f| \mathrm{d}\mu$$

Prove that there exist $c_1, c_2 \in (0, \infty)$ – which may depend on p and μ – such that

$$\forall f \in L^{p,\infty} : \quad c_1[f]_p \le \|f\|_{p,\infty} \le c_2[f]_p$$

Proof. We first show that $c_1 = 1$ suffices for the first inequality. For $f \in L^{p,\infty}$, the sets $U_t = \{|f| > t\}$ all have finite measure. If $\mu(U_t) > 0$, then

$$t\mu(U_t) \le \int_{U_t} |f| d\mu \le \|f\|_{p,\infty} \mu(U_t)^{1-1/p}$$

which implies
$$t\mu(U_t)^{1/p} \leq \|f\|_{p,\infty}$$
This holds for all $t > 0$ such that $\mu(U_t) > 0$. Observe that the latter inequality holds trivially when $\mu(U_t) = 0$, so we conclude the easy estimate
$$[f]_p \leq \|f\|_{p,\infty}$$

We now consider the reverse estimate. Let $E \in \mathcal{X}$ have $\mu(E) \in (0, \infty)$. For $t > 0$, we have the two trivial estimates
$$\mu(E \cap U_t) \leq \mu(E), \quad \mu(E \cap U_t) \leq t^{-p}[f]_p^p$$
Then, by a standard distribution function manipulation,
$$\begin{aligned}\int_E |f| &= \int_0^\infty \mu(E \cap U_t)dt \\ &\leq \int_0^{\mu(E)^{-1/p}[f]_p} \mu(E)dt + \int_{\mu(E)^{-1/p}[f]_p} t^{-p}[f]_p^p dt \\ &= \mu(E)^{1-1/p}[f]_p + \frac{1}{p-1}[f]_p\mu(E)^{1-1/p} \\ &= \frac{p}{p-1}\mu(E)^{1-1/p}[f]_p\end{aligned}$$
so that
$$\frac{1}{\mu(E)^{1-1/p}} \int_E |f| d\mu \leq \frac{p}{p-1}[f]_p$$
for all $E \in \mathcal{X}$ with $\mu(E) \in (0, \infty)$. Thus
$$[f]_p \leq \|f\|_{p,\infty} \leq \frac{p}{p-1}[f]_p$$
for all $f \in L^{p,\infty}$, as was to be shown.

□

Fall 2019 Problem 4. Let $A \subseteq \mathbb{R}$ be measurable with positive Lebesgue measure. Prove that the set $A - A = \{z - y : z, y \in A\}$ has non-empty interior. *Hint*: Consider the function $\varphi(x) = \int \chi_A(x + y)\chi_A(y)\mathrm{d}y$, where χ_A is the characteristic function of A.

Proof. We begin by claiming that, for each $f \in L^1(\mathbb{R})$, the function $\tau_\cdot f$, $t \mapsto \tau_t f$, $\tau_t f(x) = f(x + t)$, is continuous as a function $\mathbb{R} \to L^1(\mathbb{R})$. Note that this clearly holds when $f \in C_c(\mathbb{R})$ by uniform continuity. On the other hand, each individual τ_t is an isometry of L^1. As a consequence, if $f \in L^1(\mathbb{R})$ and $\varepsilon > 0$, then fix $g \in C_c(\mathbb{R})$ such that $\|f - g\|_1 < \varepsilon/4$ and let $\delta > 0$ be such that $\|\tau_t g - g\|_1 < \varepsilon/4$ for all $|t| < \delta$. Then, for any $|t| < \delta$,
$$\|\tau_t f - f\|_1 \leq \|\tau_t f - \tau_t g\|_1 + \|\tau_t g - g\|_1 + \|g - f\|_1 < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$$
so the function $t \mapsto \tau_t f$ is continuous at 0. By the group property of $\tau_\cdot$, it follows that the function is continuous on all of $\mathbb{R}$.

Next, since integration against $L^\infty(\mathbb{R})$-functions is continuous on $L^1(\mathbb{R})$, we see that the function

$$t \mapsto \int \tau_t f(y) g(y) dy$$

is continuous for each fixed $f \in L^1(\mathbb{R})$, $g \in L^\infty(\mathbb{R})$. In particular, the function φ is continuous. On the other hand,

$$\varphi(0) = \int \chi_A(y)\chi_A(y) dy = m(A) > 0$$

writing m for Lebesgue measure. Thus there is an interval $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ over which $\varphi > 0$.

Finally, note that for any $x \in \mathbb{R}$,

$$\int \chi_A(x+y)\chi_A(y) dy = \int \chi_{A-x}(y)\chi_A(y) dy = m((A-x) \cap A)$$

In particular, for all $|x| < \varepsilon$ there exists $z \in (A-x) \cap A$; that is to say, there exist $z, y \in A$ such that $y - x = z$, i.e. $x = y - z$. Thus we have shown that $A - A$ contains the interval $(-\varepsilon, \varepsilon)$, as was to be established. $\square$

Fall 2019 Problem 5. Prove the following claim: Let $\mathcal{H}$ be a Hilbert space with the scalar product of x and y by (x, y) and let $A, B : \mathcal{H} \to \mathcal{H}$ be (everywhere-defined) linear operators with

$$\forall x, y \in \mathcal{H} : \quad (Bx, y) = (x, Ay)$$

Then A and B are both bounded (and thus continuous).

Proof. For each $y \in \mathcal{H}$, write $T_y : \mathcal{H} \to \mathbb{C}$ be the linear map $x \mapsto (x, Ay)$. Then, for each fixed $x \in \mathcal{H}$,

$$|T_y(x)| = |(Bx, y)| \leq \|Bx\| \cdot \|y\|$$

so that

$$\sup_{\|y\| \leq 1} |T_y(x)| < \infty \quad \forall x \in \mathcal{H}$$

By the uniform boundedness theorem,

$$\sup_{\|y\|, \|x\| \leq 1} |T_y(x)| < \infty$$

so there is a constant $C < \infty$ so that

$$\|T_y\| \leq C\|y\|, \quad \forall y \in \mathcal{H}$$

Thus

$$\|Ay\|^2 = (Ay, Ay) = T_y(Ay) \leq C\|Ay\|\|y\|$$

which implies

$$\|Ay\| \leq C\|y\|, \quad \forall y \in \mathcal{H}$$

and hence A is bounded. By the symmetry between A and B, we may also conclude that B is bounded, as was to be shown. $\square$

Fall 2019 Problem 6. Recall that $\ell^\infty(\mathbb{N}) = \{x = \{x_n\}_{n=1}^\infty : \sup_{n\geq 1} |x_n| < \infty\}$ is a Banach space (over $\mathbb{R}$) with respect to the norm $\|x\|_\infty = \sup_{n\geq 1} |x_n|$.

(1) Prove that there exists a continuous linear functional ϕ on $\ell^\infty(\mathbb{N})$ such that

$$\phi(x) = \lim_{n\to\infty} x_n$$

whenever this limit exists.

(2) Prove that this ϕ is not unique.

Proof. (1): Let $A \subseteq \ell^\infty(\mathbb{N})$ be the set of convergent sequences. Then A is a linear subspace. Furthermore, we claim that A is closed. Suppose $k \mapsto x^{(k)}$ is a sequence in A and $y \in \ell^\infty(\mathbb{N})$ is such that $\|x^{(k)} - y\|_\infty \to 0$ as $k \to \infty$. Given $\varepsilon > 0$, write $K \in \mathbb{N}$ such that $\|x^{(k)} - y\|_\infty < \frac{\varepsilon}{4}$ for all $k \geq K$. Let $N \in \mathbb{N}$ be such that $m, n > N$ implies $|x_m^{(K)} - x_n^{(K)}| < \frac{\varepsilon}{4}$. It follows that, for any $m, n > N$,

$$|y_n - y_m| \leq |y_n - x_n^{(K)}| + |x_n^{(K)} - x_m^{(K)}| + |x_m^{(K)} - y_m| < 3\frac{\varepsilon}{4} < \varepsilon$$

Thus we have demonstrated that y is a Cauchy sequence, so $y \in A$. Thus A is closed, as was to be verified.

Write $\phi_0 : A \to \mathbb{R}$ for the function $\phi_0(x) = \lim_n x_n$. Clearly ϕ_0 is linear. Moreover,

$$|\phi_0(x)| = |\lim_n x_n| \leq \limsup_n |x_n| \leq \|x\|_\infty$$

so ϕ_0 is bounded by the global norm $\|\cdot\|_\infty$. By Hahn-Banach, it follows that ϕ_0 extends to a continuous linear functional on $\ell^\infty(\mathbb{N})$ which evaluates limits on Cauchy sequences, as was to be shown.

(2): Write A' for the linear subspace of $\ell^\infty(\mathbb{N})$ spanned by A and $b = \{b_n\}_{n\geq 1}$, with $b_n = (-1)^n$. Then, for any $x \in A'$, there by definition exists a scalar α and $y \in A$ such that

$$x = \alpha b + y$$

Then observe that, since $\{y_n\}_{n=1}^\infty$ converges, from the identity

$$x_{n+1} - x_n = 2(-1)^{n+1}\alpha + (y_{n+1} - y_n)$$

we in particular have

$$\frac{1}{2}\lim_{k\to\infty}(x_{2k+2} - x_{2k+1}) = \alpha \tag{1}$$

and

$$y = x - \frac{1}{2}\lim_{k\to\infty}(x_{2k+2} - x_{2k+1}) \tag{2}$$

Define linear maps $\phi_1, \phi_2 : A' \to \mathbb{R}$ via

$$\phi_1(\alpha b + y) = \alpha + \lim_{n\to\infty} y_n$$

$$\phi_2(\alpha b + y) = -\alpha + \lim_{n\to\infty} y_n$$

By (1) and (2), these are well-defined. They are also clearly linear. We wish to prove a bound that allows us to use Hahn-Banach again. To this end, write $y_\infty = \lim_n y_n$. Then

$$|\alpha| + |y_\infty| = \limsup_{n\to\infty} |(-1)^n\alpha + y_n| \leq \|\alpha b + y\|_\infty$$

so that, for $i = 1, 2$, we have

$$|\phi_i(\alpha b + y)| \leq |\alpha| + |y_\infty| \leq \|\alpha b + y\|_\infty$$

Thus ϕ_1, ϕ_2 both extend to bounded linear maps $\ell^\infty(\mathbb{N}) \to \mathbb{R}$ that extend the linear functional on A. Since they disagree on the element b, we conclude that the extension in part (a) is not unique. $\square$

Fall 2019 Problem 7. Let $J \subseteq \mathbb{R}$ be a compact interval, and let μ be a finite Borel measure whose support lies in J. For $z \in \mathbb{C} \setminus J$ define

$$F_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - t} \mathrm{d}\mu(t)$$

Prove that the mapping $\mu \mapsto F_\mu$ is one-to-one.

Proof. By Morera's theorem it follows immediately that each $F_\mu(z)$ is analytic on $\mathbb{C} \setminus J$. Moreover, if $K \subseteq \mathbb{C} \setminus J$ is compact, then

$$\frac{1}{z - t} = \sum_{n=0}^{\infty} t^n z^{-n-1}$$

converging uniformly over $(z, t) \in K \times J$. Consequently,

$$F_\mu(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\mathbb{R}} t^n d\mu(t)$$

By the uniqueness of power series, if $F_\mu(z) = F_\nu(z)$ for all $z \in \mathbb{C} \setminus J$ then

$$\int_{\mathbb{R}} t^n d\mu(t) = \int_{\mathbb{R}} t^n d\nu(t) \quad \forall n \geq 0$$

It trivially follows that

$$\int_{\mathbb{R}} P(t) d\mu(t) = \int_{\mathbb{R}} P(t) d\nu(t) \quad \forall P \in \mathbb{R}[t]$$

By Stone-Weierstrass, we then have

$$\int_{\mathbb{R}} f(t) d\mu(t) = \int_{\mathbb{R}} f(t) d\nu(t) \quad \forall f \in C(J)$$

By the Riesz-Markov-Kakutani representation theorem, it follows directly that $\mu = \nu$. Recalling our assumption $F_\mu = F_\nu$, we see that the mapping $\mu \mapsto F_\mu$ is one-to-one, as was to be shown. $\square$

Fall 2019 Problem 8. A function $f : \mathbb{C} \to \mathbb{C}$ is entire and has the property that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geq 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.

Proof. Let $u(z) = \frac{1}{\bar{z}}$ for $z \in \mathbb{C} \setminus \{0\}$, and define

$$g : \mathbb{C} \to \mathbb{C}, \quad g(z) = \begin{cases} f(z) & |z| \leq 1 \\ (u \circ f \circ u)(z) & |z| > 1 \end{cases}$$

Since u fixes $\{|z| = 1\}$ pointwise, it follows that g is continuous. Since each u is antiholomorphic and f is holomorphic, it follows that g is holomorphic on $\mathbb{C} \setminus \{|z| = 1\}$. By a standard Morera's theorem argument, g is entire.

If f is constant, then it is trivial to verify that we are done. Otherwise, we may find an entire function $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) \neq 0$ and $f(z) = z^k h$ for some $k \geq 0$. Consider the power series

$$g(z) = \sum_{n \geq 0} a_n z^n, \quad z \in \mathbb{C}$$

Then, for $|z| > 1$,

$$u(f(u(z))) = \sum_{n \geq 0} a_n z^n$$

so that, for $0 < |z| < 1$,

$$u(f(\bar{z})) = \sum_{n \geq 0} a_n z^{-n}$$

Recalling the factorization of f,

$$\frac{1}{z^k \overline{h(\bar{z})}} = \sum_{n \geq 0} a_n z^{-n}, \quad 0 < |z| < 1$$

from which we obtain

$$\frac{1}{h(z)} = \sum_{n \geq 0} \overline{a_n} z^{k-n}, \quad 0 < |z| < 1$$

On the other hand, h is nonvanishing near $z = 0$, so the meromorphic expansion in the preceding display has no singular terms. Thus

$$a_n = 0 \quad \forall n > k$$

which is to say

$$g(z) = \sum_{n=0}^{k} a_n z^n$$

On the other hand, g and f are two entire functions that agree on $|z| < 1$, so $g = f$ everywhere. Thus

$$f(z) = \sum_{n=0}^{k} a_n z^n$$

On the other hand, recalling that $f(z) = z^k h(z)$ with h holomorphic, it follows that $a_n = 0$ for $n < k$. Thus $f(z) = a_k z^k$; the result follows directly. $\square$

Fall 2019 Problem 9. Determine the number of zeroes of the polynomial

$$P(z) = z^6 - 6z^2 + 10z + 2$$

in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. Prove your claim.

Proof. When $|z| = 1$,

$$|z^6 - 6z^2 + 2| \leq 9 < |10z|$$

so that

$$|P(z)| \geq |10z| - |z^6 - 6z^2 + 2| > 0$$

Thus, by Rouché's theorem, P and $z \mapsto 10z$ have the same number of zeroes in $|z| \leq 1$, i.e. 1. On the other hand, for $|z| = 2$,

$$|z^6 - P(z)| = |6z^2 - 10z - 2| \leq 24 + 20 + 2 = 46 < 64 = |z^6|$$

Thus by Rouché's theorem, P and $z \mapsto z^6$ have the same number of zeroes in $\{|z| \leq 2\}$, i.e. 6. Since all but one of the zeroes of P are outside $\{|z| \leq 1\}$, it follows that P has 5 zeroes in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. $\square$

Fall 2019 Problem 10. Evaluate

$$\lim_{x\to\infty} \int_0^x \sin(t^2)\mathrm{d}t$$

Justify all steps.

Proof. We abbreviate $f(z) = e^{iz^2}$. For $R > 0$, let γ_1 be the path from 0 to R in $\mathbb{C}$, γ_2 the circular arc from R to $Re^{i\frac{\pi}{4}}$, and γ_3 the line segment from $Re^{i\frac{\pi}{4}}$ to 0. Write Γ_R for the composed path $\gamma_1 \to \gamma_2 \to \gamma_3$, i.e. the closed path traversing the previous three in a CCW way. Observe that

$$\int_{\Gamma_R} f(z)dz = 0$$

because f is entire. Note too that

$$\int_{\gamma_1} f(z)dz = \int_0^R (\cos(t^2) + i\sin(t^2))dt \tag{3}$$

and

$$\int_{\gamma_3} f(z)dz = -e^{i\frac{\pi}{4}} \int_0^R e^{-t^2}dt \tag{4}$$

We need to more carefully study the last integral. It may be expressed as

$$\int_{\gamma_2} f(z)dz = iR\int_0^{\pi/4} e^{iR^2e^{2i\theta}}e^{i\theta}d\theta$$

The magnitude of the last integrand is

$$|e^{iR^2e^{2i\theta}}e^{i\theta}| = e^{-R^2\cos(2\theta)}$$

Note that, for $0 \leq \theta \leq \frac{\pi}{4}$,

$$\cos(2\theta) \geq 1 - \frac{4}{\pi}\theta$$

and so

$$\left|\int_{\gamma_2} f(z)dz\right| \leq R\int_0^{\pi/4} e^{-R^2(1-\frac{4}{\pi}\theta)}d\theta = Re^{-R^2}\int_0^{\pi/4} e^{\frac{4}{\pi}\theta R^2}d\theta = Re^{-R^2}\frac{\pi}{4R^2}\left[e^{R^2} - 1\right]$$

which immediately implies

$$\left| \int_{\gamma_2} f(z)dz \right| \lesssim R^{-1}, \quad (R > 1) \tag{5}$$

We now put the pieces together. From 4, we see that

$$\int_{\gamma_3} f(z)dz = -e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2} + o(1)$$

(here and on, $o(1)$ will be as $R \to +\infty$). From 5,

$$\int_{\gamma_2} f(z)dz = o(1)$$

Thus

$$\begin{aligned} 0 &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz \\ &= \int_0^R (\cos(t^2) + i\sin(t^2))dt - e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2} + o(1) \end{aligned}$$

from which it follows

$$\left[\int_0^R \cos(t^2)dt - \frac{\sqrt{\pi}}{2} \cos(\frac{\pi}{4}) \right] + i \left[\int_0^R \sin(t^2)dt - \frac{\sqrt{\pi}}{2} \sin(\frac{\pi}{4}) \right] = o(1)$$

From the form of the complex norm in terms of the real and imaginary parts, it in particular follows that

$$\lim_{R\to\infty} \int_0^R \cos(t^2)dt = \frac{\sqrt{\pi}}{2} \cos(\frac{\pi}{4}) = \frac{\sqrt{2\pi}}{4}$$

as was to be calculated. □

Fall 2019 Problem 11. Find a conformal map of the domain

$$D = \{z \in \mathbb{C} : |z - 1| < \sqrt{2}, |z + 1| < \sqrt{2}\}$$

onto the open unit disk centered at the origin. It suffices to write this map as a composition of explicit conformal maps.

Proof. Let f_1 be the map

$$f_1(z) = -\frac{z - i}{z + i}$$

Then the two boundary curves of $f_1(D)$ are lines through 0, which are bisected by a line through 0 containing the point

$$f_1(0) = 1$$

i.e. $\mathbb{R}$. Thus $f_1(D)$ is an angular sector with corner at 0, containing $\mathbb{R}_{>0}$, and is symmetric about $z \mapsto \bar{z}$. It remains to identify the angle of the sector. Note that the tangent vector to $\{|z - 1| = \sqrt{2}\}$ at i is orthogonal to the displacement vector from 1 to i, and the tangent vector to $\{|z + 1| = \sqrt{2}\}$ at i is orthogonal to the displacement vector from -1 to i. Thus the internal angle of D at i is $\frac{\pi}{2}$; the same holds for $-i$. We conclude that

$$f_1(D) = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, |\mathrm{Im}(z)| \leq \mathrm{Re}(z)\}$$

Next, if $f_2(z) = iz^2$, then $f_2(f_1(D))$ is the upper half-plane. Lastly, it is standard that f_1 carries the upper half-plane to the unit disk. Thus a suitable conformal mapping is $\phi = f_1 \circ f_2 \circ f_1$. □

Fall 2019 Problem 12. Show that

$$F(z) = \int_1^\infty \frac{t^z}{\sqrt{1+t^3}}\mathrm{d}t$$

is well defined (by the integral) and analytic in $\{z \in \mathbb{C} : \mathrm{Re}(z) < \frac{1}{2}\}$, and admits a meromorphic continuation to the region $\{z \in \mathbb{C} : \mathrm{Re}(z) < \frac{3}{2}\}$.

Proof. We address the claims in order. First, if $\mathrm{Re}(z) < \frac{1}{2}$, then for $t \in (1, \infty)$,

$$\left|\frac{t^z}{\sqrt{1+t^3}}\right| \lesssim t^{-\frac{3}{2}+\mathrm{Re}(z)}$$

Since the exponent is strictly less than -1, it follows that the left-hand side of the preceding display is integrable over $(1, \infty)$. Consequently, the integral defining F converges absolutely.

We now validate that F is analytic via Morera's theorem. Let T be any triangle in $\{z \in \mathbb{C} : \mathrm{Re}(z) < \frac{1}{2}\}$. Then, by compactness of T, there is some $\varepsilon > 0$ such that, for all $z \in T$ one has $\mathrm{Re}(z) \leq \frac{1}{2} - \varepsilon$. We would like to commute integrals: note that

$$\int_T \int_1^\infty \left|\frac{t^z}{\sqrt{1+t^3}}\right| dt|dz| \lesssim \int_T \int_1^\infty t^{-\frac{3}{2}+\frac{1}{2}-\varepsilon} dt|dz| < \infty$$

so by Fubini

$$\int_T F(z)dz = \int_1^\infty \frac{1}{\sqrt{1+t^3}} \int_T t^z dz dt = 0$$

and it follows that F is analytic.

We now study the meromorphic extension. Fix $R > 0$ large. Integrating by parts, for any $\mathrm{Re}(z) < \frac{3}{2}$,

$$\begin{aligned}
&\int_1^R t^{z-\frac{3}{2}} \frac{t^{\frac{3}{2}}}{\sqrt{1+t^3}} dt \\
&= \frac{t^{z-\frac{3}{2}}}{1+t^3} \left(\frac{3}{2} t^{1/2}(1+t^3)^{1/2} - \frac{3}{2} t^{\frac{3}{2}}(1+t^3)^{-1/2} t^2\right)\Bigg|_{t=1}^R \\
&- \frac{3}{2} \frac{1}{z-\frac{1}{2}} \int_1^R \frac{t^z}{(1+t^3)^{3/2}} dt
\end{aligned}$$

Then

$$F(z) = \lim_{R\to\infty} \left[-\frac{3}{2}\frac{1}{z-\frac{1}{2}} \int_1^R \frac{t^z}{(1+t^3)^{3/2}} dt + \frac{3}{2} \frac{t^{z-1}}{(1+t^3)^{3/2}}\Bigg|_{t=1}^R\right], \quad \mathrm{Re}(z) < \frac{1}{2}$$

Write

$$g_{1,R}(z) = -\frac{3}{2}\int_1^R \frac{t^z}{(1+t^3)^{3/2}} dt, \quad g_{2,R}(z) = \frac{3}{2} \frac{t^{z-1}}{(1+t^3)^{3/2}}\Bigg|_{t=1}^R$$

Then each $g_{i,R}$ is holomorphic on $\{z \in \mathbb{C} : \mathrm{Re}(z) < \frac{3}{2}\}$ and converges locally uniformly there. It follows that F extends to a meromorphic function on $\{z \in \mathbb{C} : \mathrm{Re}(z) < \frac{3}{2}\}$, as was to be shown. □

3 Spring 2020

Spring 2020 Problem 1. Assume $f \in C_c^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} e^{-tx^2} f(x)dx = 0 \quad \text{for any} \quad t \geq 0$$

Show that $f(x) = -f(-x)$ for any $x \in \mathbb{R}$.

Proof. Taking the even part of f, that is, $\frac{f(x)+f(-x)}{2}$, we see that the claim holds if and only if

$$f \text{ even, } C_c^\infty(\mathbb{R}) \text{ orthogonal to centered Gaussians } \implies f = 0$$

Suppose f satisfies the left-hand side of the above implication. Note that

$$0 = \int_{\mathbb{R}} e^{-tx^2} f(x)dx = \sqrt{\frac{\pi}{t}} \int_{\mathbb{R}} e^{-\pi^2\xi^2/t} \hat{f}(\xi)d\xi$$

for all $t > 0$, which implies that $\hat{f}$ is also orthogonal to centered Gaussians. Since f is even and real-valued, $\hat{f}$ is also even and real-valued. Since f is compactly supported, the integral

$$\hat{f}(z) = \int_{\mathbb{R}} e^{-2\pi i z x} f(x)dx$$

is well-defined and continuous for $z \in \mathbb{C}$. If Δ is any triangle in $\mathbb{C}$, then Fubini provides

$$\begin{aligned}
\int_\Delta \hat{f}(z)dz &= \int_\Delta \int_{\mathbb{R}} e^{-2\pi i z x} f(x)dxdz \\
&= \int_{\mathbb{R}} \int_\Delta e^{-2\pi i z x} f(x)dzdx \quad \text{since the integrand is continuous and compactly-supported} \\
&= 0 \quad \text{since } e^{-2\pi i z x} \text{ is analytic in } z \text{ for each } x \in \mathbb{R}
\end{aligned}$$

so by Morera we have that $\hat{f}$ extends to an entire function. Thus $\hat{f}$ on $\mathbb{R}$ is given by a convergent real power series

$$\hat{f}(\xi) = \sum_{n=0}^\infty a_n \xi^n$$

and, since $\hat{f}$ is even, $a_n = 0$ for all odd n. But then, for any $t > 0$,

$$0 = \int_{\mathbb{R}} e^{-t\xi^2} \hat{f}(\xi)d\xi = \sum_{n=0}^\infty a_n \int_{\mathbb{R}} e^{-t\xi^2} \xi^n d\xi = \sqrt{\frac{\pi}{t}} \sum_{k=0}^\infty a_{2k} \frac{(2k-1)!!}{(2t)^k}$$

Thus the real power series

$$\sum_{k=0}^\infty a_{2k} \frac{(2k-1)!!}{(2t)^k}$$

which converges uniformly in a neighborhood of ∞, and hence defines an analytic function there, is identically 0 on a non-discrete set, and hence has zero coefficients. Thus each a_n is equal to 0, which implies that f was zero from the start.

$\square$

Spring 2020 Problem 2. [1] Assume $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of differentiable functions satisfying

$$\int_{\mathbb{R}} |f_n(x)| dx \leq 1 \quad \text{and} \quad \int_{\mathbb{R}} |f_n'(x)| dx \leq 1.$$

Assume also that for any $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that

$$\sup_n \int_{|x| \geq R(\varepsilon)} |f_n(x)| dx < \varepsilon$$

Show that there exists a subsequence of $\{f_n\}$ that converges in $L^1(\mathbb{R})$.

Proof. Note that the second condition implies that the $\{f_n\}$ have total variation bounded by 1. Since each f_n is absolutely integrable, $|f_n| \leq 1$ everywhere since otherwise the total variation condition would imply that $|f_n| > \varepsilon > 0$ everywhere, contradicting integrability. Thus the $\{f_n\}$ are uniformly bounded.

Now let $\{\phi_\varepsilon\}_\varepsilon$ be approximations to the identity. Then, for $I \subseteq \mathbb{R}$ compact,

$$\begin{aligned}
\|f_n - f_n * \phi_\varepsilon\|_{L^1(I)} &= \int_I \left| \int_{\mathbb{R}} (f_n(x) - f_n(x-y))\phi_\varepsilon(y) dy \right| dx \\
&= \int_I \left| \int_{\mathbb{R}} (f_n(x) - f_n(x - \varepsilon y))\phi(y) dy \right| dx \\
&\leq \int_{\mathbb{R}} \int_I |f_n(x) - f_n(x - \varepsilon y)| dx \phi(y) dy \\
&\leq \varepsilon |I|
\end{aligned}$$

Thus, for each $\varepsilon > 0$, we choose $I \subseteq \mathbb{R}$ compact so that all f_n satisfy

$$\sup_n \int_{\mathbb{R} \setminus I} |f_n| < \varepsilon$$

and for this choice of I, choose $\delta_0 > 0$ such that

$$\|f_n - f_n * \phi_\delta\|_{L^1(I)} < \varepsilon \quad \forall \delta_0 \geq \delta > 0$$

For each such $\delta > 0$, the sequence $f_n * \phi_\delta$ is uniformly bounded and equicontinuous: first, by Young,

$$\|f_n * \phi_\delta\|_{L^\infty(I)} \leq \|f_n\|_{L^\infty(\mathbb{R})} \|\phi_\delta\|_{L^1(\mathbb{R})} \leq 1$$

Secondly,

$$\|(f_n * \phi_\delta)'\|_{L^\infty(I)} = \|f_n * \phi_\delta'\|_{L^\infty(I)} \leq \|f_n\|_{L^\infty(\mathbb{R})} \|\phi_\delta'\|_{L^1(\mathbb{R})} \leq \delta^{-2} \|\phi'\|_{L^1(\mathbb{R})} < \infty$$

with the latter expression independent of n. Thus the family $\{f_n * \phi_\delta\}_n$ is uniformly bounded and equicontinuous for each $\delta > 0$, and hence by Arzelà-Ascoli there is a continuous function f_δ on I and a subsequence n_k such that $f_{n_k} * \phi_\delta \to f_\delta$ uniformly on I. As such, there is a $K \in \mathbb{N}$ such that, for all $k > K$,

$$\|f_{n_k} * \phi_\delta - f_\delta\|_{L^1(I)} < \varepsilon$$

[1] keyword: Helly's selection theorem

All together, we see that, for $j, k > K$,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|_{L^1(I)} &\le \|f_{n_j} - f_{n_j} * \phi_\delta\|_{L^1(I)} + \|f_{n_j} * \phi_\delta - f_\delta\|_{L^1(I)} \\ &\quad + \|f_\delta - f_{n_k} * \phi_\delta\|_{L^1(I)} + \|f_{n_k} * \phi_\delta - f_{n_k}\|_{L^1(I)} \\ &< 4\varepsilon\end{aligned}$$

and hence

$$\|f_{n_j} - f_{n_k}\|_{L^1(I)} < 5\varepsilon$$

for sufficiently large j, k. Thus we may construct the convergent subsequence as desired.

□

Spring 2020 Problem 3. Prove that $L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ is a Borel subset of $L^3(\mathbb{R}^n)$.

Proof. Note that, for $f \in L^3(\mathbb{R}^n)$,

$$f \in L^\infty(\mathbb{R}^n) \iff \exists K \in \mathbb{N} \text{ such that } m(\{|f| > K\}) = 0$$

We claim that

$$m(\{|f| > K\}) = 0 \iff \int_q^r |f| \le K(r-q)\, \forall q < r \in \mathbb{Q} \tag{6}$$

The forward implication is clear. For the reverse implication, suppose that there is some bounded set $S \subseteq \mathbb{R}$ with $m(S) > 0$ such that $|f| > K + \varepsilon$ on S, where $\varepsilon > 0$. Then for every $\delta > 0$, the definition of Lebesgue measure (with some minor tweaks) supplies a finite disjoint union of open intervals with rational endpoints

$$U = (q_1, r_1) \cup \cdots \cup (q_n, r_n)$$

such that

$$S \subseteq U, m(U) < m(S) + \delta$$

Then we have

$$\int_U |f| = \int_S |f| + \int_{U\setminus S} |f| > (K+\varepsilon)m(S) = (K+\varepsilon)\sum_{j=1}^n (r_j - q_j)$$

If the right-hand side of 6 still holds, the above provides

$$K\sum_{j=1}^n (r_j - q_j) + \int_{U\setminus S} |f| > (K+\varepsilon)\sum_{j=1}^n (r_j - q_j)$$

or

$$\int_{U\setminus S} |f| > \varepsilon \sum_{j=1}^n (r_j - q_j) = \varepsilon m(S) > 0$$

However, the left-hand side of the above can be expanded via Hölder as

$$\int_{U\setminus S} |f| = \|f\chi_{U\setminus S}\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^3(\mathbb{R}^n)} \|\chi_{U\setminus S}\|_{L^{3/2}(\mathbb{R}^n)} = \|f\|_{L^3(\mathbb{R}^n)} m(U \setminus S)^{2/3} < \|f\|_{L^3(\mathbb{R}^n)} \delta^{2/3}$$

so we conclude that, for our given $f \in L^3(\mathbb{R}^n)$, there is some $m(S) > 0$ and $\varepsilon > 0$ such that for every $\delta > 0$ we have

$$\|f\|_{L^3(\mathbb{R}^n)}\delta^{2/3} > \varepsilon m(S) > 0$$

Since $\|f\|_{L^3(\mathbb{R}^n)} < +\infty$, we may send $\delta \to 0$ to obtain a contradiction.

Thus we have shown 6. Since each $\chi_{(q,r)}$ belongs to $L^{3/2}(\mathbb{R}^n) = (L^3(\mathbb{R}^n))^*$, we see that integrating f against $\chi_{(q,r)}$ is a continuous functional on $L^3(\mathbb{R}^n)$. Thus the collection of f satisfying the RHS of 6 is Borel; taking a countable union over $K \in \mathbb{N}$ provides that $L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ is a Borel subset of $L^3(\mathbb{R}^n)$, as desired.

□

Spring 2020 Problem 4. [2] Fix $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_0^2 f(x)\sin(x^n)dx = 0$$

Proof. For each n,

$$\int_0^2 f(x)\sin(x^n)dx = \int_0^1 f(x)\sin(x^n)dx + \int_1^2 f(x)\sin(x^n)dx$$

We analyze each term separately. Note that

$$\sin(x^n) \xrightarrow{n} 0 \quad \text{pointwise for } x \in (0,1)$$

and so

$$f(x)\sin(x^n) \xrightarrow{n} 0 \quad \text{pointwise for } x \in (0,1)$$

Thus, by DCT, since $f(x)\sin(x^n) \leq |f(x)| \in L^1(\mathbb{R})$ for each n,

$$\lim_{n\to\infty} \int_0^1 f(x)\sin(x^n)dx = \int_0^1 \lim_{n\to\infty} f(x)\sin(x^n)dx = 0$$

which is the desired result for the first term. Note that

$$\frac{1}{inx^{n-1}}\frac{d}{dx}e^{ix^n} = e^{ix^n}$$

For the second term, assuming first that $f \in C_c^\infty((1,2))$,

$$\begin{aligned}
\int_1^2 f(x)\sin(x^n)dx &= \mathrm{Im}\left[\int_1^2 f(x)e^{ix^n}dx\right] \\
&= \mathrm{Im}\left[\int_1^2 \frac{f(x)}{inx^{n-1}}\frac{d}{dx}e^{ix^n}dx\right] \\
&= -\mathrm{Im}\left[\int_1^2 e^{ix^n}\frac{d}{dx}\left(\frac{f(x)}{inx^{n-1}}\right)dx\right] \\
&= -\mathrm{Im}\left[\int_1^2 e^{ix^n}\frac{f'(x)}{inx^{n-1}}dx + \frac{(n-1)}{in}\int_1^2 e^{ix^n}\frac{f(x)}{x^n}\right]dx \\
&= \int_1^2 \cos(x^n)\frac{f'(x)}{nx^{n-1}}dx + \frac{(n-1)}{n}\int_1^2 \cos(x^n)\frac{f(x)}{x^n}
\end{aligned}$$

[2] keyword: oscillatory integral

and hence, by the triangle inequality and DCT,

$$\left|\int_1^2 f(x)\sin(x^n)dx\right| \leq \left|\int_1^2 \cos(x^n)\frac{f'(x)}{nx^{n-1}}dx\right| + \left|\frac{(n-1)}{n}\int_1^2 \cos(x^n)\frac{f(x)}{x^n}dx\right|$$
$$\stackrel{n}{\to} 0$$

Thus we have the desired limit in the $\int_1^2$ term for all $f \in C_c^\infty(1,2)$. By a standard fact, for general $f \in L^1((1,2))$ we may find a sequence $\{f_n\}_{n=1}^\infty$ in $C_c^\infty((1,2))$ such that $f_n \to f$ in L^1. For each $\varepsilon > 0$, let $j \in \mathbb{N}$ be such that $\|f_j - f\|_{L^1} < \varepsilon/2$. By the above, there is some $n \in \mathbb{N}$ such that, for all $k \geq n$,

$$\left|\int_1^2 f_j(x)\sin(x^k)dx\right| < \varepsilon/2$$

All together we have

$$\begin{aligned}\left|\int_1^2 f(x)\sin(x^k)dx\right| &\leq \left|\int_1^2 f_j(x)\sin(x^k)dx\right| + \left|\int_1^2 [f_j(x) - f(x)]\sin(x^k)dx\right| \\ &< \varepsilon/2 + \int_1^2 |f_j(x) - f(x)|dx \quad \text{since } |\sin(t)| \leq 1 \text{ everywhere} \\ &< \varepsilon\end{aligned}$$

for all $k \geq n$; letting $\varepsilon \to 0$ we obtain the desired

$$\int_1^2 f(x)\sin(x^n)dx \stackrel{n}{\to} 0$$

which together with the limit on for $\int_0^1$ provides the desired result.

□

Spring 2020 Problem 5. Rigorously determine the infimum of

$$\int_{-1}^1 |P(x) - |x||^2 dx$$

over all choices of polynomials $P \in \mathbb{R}[x]$ of degree not exceeding three.

Proof. Some details omitted. Write a general degree ≤ 3 polynomial as $P_{abcd}(x) = ax^3 + bx^2 + cx + d$. We claim first that, for any particular choice of (a, b, c, d),

$$E(a,b,c,d) = \int_{-1}^1 |P_{abcd}(x) - |x||^2 \geq \int_{-1}^1 |P_{0b0d}(x) - |x||^2 = E(0,b,0,d) \tag{7}$$

Differentiating E by a, c we see that the function

$$(a,c) \mapsto E(a,b,c,d)$$

has a unique critical point at $a = c = 0$. At this point, the Hessian of E in a, c is

$$H_{a,c}E = \begin{bmatrix} 4/7 & 4/5 \\ 4/5 & 4/3 \end{bmatrix}$$

which has positive determinant and trace, hence is positive definite. Thus (7) holds and we may restrict attention to even polynomials.

Now, differentiating E against b, d,

$$\partial_b E = 4b/5 + 4d/3 - 1, \quad \partial_d E = 4b/3 + 4d - 2$$

which defines a vector field in the b, d plane. The inner-product of this vector field with an outer-pointing vector field is given by

$$D_{b,d}E_{(b,d)} \cdot \frac{1}{\sqrt{2}}(b, d) = \frac{1}{\sqrt{2}}\left(\frac{4b^2}{5} + \frac{8bd}{3} + 4d^2 - b - 2d\right)$$

which is positive for sufficiently large $\|(b, d)\|$, which implies that E achieves a global minimum somewhere. This happens when $D_{b,d}E_{(b,d)} = 0$, or when

$$b = \frac{15}{16}, d = \frac{3}{16}$$

and here we achieve

$$\inf E = E(0, \frac{15}{16}, 0, \frac{3}{16}) = \frac{1}{96}$$

as the infimum value. $\square$

Spring 2020 Problem 6. Let us define a sequence of linear functionals on $L^\infty(\mathbb{R})$ as follows:

$$L_n(f) = \frac{1}{n!}\int_0^\infty x^n e^{-x} f(x)dx.$$

(a) Prove that no subsequence of this sequence converges weak-$*$.
(b) Explain why this does not contradict the Banach-Alaoglu Theorem.

Proof. (a): Suppose $\{L_{n_k}\}_k$ is a subsequence; we show that this sequence does not converge weak-$*$'ly. Since each integrand $\frac{1}{n!}x^n e^{-x}$ converges locally uniformly to 0, we may choose a sequence of compact intervals $I_k \subseteq \mathbb{R}$ satisfying

$$\frac{1}{n_k!}\int_{\mathbb{R}_+ \setminus I_k} x^{n_k} e^{-x} dx < \frac{1}{10}$$

and

$$\inf I_k \overset{k\to\infty}{\longrightarrow} \infty$$

Choose a subsequence $\{I_{k_j}\}_j$ whose intervals are pairwise disjoint. Then

$$f = \sum_{j=1}^\infty (-1)^j \chi_{I_{k_j}}$$

is in $L^\infty(\mathbb{R})$. Then, for each j,

$$L_{n_{k_j}}(f) = L_{n_{k_j}}((-1)^j \chi_{I_{k_j}}) + L_{n_{k_j}}(\sum_{\ell\neq j}(-1)^\ell \chi_{I_{k_\ell}}) = (-1)^j + O(\frac{1}{10})$$

where the implicit constant is at most 2. Thus the sequence

$$\{L_{n_k}(f)\}_k$$

is not Cauchy, and so does not converge. Thus $\{L_{n_k}\}_k$ does not converge weak-$*$'ly, as desired.

(b): Since $L^\infty(\mathbb{R})$ is non-separable, Banach-Alaoglu only shows that the unit ball of $(L^\infty(\mathbb{R}))^*$ is compact, not sequentially compact. $\square$

Spring 2020 Problem 7. [3] Let $\mathcal{F}_M$ be the set of functions holomorphic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ that satisfy

$$\int_0^{2\pi} |f(e^{it})|dt \leq M < \infty.$$

Show that every sequence $\{f_n\}$ in $\mathcal{F}_M$ contains a subsequence that converges uniformly on compact subsets of $\mathbb{D}$.

Proof. Note first that $\mathcal{F}_M$ is locally uniformly bounded and locally equicontinuous: for $K \subseteq \mathbb{D}$ a compact ball, set $r = \text{dist}(K, \mathbb{D}) > 0$. Then, for any $f \in \mathcal{F}_M$, and any $z \in K$,

$$\begin{aligned} |f(z)| &= \left| \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(w)}{w - z} dw \right| \\ &\leq \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{|f(w)|}{|w - z|} ds(w) \\ &\leq \frac{1}{2\pi r} \int_{\partial\mathbb{D}} |f(w)| ds(w) \leq \frac{M}{2\pi r} \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(w)}{(w - z)^2} dw \right| \\ &\leq \frac{M}{2\pi r^2} \end{aligned}$$

which implies, for any $w \in K$,

$$|f(z) - f(w)| \leq |z - w||f'(t)| \leq \frac{M}{2\pi r^2}|z - w|$$

where t is some point in K on the segment connecting z and w.

Thus, for each $k \geq 2$ natural, the sequence $\{f_n\}$ is uniformly bounded and equicontinuous on $\overline{B(0, 1 - \frac{1}{k})}$. By Arzelà-Ascoli, we may iteratively refine $\{f_n\}$ (keeping the first k terms on each step) to be uniformly convergent on each of these balls; this provides the locally uniformly convergent subsequence as desired. $\square$

[3] keyword: normal family

Spring 2020 Problem 8. For each $z \in \mathbb{C}$, let

$$F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}.$$

(a) Show that F is an entire function and satisfies $|F(z)| \le e^{|z|}$.
(b) Show that there is an infinite collection of numbers $a_n \in \mathbb{C}$, so that

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

and the product converges uniformly on compact subsets of $\mathbb{C}$.

Proof. (a): Note that

$$|F(z)| \le \sum_{n=0}^{\infty} \frac{|z|^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!}\binom{2n}{n} 2^{-2n} \le \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!} \le e^{|z|}$$

using the fact that

$$\sum_{j=0}^{2n} \binom{2n}{j} = 2^{2n}$$

which implies that F converges absolutely on $\mathbb{C}$, hence is an entire function satisfying the desired estimate.

(b): Define the function

$$G(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^{2n}(n!)^2}$$

Thus $F(z) = G(z^2)$. If $\{a_n\}$ are the zeros of F (n ranging over $\mathbb{Z} \setminus \{0\}$ and $a_{-n} = -a_n$), then $a_n^2 =: b_n$ are the zeros of G (here n ranges over $\mathbb{N}$). From $|F(z)| \le e^{|z|}$ we see that F has order ≤ 1, and thus

$$\sum_{n \ne 0} \frac{1}{|a_n|^2} < \infty$$

which implies

$$\sum_{n=1}^{\infty} \frac{1}{|b_n|} < \infty$$

so that G has order < 1. Hadamard's canonical representation then provides

$$G(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{b_n}\right)$$

and hence

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

converging locally uniformly in $\mathbb{C}$. $\square$

Spring 2020 Problem 9. Let $f \in L^1(\mathbb{C}) \cap C^1(\mathbb{C})$. Show that the integral

$$u(z) = -\frac{1}{2\pi} \iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\lambda(\zeta)$$

defines a C^1 function on the whole complex plane that satisfies

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) u(x+iy) = f(x+iy)$$

In this problem, $d\lambda(\zeta)$ denotes (planar) Lebesgue measure on $\mathbb{C}$ and C^1 is meant in the real-variables sense.

Proof. Since f is L^1 and $\frac{1}{\zeta - z}$ is bounded near ∞, the integrand is integrable near ∞. Since f is C^1 and $\frac{1}{\zeta - z}$ is locally integrable, the integrand is locally integrable. Thus the integrand is globally integrable and so u is well-defined for every $z \in \mathbb{C}$.

Note that, by Cauchy-Pompeiu,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi} \iint_D (\partial_x + i\partial_y) f \frac{1}{\zeta - z} d\lambda(\zeta)$$

for any domain D containing z. Since f is L^1, there is a sequence of radii R_j such that

$$\int_{|z-\zeta|=R_j} |f| \frac{ds}{2\pi R_j} = \text{avg}_{|z-\zeta|=R_j} |f| \to 0$$

If this weren't true, then the above integral would be above $\varepsilon > 0$ for all sufficiently large $R > 0$; however, we would then have

$$\begin{aligned}
\infty &> \int_{|\zeta - z| \geq 1} \left| \frac{f(\zeta)}{\zeta - z} \right| d\lambda(\zeta) \\
&= \int_0^{2\pi} \int_1^\infty \frac{|f(z + Re^{i\theta})|}{R} R dR d\theta \\
&= \int_1^\infty 2\pi \text{avg}_{|z-\zeta|=R} |f| dR \\
&\geq 2\pi\varepsilon \int_{R_0}^\infty dR = \infty
\end{aligned}$$

a contradiction. Thus we have the desired sequence R_j, and so

$$\left| \frac{1}{2\pi i} \int_{|z-\zeta|=R_j} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq \frac{1}{2\pi R_j} 2\pi R_j \text{avg}_{|z-\zeta|=R_j} |f| \to 0$$

Setting $D = D_j = \{z : |z - \zeta| \leq R_j\}$ and taking a limit, we find

$$f(z) = \lim_{j\to\infty} -\frac{1}{2\pi} \iint_{D_j} (\partial_x + i\partial_y) f \frac{1}{\zeta - z} d\lambda(\zeta) = -\frac{1}{2\pi} \iint_{\mathbb{C}} (\partial_x + i\partial_y) f \frac{1}{\zeta - z} d\lambda(\zeta)$$

where the latter integral is in the sense of principal value.

Assuming first that f is compactly supported,

$$\begin{aligned}
\frac{\partial}{\partial x}u(x+iy) &= \lim_{\mathbb{R}\ni h\to 0} -\frac{1}{2\pi h}\iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta-(z+h)} - \frac{f(\zeta)}{\zeta - z}d\lambda(\zeta)\\
&= \lim_{\mathbb{R}\ni h\to 0} -\frac{1}{2\pi h}\iint_{\mathbb{C}} \frac{f(\zeta)}{(\zeta-h)-z} - \frac{f(\zeta)}{\zeta - z}d\lambda(\zeta)\\
&= \lim_{\mathbb{R}\ni h\to 0} -\frac{1}{2\pi h}\iint_{\mathbb{C}} \frac{f(\zeta+h)}{\zeta-z} - \frac{f(\zeta)}{\zeta - z}d\lambda(\zeta) \quad \text{by a change of variables}\\
&= \lim_{\mathbb{R}\ni h\to 0} -\frac{1}{2\pi}\iint_{\mathbb{C}} \frac{f(\zeta+h)-f(\zeta)}{h}\frac{1}{\zeta - z}d\lambda(\zeta)\\
&= -\frac{1}{2\pi}\iint_{\mathbb{C}} \frac{\partial_x f(\zeta)}{\zeta - z}d\lambda(\zeta)
\end{aligned}$$

where we justify the last exchange of integral and limit by DCT, since

$$\left|\frac{f(\zeta+h)-f(\zeta)}{h}\frac{1}{\zeta-z}\right| \leq \sup_{w\in\text{supp}(f)}|f'(w)|\frac{1}{|\zeta-z|}\chi_{B_R(0)}(\zeta)$$

for sufficiently large R and small h. Similarly,

$$\frac{\partial}{\partial y}u(x+iy) = -\frac{1}{2\pi}\iint_{\mathbb{C}}\frac{\partial_y f(\zeta)}{\zeta-z}d\lambda(\zeta)$$

so that

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u(x+iy) = -\frac{1}{2\pi}\iint_{\mathbb{C}}\frac{(\partial_x+i\partial_y)f(\zeta)}{\zeta-z}d\lambda(\zeta) = f(x+iy)$$

by the Cauchy-Pompeiu calculation above. If χ_j is a sequence of smooth cutoff functions satisfying

$$0\leq \chi_j\leq\chi_{j+1}, \chi_j \xrightarrow{j} 1, \chi_j\equiv 1 \quad \text{on } B_j(z), \text{supp}(\chi_j)\subsetneq B_{j+1}(z)$$

then, setting u_j to be the function constructed in the problem using the function $f\chi_j$,

$$\begin{aligned}
\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u_j(x+iy) &= -\frac{1}{2\pi}\iint_{\mathbb{C}}\frac{(\partial_x+i\partial_y)[f(\zeta)\chi_j(\zeta)]}{\zeta-z}d\lambda(\zeta)\\
&= -\frac{1}{2\pi}\iint_{\mathbb{C}}\frac{f(\zeta)(\partial_x+i\partial_y)\chi_j(\zeta)}{\zeta-z}d\lambda(\zeta)\\
&\quad -\frac{1}{2\pi}\iint_{\mathbb{C}}\frac{\chi_j(\zeta)(\partial_x+i\partial_y)f(\zeta)}{\zeta-z}d\lambda(\zeta)\\
&= \chi_j(x+iy)f(x+iy)
\end{aligned}$$

As $j\to\infty$, the right-hand side of the above has local uniform limit $f(x+iy)$. Thus

$$\lim_{j\to\infty}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u_j(x+iy) = f(x+iy)$$

whereas

$$u_j\to u \quad \text{pointwise}$$

Since $\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u_j$ converges locally uniformly, it follows that

$$f(x+iy)=\lim_{j\to\infty}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u_j(x+iy)=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u(x+iy)$$

as desired. □

Spring 2020 Problem 10. Evaluate the improper Riemann integral

$$\int_0^\infty \frac{x^2-1}{x^2+1}\frac{\sin x}{x}dx.$$

Justify all manipulations.

Proof. Since the integrand is even,

$$\int_0^\infty \frac{x^2-1}{x^2+1}\frac{\sin x}{x}dx=\frac{1}{2}\int_{\mathbb{R}}\frac{x^2-1}{x^2+1}\frac{\sin x}{x}dx$$

Let C_R denote the upper half circle with radius R, that is, the circular arc in the upper half plane connecting R to $-R$. Let D_R denote the curve formed by a straight line from $-R$ to $-\frac{1}{R}$, travels a half-circle in the upper half plane to $\frac{1}{R}$, and then travels by a straight line segment to R. Let Γ_R denote the curve $C_R\cup D_R$. Then for $R\geq 2$ the residue theorem provides

$$\begin{aligned}\int_{\Gamma_R}\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz&=2\pi i\text{Res}\left[\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz,i\right]\\&=2\pi i\frac{-2}{2i}\frac{e^{-1}}{i}\\&=2\pi e^{-1}i\end{aligned}$$

By Jordan's lemma,

$$\left|\int_{C_R}\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz\right|\leq\pi\text{max}_{z\in C_R}\left|\frac{z^2-1}{z(z^2+1)}\right|\leq\frac{\pi}{R}\cdot 100$$

so that

$$\lim_{R\to\infty}\int_{C_R}\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz=0$$

By the fractional residue theorem,

$$\begin{aligned}
\lim_{R\to\infty}\int_{D_R}\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz &= -\lim_{R\to\infty}\int_{|z|=\frac{1}{R},\mathrm{Im}(z)>0}\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz \\
&\quad+\lim_{R\to\infty}\int_{(-R,-\frac{1}{R})\cup(\frac{1}{R},R)}\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz \\
&= -\frac{1}{2}2\pi i\mathrm{Res}\left[\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}dz,0\right] \\
&\quad+\lim_{R\to\infty}\int_{(-R,-\frac{1}{R})\cup(\frac{1}{R},R)}\mathrm{Re}\left(\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}\right)dz \\
&\quad+i\lim_{R\to\infty}\int_{(-R,-\frac{1}{R})\cup(\frac{1}{R},R)}\mathrm{Im}\left(\frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}\right)dz \\
&= \pi i+0 \quad \text{(since the integrand is odd)} \\
&\quad+i\lim_{R\to\infty}\int_{(-R,-\frac{1}{R})\cup(\frac{1}{R},R)}\frac{x^2-1}{x^2+1}\frac{\sin x}{x}dx \\
&= \pi i+i\int_{\mathbb{R}}\frac{x^2-1}{x^2+1}\frac{\sin x}{x}dx
\end{aligned}$$

Together we have

$$\begin{aligned}
\int_0^\infty\frac{x^2-1}{x^2+1}\frac{\sin x}{x}dx &= \frac{1}{2}\int_{\mathbb{R}}\frac{x^2-1}{x^2+1}\frac{\sin x}{x}dx \\
&= \frac{1}{2i}\left[2\pi e^{-1}i-\pi i\right] \\
&= \pi e^{-1}-\frac{\pi}{2}
\end{aligned}$$

□

Spring 2020 Problem 11. Let $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$ and let $K\subsetneq\mathbb{T}$ be a compact proper subset.
(a) Show that there is a sequence of polynomials $P_n(z)$ so that $P_n(z)\to\bar{z}$ uniformly on K.
(b) Show that there is *no* sequence of polynomials $P_n(z)$ for which $P_n(z)\to\bar{z}$ uniformly on $\mathbb{T}$.

Proof. (a): Since $\mathbb{C}\setminus K$ contains $\mathbb{D}$, $\mathbb{C}\setminus\overline{\mathbb{D}}$, and some point of $\partial\mathbb{D}$, we see that $\mathbb{C}\setminus K$ is path connected and hence connected. By Runge's theorem, we may find a sequence of polynomials $\{P_n\}$ such that $P_n(z)\to\frac{1}{z}$ uniformly on K. Since $\overline{z}=\frac{1}{z}$ on $K\subseteq\partial\mathbb{D}$, this is the desired result.

(b): Suppose for the sake of contradiction that P_n is a sequence of polynomials converging uniformly on $\mathbb{T}$ to $\overline{z}$. Then, for each $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$,

$$0=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{P_n(w)}{w-z}dw=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\bar{w}}{w-z}dw=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{1}{w(w-z)}dw=-\frac{1}{z}$$

a contradiction. □

Spring 2020 Problem 12. [4] Let u be a continuous subharmonic function on $\mathbb{C}$ that satisfies

$$\limsup_{|z|\to\infty} \frac{u(z)}{\log|z|} \leq 0$$

Show that u is constant on $\mathbb{C}$.

Proof. Since subharmonic functions are preserved by conformal changes-of-coordinate, the function

$$v(z) := u\left(\frac{1}{z}\right)$$

is subharmonic on $\mathbb{C} \setminus \{0\}$ and satisfies

$$v(z) = o(\log|z|) \quad \text{as } z \to 0$$

For each $\varepsilon > 0$, the function $v(z) - \varepsilon \log|z|$ satisfies the maximum principle on $\mathbb{D} \setminus \{0\}$. By the decay estimate on v at 0,

$$v(z) - \varepsilon \log|z| \to -\infty \quad \text{as } z \to 0$$

and hence, for any $z \in \mathbb{D} \setminus \{0\}$,

$$v(z) - \varepsilon \log|z| \leq \max_{z\in\partial\mathbb{D}} v(z)$$

Since $\varepsilon > 0$ was arbitrary, we conclude

$$v(z) \leq \max_{z\in\partial\mathbb{D}} v(z)$$

for all $z \in \mathbb{D} \setminus \{0\}$. Thus v is bounded above near 0, so u is bounded above near ∞. Since u was assumed to be continuous, it is locally bounded, and hence is globally bounded above. By a standard fact, globally bounded above subharmonic functions on $\mathbb{C}$ are constant (by e.g. the super-averaging principle) and hence u is constant.

□

[4] keyword: Phragmén-Lindelöf

4 Fall 2020

Fall 2020 Problem 1. (a) Suppose $f : [0,1] \times [0,\infty) \to [0,1]$ is continuous. Prove that $F : [0,1] \to [0,1]$ defined by

$$F(x) = \limsup_{y\to\infty} f(x,y)$$

is Borel measurable.
(b) Show that for any Borel set $E \subseteq [0,1]$ there is a choice of continuous function $f : [0,1]\times\mathbb{R} \to [0,1]$ so that F agrees almost everywhere with the indicator function of E.

Proof. (a): Since half-open intervals of the form (a,∞) with $a \in \mathbb{R}$ generate as a σ-algebra the full Borel σ-algebra $\mathcal{B}$, it suffices to show that

$$F^{-1}((a,\infty)) \in \mathcal{B}$$

for arbitrary $a \in \mathbb{R}$. The left-hand side may be written as

$$\begin{aligned} F^{-1}((a,\infty)) &= \{x \in [0,1] : \exists n \in \mathbb{N}\, \forall q \in \mathbb{Q}\cap[0,\infty) \exists p \in \mathbb{Q}\cap(q,\infty) \text{ s.t. } f(x,p) \in (a+\frac{1}{n},\infty)\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{q\in\mathbb{Q}\cap[0,\infty)} \bigcup_{p\in\mathbb{Q}\cap(q,\infty)} \{x \in [0,1] : f(x,p) \in (a+\frac{1}{n},\infty)\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{q\in\mathbb{Q}\cap[0,\infty)} \bigcup_{p\in\mathbb{Q}\cap(q,\infty)} f(\cdot,p)^{-1}(a+\frac{1}{n},\infty) \end{aligned}$$

Since f is continuous, the latter set is Borel; hence F is Borel, as desired.

(b): (Currently in the works)

We claim that, if $A_1, A_2, \ldots$ are Borel subsets of $[0,1]$ and $f_1, f_2, \ldots : [0,1]\times[0,\infty) \to [0,1]$ are continuous such that

$$\limsup_{y\to\infty} f_n(x,y) = 1_{A_n}(x) \quad \text{a.e. } x \in [0,1]$$

for each n, then there exists $f : [0,1]\times[0,\infty) \to [0,1]$ continuous such that

$$\limsup_{y\to\infty} f(x,y) = 1_{\bigcap_n \bigcup_{k\ge n} A_k}(x) \quad \text{a.e. } x \in [0,1]$$

Note that, for each $n \in \mathbb{N}$,

$$[0,1]\setminus A_n = \bigcup_{r\in\mathbb{N}} \{x \in [0,1] : \sup_{y>r} f_n(x,y) \le 1/2n\}$$

modulo nullsets. By continuity from below of measures, there exists some r_n such that

$$B_n := \{x \in [0,1] : \sup_{y>r_n} f_n(x,y) \le 1/2n\}$$

is a subset (mod null) of $[0,1]\setminus A_n$, and satisfies

$$m([0,1]\setminus(A_n \cup B_n)) < 2^{-n}.$$

Note next that

$$\sup_{y>r_n} f(x,y) = 1 \quad \text{for a.e. } x \in A_n$$

so that, modulo nullsets, we have

$$A_n = \bigcup_{s > r_n} \{x \in [0,1] : \sup_{r_n < y < s} f(x,y) > 1 - 1/2n\}$$

By continuity from above of finite measures, there is some $s_n > r_n$ such that

$$C_n := \{x \in [0,1] : \sup_{r_n < y < s_n} f(x,y) > 1 - 1/2n\}$$

has

$$m(A_n \setminus C_n) < 2^{-n}$$

and $C_n \setminus A_n$ is null.

Next, we claim that

$$\bigcap_n \bigcup_{k \geq n} A_k = \bigcap_n \bigcup_{k \geq n} C_k$$

mod nullsets. Note that

$$\begin{aligned}
m\left(\bigcap_n \bigcup_{k \geq n} A_k\right) &= \lim_{n \to \infty} m\left(\bigcup_{k \geq n} A_k\right) \\
&= \lim_{n \to \infty} m\left(\bigcup_{k \geq n} C_k \cup \bigcup_{k \geq n} A_k \setminus C_k\right) \\
&\leq \lim_{n \to \infty} \left[2^{-n+1} + m\left(\bigcup_{k \geq n} C_k\right)\right] \\
&= m\left(\bigcap_n \bigcup_{k \geq n} C_k\right)
\end{aligned}$$

The reverse inequality is trivial, so in fact

$$m\left(\bigcap_n \bigcup_{k \geq n} C_k\right) = m\left(\bigcap_n \bigcup_{k \geq n} A_k\right)$$

From the equality

$$m\left(\bigcap_n \bigcup_{k \geq n} C_k\right) + m\left(\left\{\bigcap_n \bigcup_{k \geq n} A_k\right\} \setminus \left\{\bigcap_n \bigcup_{k \geq n} C_k\right\}\right) = m\left(\bigcap_n \bigcup_{k \geq n} A_k\right)$$

we conclude that

$$m\left(\left\{\bigcap_n \bigcup_{k \geq n} A_k\right\} \Delta \left\{\bigcap_n \bigcup_{k \geq n} C_k\right\}\right)$$

so that

$$\bigcap_n \bigcup_{k \geq n} C_k = \bigcap_n \bigcup_{k \geq n} A_k$$

modulo nullsets.

Next, we claim that

$$\bigcup_n \bigcap_{k\geq n} [0,1] \setminus A_k = \bigcup_n \bigcap_{k\geq n} B_k$$

modulo nullsets. Since $B_k \setminus ([0,1] \setminus A_k)$ is null for all k, it follows that

$$\left\{\bigcup_n \bigcap_{k\geq n} B_k\right\} \setminus \left\{\bigcup_n \bigcap_{k\geq n} [0,1] \setminus A_k\right\}$$

is null. We consider the other difference. We compute

$$\begin{aligned}
m\left(\bigcup_n \bigcap_{k\geq n} [0,1] \setminus A_k\right) &= \lim_{n\to\infty} m\left(\bigcap_{k\geq n} [0,1] \setminus A_k\right) \\
&= \lim_{n\to\infty} m\left(\bigcap_{k\geq n} B_k \cup ([0,1] \setminus (A_k \cup B_k))\right) \\
&= \lim_{n\to\infty} \lim_{\ell\to\infty} m\left(\bigcap_{k=n}^{\ell} B_k \cup ([0,1] \setminus (A_k \cup B_k))\right) \\
&\leq \lim_{n\to\infty} \lim_{\ell\to\infty} m\left(\bigcap_{k=n}^{\ell} B_k\right) + \sum_{k=n}^{\ell} 2^{-n} \\
&= \lim_{n\to\infty} m\left(\bigcap_{k=n}^{\infty} B_k\right) + 2^{-n+1} \\
&= m\left(\bigcup_n \bigcap_{k\geq n} B_k\right)
\end{aligned}$$

so that we may conclude

$$m\left(\bigcup_n \bigcap_{k\geq n} [0,1] \setminus A_k\right) = m\left(\bigcup_n \bigcap_{k\geq n} B_k\right)$$

Lastly, we write

$$\begin{gathered}
N_{n,0} = B_n \setminus ([0,1] \setminus A_n) \\
N_{n,1} = C_n \setminus A_n \\
N_{n,2} = \{x \in A_n : \limsup_{y\to\infty} f_n(x,y) \neq 1\} \\
N_{n,3} = \{x \in [0,1] \setminus A_n : \limsup_{y\to\infty} f_n(x,y) \neq 0\} \\
N_{\infty,1} = \left\{\bigcap_n \bigcup_{k\geq n} A_k\right\} \Delta \left\{\bigcap_n \bigcup_{k\geq n} C_k\right\} \\
N_{\infty,2} = \left\{\bigcup_n \bigcap_{k\geq n} [0,1] \setminus A_k\right\} \Delta \left\{\bigcup_n \bigcap_{k\geq n} B_k\right\}
\end{gathered}$$

and

$$N = N_{\infty,1} \cup N_{\infty,2} \cup \bigcup_{n=1}^{\infty} (N_{n,0} \cup N_{n,1} \cup N_{n,2} \cup N_{n,3})$$

By assumption, each of the sets in the above union is null, so N is null.

We now construct f. For $n \in \mathbb{N}$ and $t \in [1/4, 3/4]$, we set

$$f(x, n+t) = f_n(x, 2(t - 1/4)(s_n - r_n) + r_n)$$

and, for $0 \leq t \leq 1/4$,

$$f(x, n+t) = 4t f_n(x, r_n)$$

and, for $3/4 \leq t < 1$,

$$4(t - 3/4) f(x, s_n)$$

and $f(x, y) = 0$ for $y < 1$. Observe then that we have arranged for f to be a continuous function over the desired domain, taking values in $[0, 1]$. We claim that f has the appropriate limsup almost everywhere. Suppose $x \in \bigcap_n \bigcup_{k \geq n} A_k$ with $x \notin N$. Then $x \in \bigcap_n \bigcup_{k \geq n} C_k$, so $x \in \bigcup_{k \geq n} C_k$ for all n, so there are $n_1 < n_2 < \cdots \in \mathbb{N}$ such that $x \in C_{n_k}$ for all k. Thus

$$\sup_{r_{n_k} < y < s_{n_k}} f_{n_k}(x, y) > 1 - 1/2n_k$$

so that

$$\sup_{n_k < y < n_k + 1} f(x, y) > 1 - 1/2n_k$$

and hence

$$\limsup_{y \to \infty} f(x, y) = 1$$

Assume now that $x \notin \bigcap_n \bigcup_{k \geq n} A_k$ and $x \notin N$. Then $x \in \bigcup_n \bigcap_{k \geq n} B_k$, so $x \in \bigcap_{k \geq n} B_k$ for some $n \in \mathbb{N}$. Thus

$$\sup_{y > r_k} f_n(x, y) \leq 1/2k$$

and so (by the construction of f)

$$\sup_{y > n_k} f(x, y) \leq 1/2k$$

from which

$$\limsup_{y \to \infty} f(x, y) = 0$$

Thus we have demonstrated

$$\limsup_{y \to \infty} f(x, y) = 1_{\bigcap_n \bigcup_{k \geq n} A_k}(x)$$

for a.e. $x \in [0, 1]$. We now study the family

$$\mathcal{E} = \{A \subseteq [0,1] : A \text{ Borel and } \exists f : [0,1] \times [0, \infty) \to [0,1] \text{ continuous} \\ \text{s.t. } 1_A(x) = \limsup_{y \to \infty} f(x, y) \text{ for a.e. } x \in [0,1]\}$$

By the previous portion of the argument, $\mathcal{E}$ is closed under countable unions and countable intersections.

We now demonstrate that $\mathcal{E}$ contains intervals. For brevity, we only consider the case of closed intervals. Suppose $a < b \in [0,1]$. Then the function

$$f(x,y) = \begin{cases} 0 & x \notin (a - \frac{1}{1+y}, b + \frac{1}{1+y}) \\ (1+y)(x - (a - \frac{1}{1+y})) & x \in [a - \frac{1}{1+y}, a] \\ 1 & x \in (a,b) \\ -(1+y)(x-b) & x \in [b, b + \frac{1}{1+y}] \end{cases}$$

is clearly continuous on $[0,1] \times [0,\infty)$ and satisfies

$$\lim_{y\to\infty} f(x,y) = \chi_{[a,b]}$$

as claimed.

Finally, observe that $\mathcal{E}$ contains the ring of finite unions of intervals. Since $\mathcal{E}$ is a monotone class, $\mathcal{E}$ contains the σ-algebra generated by intervals, i.e. $\mathcal{E}$ contains all Borel sets, as was to be shown.

$\square$

Fall 2020 Problem 2. Show that there is a constant $c \in \mathbb{R}$ so that

$$\lim_{n\to\infty} \int_0^1 f(x)\cos(\sin(n\pi x))dx = c\int_0^1 f(x)dx$$

for every $f \in L^1([0,1])$. The limit is taken over those $n \in \mathbb{N}$.

Proof. We first show that

$$\int_0^1 \cos(\sin(r\pi x))dx \overset{r\to\infty}{\longrightarrow} c$$

for some constant $c \in \mathbb{R}$. To do this, define the functions

$$G(r) := \int_0^1 \cos(\sin(\pi r x))dx = \frac{1}{r}\int_0^r \cos(\sin(\pi x))dx$$

and

$$F(r) = \int_0^r \cos(\sin(\pi x))dx = rG(r)$$

for $0 < r \in \mathbb{R}$. Note that

$$\begin{aligned} F(r+1) &= F(r) + \int_r^{r+1} \cos(\sin(\pi x))dx \\ &= F(r) + \int_0^1 \cos(\sin(\pi(x - \lfloor r \rfloor)))dx \\ &= F(r) + \int_0^1 \cos(\pm\sin(\pi x))dx \\ &= F(r) + F(1) \end{aligned}$$

from which we conlude

$$F(r) = \lfloor r \rfloor F(1) + F(r - \lfloor r \rfloor)$$

whenever $r \notin \mathbb{N}$, and

$$F(n) = nF(1)$$

for $n \in \mathbb{N}$. Thus

$$G(n) = \frac{F(n)}{n} = F(1)$$

and, for $r \notin \mathbb{N}$,

$$G(r) = \frac{F(r)}{r} = F(1) - \frac{(\lfloor r \rfloor - r)F(1)}{r} + \frac{F(r - \lfloor r \rfloor)}{r}$$

from which we easily see that

$$G(r) \overset{r\to\infty}{\longrightarrow} F(1) \in \mathbb{R}$$

which is our c. Thus, for any interval $[a,b] \subseteq [0,1]$ of positive length,

$$\begin{aligned}\int_0^1 \chi_{[a,b]}(x)\cos(\sin(n\pi x))dx &= \int_0^1 (\chi_{[0,b]} - \chi_{[0,a]})\cos(\sin(n\pi x))dx \\ &= bG(bn) - aG(an) \\ &\overset{n\to\infty}{\longrightarrow} (b-a)c = c\int \chi_{[a,b]}\end{aligned}$$

if $a \neq 0$, and

$$\int_0^1 \chi_{[0,b]}(x)\cos(\sin(n\pi x))dx = bG(bn) \overset{n\to\infty}{\longrightarrow} bc = c\int \chi_{[0,b]}$$

Summing, we see that for any simple function f,

$$\lim_{n\to\infty}\int_0^1 f(x)\cos(\sin(n\pi x))dx = c\int_0^1 f(x)dx$$

Now suppose $f \in L^1([0,1])$ and $f_1, f_2, \ldots \overset{L^1}{\to} f$ are simple functions. Let $\varepsilon > 0$, and fix $k \in \mathbb{N}$ such that $\|f_k - f\|_{L^1} < \varepsilon/3$ and $\|f_k - f\|_{L^1} < \varepsilon/(3c)$. Lastly, pick $N \in \mathbb{N}$ such that, for all $n > N$,

$$\left|\int_0^1 f_k(x)\cos(\sin(n\pi x))dx - c\int_0^1 f_k(x)dx\right| < \varepsilon/3 \tag{8}$$

Then we have, for such n,

$$\begin{aligned}\int_0^1 f(x)\cos(\sin(n\pi x))dx &= \int_0^1 f_k(x)\cos(\sin(n\pi x))dx + \int_0^1 [f - f_k](x)\cos(\sin(n\pi x))dx \\ &= c\int_0^1 f_k(x)dx + \left(\int_0^1 f_k(x)\cos(\sin(n\pi x))dx - \int_0^1 f_k(x)dx\right) \\ &+ \int_0^1 [f - f_k](x)\cos(\sin(n\pi x))dx \\ &= c\int_0^1 f(x)dx + c\int_0^1 [f - f_k](x)dx \\ &+ \left(\int_0^1 f_k(x)\cos(\sin(n\pi x))dx - \int_0^1 f_k(x)dx\right) \\ &+ \int_0^1 [f - f_k](x)\cos(\sin(n\pi x))dx \\ &= c\int_0^1 f(x)dx + I + II + II\end{aligned}$$

Since $\|f_k - f\|_{L^1} < \varepsilon/(3c)$, we see that $|I| < \varepsilon/3$. Since $\|f_k - f\|_{L^1} < \varepsilon/3$ and $|\cos(\sin(n\pi x))| \leq 1$ everywhere, we see that $|III| < \varepsilon/3$. Lastly, by 8, $|II| < \varepsilon/3$. Thus, for all $n > N$,

$$\left|\int_0^1 f(x)\cos(\sin(n\pi x))dx - c\int_0^1 f(x)dx\right| < \varepsilon$$

and so $\int_0^1 f(x)\cos(\sin(n\pi x))dx \xrightarrow{n} c\int_0^1 f(x)dx$ for arbitrary $f \in L^1([0,1])$, as desired. $\square$

Fall 2020 Problem 3. [5] Let $d\mu_n$ be a sequence of probability measures on $[0,1]$ so that

$$\int f(x)d\mu_n(x)$$

converges for every continuous function $f : [0,1] \to \mathbb{R}$.
(a) Show that

$$\iint g(x,y)d\mu_n(x)d\mu_n(y)$$

converges for every continuous function $g : [0,1]^2 \to \mathbb{R}$.
(b) Show by example that under the above hypotheses, it is possible that

$$\iint_{0\leq x\leq y\leq 1} d\mu_n(x)d\mu_n(y)$$

does not converge.

Proof. (a): We first show this in the case $g(x,y) = f(x)f'(y)$ for continuous functions f, f' on $[0,1]$. In this case,

$$\iint g(x,y)d\mu_n(x)d\mu_n(y) = \iint f(x)f'(y)d\mu_n(x)d\mu_n(y) = \left(\int_0^1 f(x)d\mu_n(x)\right)\left(\int_0^1 f'(y)d\mu_n(y)\right)$$

is a product of convergent sequences, hence converges.

We claim that the algebra $\mathcal{A}$ generated by such products are dense in $C([0,1]^2)$ with respect to the L^∞ norm. To show this we use Stone-Weierstrass: since $[0,1]^2$ is a compact metric space, it suffices to show that $\mathcal{A}$ contains the constants and separates points. Clearly we have all constants. Suppose $(x,y),(x',y') \in [0,1]^2$ are distinct, say $x \neq x'$. Then let f be continuous on $[0,1]$ such that $f(x) = 1, f(x') = 0$. Then $g(x,y) = f(x)$ is a product of two continuous functions on $[0,1]$ which separates (x,y) from (x',y'); thus $\mathcal{A}$ is dense in $C([0,1]^2)$.

Since $\mathcal{A}$ is in fact linearly generated by such tensor product pairs, we see from the computation

$$\iint ag(x,y) + bg'(x,y)d\mu_n(x)d\mu_n(y) = a\iint g(x,y)d\mu_n(x)d\mu_n(y) + b\iint g'(x,y)d\mu_n(x)d\mu_n(y)$$

that the desired convergence holds for all elements of $\mathcal{A}$.

We conclude by approximation: suppose $g \in C([0,1]^2)$ and let $g_1, g_2, \ldots \in \mathcal{A}$ such that $g_j \xrightarrow{L^\infty} g$. If $\varepsilon > 0$, fix $j \in \mathbb{N}$ such that $\|g_j - g\|_{L^\infty} < \varepsilon/3$. Then let $N \in \mathbb{N}$ such that

$$\left|\iint g_j(x,y)d\mu_n(x)d\mu_n(y) - \iint g_j(x,y)d\mu_m(x)d\mu_m(y)\right| < \varepsilon/3$$

[5] keyword: Stone-Weierstrass

for all $n, m > N$. Then, for such n, m,

$$\begin{aligned}
&\left|\iint g(x,y)d\mu_n(x)d\mu_n(y) - \iint g(x,y)d\mu_m(x)d\mu_m(y)\right| \\
&\le \left|\iint g(x,y)d\mu_n(x)d\mu_n(y) - \iint g_j(x,y)d\mu_n(x)d\mu_n(y)\right| \\
&+ \left|\iint g_j(x,y)d\mu_n(x)d\mu_n(y) - \iint g_j(x,y)d\mu_m(x)d\mu_m(y)\right| \\
&+ \left|\iint g_j(x,y)d\mu_m(x)d\mu_m(y) - \iint g(x,y)d\mu_m(x)d\mu_m(y)\right| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

from which we conclude that

$$n \mapsto \iint g(x,y)d\mu_n(x)d\mu_n(y)$$

is Cauchy, hence convergent. Thus the result holds for every continuous real-valued g, as desired.

(b): Define μ_n as

$$\mu_n = \begin{cases} \delta_1 & n \text{ odd} \\ \sum_{k=1}^n \frac{1}{n}\delta_{1-k/n^2} & n \text{ even} \end{cases}$$

Then clearly every continuous function $[0,1] \to \mathbb{R}$ satisfies

$$\int f(x)d\mu_n(x) \to f(1)$$

as $n \to \infty$; however,

$$\iint_{0\le x\le y\le 1} d\mu_n(x)d\mu_n(y) = 1$$

for n odd, whereas for n even

$$\begin{aligned}
\iint_{0\le x\le y\le 1} d\mu_n(x)d\mu_n(y) &= 1/2 + \sum_{k=1}^n \frac{1}{n^2} \\
&= 1/2 + 1/n
\end{aligned}$$

which has limit $1/2$. Thus the above sequence fails to converge over $n \in \mathbb{N}$. $\square$

Fall 2020 Problem 4. Let X be a separable Banach space over $\mathbb{R}$ and let $F : X \to \mathbb{R}$ be norm-continuous and convex. Suppose now that a sequence x_n in X converges weakly to $x \in X$. Show that

$$F(x) \le \sup_n F(x_n)$$

Proof. By a standard fact of functional analysis, there are convex linear combinations

$$y_n = t_1^n x_1 + \ldots + t_n^n x_n$$

such that $y_n \to x$ strongly. Since F is (strongly) continuous, $F(y_n) \to F(x)$. By convexity,

$$F(y_n) \le t_1^n F(x_1) + \ldots + t_n^n F(x_n) \le \sup_{1\le j\le n} F(x_n)$$

For every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that, for each $n > N$,

$$F(x) \leq F(y_n) - \varepsilon \leq \sup_{1 \leq j \leq n} F(x_n) - \varepsilon \leq \sup_n F(x_n) - \varepsilon$$

Sending ε to 0, we conclude that

$$F(x) \leq \sup_n F(x_n)$$

as desired. $\square$

Fall 2020 Problem 5. [6] Suppose $f \in L^1([0,1])$ has the property that

$$(*) \qquad \int_E |f(x)| dx \leq \sqrt{|E|}$$

for every Borel $E \subseteq [0,1]$. Here $|E|$ denotes the Lebesgue measure of E.
(a) Show that $f \in L^p([0,1])$ for all $p < 2$.
(b) Give an example of an f satisfying $(*)$ that is not in $L^2([0,1])$.

Proof. (a): Fix $p < 2$. For $n \in \mathbb{Z}$, write $L_n = \{x \in [0,1] : 2^n \leq |f(x)| < 2^{n+1}\}$. Then we have

$$2^n |L_n| \leq \int_{L_n} |f(x)| dx \leq |L_n|^{1/2}$$

so that $2^n |L_n|^{1/2} \leq 1$. Let $q \in (2p-2, 2)$; then $2^{qn} |L_n|^{\frac{q}{2}} \leq 1$ as well. We may also write

$$\|f\|_1 \geq \sum_{n \in \mathbb{Z}} 2^n |L_n|$$

so that the right-hand side is finite.

Finally, we compute:

$$\begin{aligned}
\int_{[0,1]} |f(x)|^p dx &\leq 2^p \sum_{n \in \mathbb{Z}} 2^{np} |L_n| \\
&\leq \frac{2^{2p}}{2^p - 1} + 2^p \sum_{n \geq 1} 2^{n(p-1-\frac{q}{2})} \cdot 2^{n(1-\frac{q}{2})} |L_n|^{1-\frac{q}{2}} \cdot 2^{qn} |L_n|^{\frac{q}{2}} \\
&\leq \frac{2^{2p}}{2^p - 1} + 2^p \left(\sum_{n \geq 1} 2^{\frac{n(2p-2-q)}{q}} \right)^{\frac{q}{2}} \left(\sum_{n \geq 1} 2^n |L_n| \right)^{1-\frac{q}{2}}
\end{aligned}$$

Observe that the first series in the latter display is a geometric series with common ratio in $(0,1)$, hence converges. The second series in the latter display is finite, by the previous comparison to $\|f\|_1$. Thus, $f \in L^p([0,1])$, as claimed.

(b): Let $f(x) = \frac{1}{4\sqrt{x}}$. Note that $f \in L^1$ but not L^2, so it remains to show that f satisfies $(*)$. Suppose $E \subseteq [0,1]$ is Borel. We will write $|E| = \lambda(E)$. Then

$$\begin{aligned}
\int_E |f(x)| dx &= \int_{E \cap [0,|E|]} |f(x)| dx + \int_{E \cap (|E|,1]} |f(x)| dx \\
&\leq \int_0^{|E|} \frac{1}{4\sqrt{x}} dx + \frac{|E|}{4\sqrt{|E|}} \\
&= \frac{1}{2}\sqrt{|E|} + \frac{1}{4}\sqrt{|E|} \leq \sqrt{|E|}
\end{aligned}$$

[6] keyword: dyadic decomposition; Lorentz spaces

as claimed. $\square$

Fall 2020 Problem 6. Prove that the following inequality is valid for all odd C^1 functions $f : [-1, 1] \to \mathbb{R}$:

$$\int_{-1}^{1} |f(x)|^2 dx \leq \int_{-1}^{1} |f'(x)|^2 dx$$

By odd, we mean that $f(-x) = -f(x)$.

Proof. We compute

$$\begin{aligned}
\int_{-1}^{1} f(x)^2 dx &= \int_{-1}^{1} \left(\int_{0}^{x} f'(t) dt \right)^2 dx \\
&\leq \left[\int_{-1}^{1} |f'(t)||1-t|^{1/2} dt \right]^2 \quad \text{by Minkowski} \\
&\leq \left[\int_{-1}^{1} f'(t)^2 dt \right] \left[\int_{-1}^{1} |1-t| dt \right] \quad \text{by Hölder} \\
&= \int_{-1}^{1} f'(t)^2 dt
\end{aligned}$$

as desired. $\square$

Fall 2020 Problem 7. Let $\Delta_j = \{z : |z - a_j| \leq r_j\}$, $1 \leq j \leq n$ be a collection of disjoint closed disks, with radii $r_j \geq 0$, all contained in the open unit disk $\mathbb{D}$ of the complex plane. Let $\Omega = \mathbb{D} \setminus (\cup_j \Delta_j)$, and let $u : \Omega \to \mathbb{R}$ be harmonic. Prove that there exist real numbers $c_1, \ldots, c_n$ such that

$$u(z) - \sum_{j=1}^{n} c_j \log |z - a_j|$$

is the real part of a (single valued) analytic function on Ω. Show also that the choice of $c_1, \ldots, c_n$ is unique.

Proof. Consider the one-form

$$v(z) = \frac{\partial u}{\partial z} dz$$

on Ω, where $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ is a Wirtinger derivative. Since u is harmonic, v is holomorphic (in the sense that the coefficient function is holomorphic). For each $1 \leq j \leq n$, set C_j to be a counterclockwise loop in Ω around Δ_j that doesn't enclose any other Δ_k; by standard algebraic topology, $\{C_j\}_{1 \leq j \leq n}$ determine a basis for $H_1(\Omega; \mathbb{R})$.

Now, for each j, set

$$c_j = \frac{1}{\pi i} \int_{C_j} v$$

A standard calculation shows

$$\int_{C_j} c_j \frac{\partial}{\partial z} \log |z - a_j| dz = \pi i c_j = \int_{C_j} v$$

so that

$$w(z) := u(z) - \sum_{j=1}^{n} c_j \log|z - a_j|$$

is a harmonic function on Ω satisfying

$$\int_{C_j} \frac{\partial}{\partial z} w(z) dz = 0$$

for all j. Since $\{C_j\}$ generate all of $H_1(\Omega; \mathbb{R})$, we see that, for $h(z) = \frac{\partial}{\partial z} w(z)$,

$$h \text{ holomorphic on } \Omega \quad \text{and} \quad \int_\gamma h(z) dz = 0$$

for all piecewise smooth loops γ in Ω. Thus, for fixed $z_0 \in \Omega$, the path integral

$$g(z) = w(z_0) + \int_{z_0}^{z} h(t) dt$$

is well-defined (that is, independent of path chosen) and is an analytic antiderivative for h on Ω.

Thus we find

$$\frac{\partial}{\partial z} g(z) = h(z) = \frac{\partial}{\partial z} w(z), \quad g(z_0) = w(z_0)$$

and so w is equal to $g + a$ where a is a conjugate-analytic function on Ω vanishing at z_0; since w is purely real-valued, $\mathrm{Im}(a) = -\mathrm{Im}(g)$ and so $\bar{a}$ is an analytic function on Ω whose imaginary part agrees with that of g everywhere on Ω. Thus a and g agree up to a real additive constant; since $a(z_0) = 0$ we see that $w = g + \bar{g} - g(z_0)$. Thus $w = 2\mathrm{Re}(g) - g(z_0)$, so w is indeed the real part of an analytic function on Ω.

Finally, if

$$u(z) - \sum_{j=1}^{n} d_j \log|z - a_j| = \mathrm{Re}(q)$$

for constants $d_1, \ldots, d_n$ and analytic function q on Ω, then

$$\int_{C_j} \frac{\partial q}{\partial z} dz = 0$$

for each z, by e.g. examining the Laurent series about each a_j. Taking real parts,

$$\int_{C_j} \frac{\partial u}{\partial z} dz = d_j \pi i$$

which implies that $d_j = c_j$ from before, so these constants are unique. □

Fall 2020 Problem 8. [7] Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and satisfy $f(\frac{1}{2}) = f(-\frac{1}{2}) = 0$. Show that

$$|f(0)| \leq \frac{1}{4}.$$

[7] keyword: Blaschke factors

Proof. Note that $\tilde{f}$, defined by

$$\tilde{f}(z) = \frac{1 - z/2}{z - 1/2}\frac{1 + z/2}{z + 1/2} f(z)$$

is analytic in $\mathbb{D}$ and takes values in $\overline{\mathbb{D}}$, by standard Blaschke factor theory. Thus

$$|f(0)| = \left|\frac{1/2}{1}\right|\left|\frac{1/2}{1}\right||\tilde{f}(0)| \leq \frac{1}{4}$$

as desired. □

Fall 2020 Problem 9. Consider the following region in the complex plane:

$$\Omega = \{x + iy : 0 < x < \infty \text{ and } 0 < y < \frac{1}{x}\}.$$

Exhibit an explicit conformal mapping f of Ω onto $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Proof. We first claim that, for $f_1(z) = z^2$,

$$f_1(\Omega) = \{z \in \mathbb{C} : \text{Im}(z) \in (0, 2)\}$$

and that f_1 is a conformal map between these two domains. To show this, note first that Ω doesn't contain any pairs z_1, z_2 with $z_2 = -z_1$; since f_1 is clearly analytic, f_1 is a conformal map $\Omega \to f_1(\Omega)$.

We show that its image is as claimed. First note that, for any $x + iy$ with $y < 1/x$,

$$f_1(x + iy) = x^2 - y^2 + i2xy, \quad 2xy < 2$$

so $f_1(\Omega) \subseteq \{z : \text{Im}(z) < 2\}$. Similarly, since $x > 0, y > 0$ for all $x + iy \in \Omega$, $f_1(\Omega) \subseteq \{z : \text{Im}(z) > 0$, so we conclude that

$$f_1(\Omega) \subseteq \{z \in \mathbb{C} : \text{Im}(z) \in (0, 2)\}$$

We claim that every point of this domain lies in the image of f_1. If $\sqrt{\cdot}$ denotes the branch of the inverse of $z \mapsto z^2$ for which $\sqrt{i} \in \Omega$, we see that

$$w = \sqrt{re^{i\theta}} = \sqrt{r(\cos\theta + i\sin\theta)} = \sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$$

satisfies

$$\text{Re}(w), \text{Im}(w) > 0 \quad \text{and } \text{Im}(w)\text{Re}(w) = r\cos(\theta/2)\sin(\theta/2) = \frac{r}{2}\sin(\theta) < 1$$

whenever $r\sin(\theta) = \text{Im}(w) < 2$, i.e. when w is in the putative image of f_1. Thus $f_1(\Omega) = \{z \in \mathbb{C} : \text{Im}(z) \in (0, 2)\}$, as claimed.

The remainder of the problem is routine: $\Omega_1 := \frac{\pi}{2}f_1(\Omega)$ satisfies

$$z \mapsto \exp(z) : \Omega_1 \to \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

conformally, and finally the Möbius transformation

$$z \mapsto \frac{z - i}{z + i}$$

carries $\mathbb{H}$ onto $\mathbb{D}$ conformally. Thus the composition

$$z \mapsto \frac{e^{\frac{\pi}{2}z^2} - i}{e^{\frac{\pi}{2}z^2} + i}$$

maps Ω conformally onto $\mathbb{D}$, as required. □

Fall 2020 Problem 10. Let $K \subseteq \mathbb{C}$ be a compact set of positive area but empty interior and define a function $F : \mathbb{C} \to \mathbb{C}$ via

$$F(z) = \iint_K \frac{1}{w-z} d\mu(w),$$

where $d\mu$ denotes (planar) Lebesgue measure on $\mathbb{C}$.
(a) Prove that $F(z)$ is bounded and continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus K$.
(b) Prove that $\{F(z) : z \in \mathbb{C}\} = \{F(z) : z \in K\}$.
Hint: If $a \in F(\mathbb{C}) \setminus F(K)$ and $F^{-1}(a) = \{z_1, \ldots, z_n\} \subseteq \mathbb{C} \setminus K$, then the argument principle can be applied to $G(z) = \frac{F(z)-a}{\prod_j (z-z_j)}$ to get a contradiction.

Proof. (a): We use the fact that translation is continuous in $L^p(\mathbb{C})$, for every $1 \le p < \infty$. If $\tau_\varepsilon f(\cdot) = f(\cdot - \varepsilon)$, we note that

$$\begin{aligned}
|(\tau_\varepsilon F - F)(z)| &\le \|(\tau_\varepsilon \frac{1}{\cdot - z} - \frac{1}{\cdot - z})\chi_K\|_{L^1(\mathbb{C})} \\
&= \|\frac{1}{\cdot - z}(\tau_{-\varepsilon}\chi_K - \chi_K)\|_{L^1(\mathbb{C})} \\
&\le \|\frac{1}{\cdot - z}\chi_{B(0,R)}\|_{L^{3/2}(\mathbb{C})}\|\tau_{-\varepsilon}\chi_K - \chi_K\|_{L^3(\mathbb{C})} \quad \text{for sufficiently large } R > 0 \\
&\overset{\varepsilon \to 0}{\longrightarrow} 0
\end{aligned}$$

where $R > 0$ is sufficiently large so that $K \subseteq B(0, R-1)$, and where the limit holds because $w \mapsto \frac{1}{w-z}$ is locally $L^{3/2}$ and translation is continuous in $L^3(\mathbb{C})$. Thus

$$F(z - \varepsilon) \to F(z)$$

as $\mathbb{C} \ni \varepsilon \to 0$, for arbitrary $z \in \mathbb{C}$; that is, F is continuous on all of $\mathbb{C}$.

Next, we argue that F is bounded on $\mathbb{C}$: if $R > 0$ is sufficiently large so that $K \subseteq B(0, R/2)$, then for every $z \notin B(0, R)$

$$|F(z)| \le \frac{2}{R}\mu(K)$$

which implies that $F(z) \to 0$ as $z \to \infty$; thus F extends to a continuous function on the compact space $\mathbb{C} \cup \{\infty\}$, so F is bounded.

If Δ is a triangle in $\mathbb{C} \setminus K$ which does not bound any part of K, we compute

$$\begin{aligned}
\int_\Delta F(z)dz &= \int_\Delta \iint_K \frac{1}{w-z} d\mu(w)dz \\
&= \iint_K \int_\Delta \frac{1}{w-z} dz d\mu(w) \quad \text{by Fubini, since } \Delta \text{ and } K \text{ are compact} \\
&= 0 \quad \text{since } w \in K \text{ and } \Delta \text{ doesn't enclose any of } K
\end{aligned}$$

and so by Morera's theorem we conclude that F is analytic on $\mathbb{C} \setminus K$.

(b): Note that, if $a \in F(\mathbb{C}) \setminus F(K)$, then $F^{-1}(a)$ is finite (since otherwise there would be an accumulation point inside $\mathbb{C} \setminus K$, which would imply that F is constant, which contradicts $a \in F(\mathbb{C}) \setminus F(K)$). Thus we may assume $F^{-1}(a) = \{z_1, \ldots, z_n\}$ for some $n > 0$, listed with multiplicity; we will reach a contradiction from this. We assume for simplicity that $0 \notin K$; by translating, the general case will follow. Since $a \notin F(K)$, and $F(K)$ is compact, we see that

$$G(z) = \frac{F(z) - a}{\prod(z - z_j)}$$

is continuous on all of $\mathbb{C} \cup \{\infty\}$ with $G(\infty) = 0$, hence bounded. Furthermore, since the z_j were listed with multiplicity, ∞ is the unique zero of G. Thus, setting

$$H(z) = G(1/z)$$

the argument principle provides

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{H'(z)}{H(z)} dz = n$$

for sufficiently small $\varepsilon > 0$, and, changing variables,

$$\frac{1}{2\pi i} \int_{|z|=1/\varepsilon} \frac{G'(z)}{G(z)} dz = -n$$

Thus $G(\{|z| = \frac{1}{\varepsilon}\})$ is a (rectifiable) curve that has winding number $-n \neq 0$ with respect to 0. For each $0 < r \leq \frac{1}{\varepsilon}$, let γ_r denote the curve $\{|z| = r\}$, traced counterclockwise. Since G doesn't vanish anywhere in $\mathbb{C}$, the winding number of $G \circ \gamma_r$ with respect to 0 is well-defined for all r, though we may need to understand "winding number" as coming from the identification $\pi_1 \mathbb{C} \setminus \{0\} \cong \mathbb{Z}$ due to lack of regularity. It is also continuous and integer-valued, so in particular is equal to $-n$ for all r. Since $0 \notin K$, we have that $B(0, r) \subseteq K^c$ for sufficiently small $r > 0$. Since G is analytic on K^c, we reach a contradiction from the conclusion that $G \circ \gamma_r$ has winding number $-n < 0$.

□

Fall 2020 Problem 11. Let $\{f_n\}$ be a sequence of analytic functions on a (connected) domain Ω such that $|f_n(z)| \leq 1$ for all n and all $z \in \Omega$. Suppose the sequence $\{f_n(z)\}$ converges for infinitely many z in a compact subset K of Ω. Prove that $\{f_n(z)\}$ converges for all $z \in \Omega$.

Proof. We claim that there is some analytic function f on Ω such that every subsequence of $\{f_n\}$ has a further subsequence which converges locally uniformly to f; the result follows. Since $|f_n(z)| \leq 1$ uniformly, $\{f_n\}$ is a normal family, so every subsequence has a locally uniformly convergent further subsequence. Suppose f^1, f^2 are two such limit functions; they are clearly analytic. Since the collection of points $z \in K$ such that $f_n(z)$ converges is infinite, and that limit value must equal $f^1(z) = f^2(z)$, we see that $f^1(z) = f^2(z)$ for infinitely many points of K. Since K is compact, $\{z : f^1(z) = f^2(z)\}$ has an accumulation point inside of Ω. Thus by the uniqueness principle $f^1 = f^2$ on Ω, and so any subsequence of the $\{f_n\}$ refines to a further subsequence which converges to the same limit function f. □

Fall 2020 Problem 12. Let $\Omega = \{z \in \mathbb{C} : -2 < \operatorname{Im} z < 2\}$. Show that there is a finite constant C so that

$$|f(0)|^2 \leq C \int_{-\infty}^{\infty} [|f(x+i)|^2 + |f(x-i)|^2] dx$$

for every holomorphic $f : \Omega \to \mathbb{D}$ for which the right-hand side is finite.

Proof. By Cauchy's integral theorem, for each $R > 0$,

$$f(0) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(z)}{z} dz$$

where Γ_R is the counterclockwise-oriented rectangle in Ω with top and bottom along the lines $\{\text{Im}(z) = \pm 1\}$ and left/right edges along the lines $\{\text{Re}(z) = \pm R\}$. Let L_R and R_R denote the left/right hand sides of this rectangle; then

$$\left| \int_{L_R} \frac{f(z)}{z} dz \right| \leq \frac{2}{R} \overset{R\to\infty}{\longrightarrow} 0$$

and similarly for R_R. Thus

$$f(0) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{f(t+i)}{t+i} + \frac{f(-t-i)}{-t-i} dt$$

By Hölder's inequality,

$$\begin{aligned} |f(0)| &\leq \frac{1}{2\pi} \lim_{R\to\infty} \left(\int_{-R}^{R} |f(t+i)|^2 + |f(-t-i)|^2 dt \right)^{1/2} \left(\int_{-R}^{R} \frac{2}{|t+i|^2} dt \right)^{1/2} \\ &= C \left(\int_{-R}^{R} |f(t+i)|^2 + |f(-t-i)|^2 dt \right)^{1/2} \end{aligned}$$

where

$$C = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} \frac{1}{1+x^2} dx \right)^{1/2} = \frac{1}{\sqrt{2\pi}}$$

Thus

$$|f(0)|^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} [|f(x+i)|^2 + |f(x-i)|^2] dx$$

as desired. □

5 Spring 2021

Spring 2021 Problem 1. Let μ be a positive Borel probability measure on $[0,1]$ and let

$$C = \sup\left\{\mu(E) : E \subseteq [0,1] \text{ with } |E| = \frac{1}{2}\right\}$$

where $|E|$ denotes the Lebesgue measure of E. Show that there exists a Borel set $F \subseteq [0,1]$ such that

$$|F| = \frac{1}{2} \quad \text{and} \quad \mu(F) = C$$

Hint. When $d\mu = f dx$, one can sometimes take $F = \{x \in [0,1] : f(x) > \lambda\}$, for a suitable $\lambda \geq 0$.

Proof. Throughout, we make implicit use of the well-known fact that, if $0 < c < |A|$ with A Borel, then there is a Borel subset $B \subseteq A$ with $|B| = c$. By the Lebesgue decomposition theorem, we may write $d\mu = f dx + \mu_1$, where $f \geq 0$ is Borel measurable and $\mu_1 \perp \mu$ is a positive Borel measure. Denote by X some Borel set satisfying

$$\mu_1(X) = 0, \qquad \int_{[0,1]\setminus X} f(x)dx = 0$$

For each $c \geq 0$ we denote

$$E_c := \{f(x) > c\}$$

Set

$$\lambda := \inf\{c > 0 : |E_c| \leq \frac{1}{2}\}$$

By Markov, since $\int |f|dx = \int f dx \leq \int \mu = 1$, we have that $\lambda \in [0,2]$. Note that

$$E_\lambda = \bigcup_{n=1}^{\infty} E_{\lambda + \frac{1}{n}}, \qquad E_\lambda \cup \{f(x) = \lambda\} = \bigcap_{n=1}^{\infty} E_{\lambda - \frac{1}{n}}$$

and so

$$|E_\lambda| = \lim_{n\to\infty} |E_{\lambda+\frac{1}{n}}| \leq \frac{1}{2}$$

and

$$|E_\lambda| + |\{f(x) = \lambda\}| = \lim_{n\to\infty} |E_{\lambda-\frac{1}{n}}| \geq \frac{1}{2}$$

Thus there is some Borel set $A \subseteq \{f(x) = \lambda\}$ such that $|E_\lambda \cup A| = \frac{1}{2}$

Finally, we set

$$F = E_\lambda \cup A \cup X$$

Note that $|F| = \frac{1}{2}$, since $X \setminus [E_\lambda \cup A]$ is contained in a set of Lebesgue measure 0. We claim that any other $E \subseteq [0,1]$ with $|E| = \frac{1}{2}$ satisfies $\mu(E) \leq \mu(F)$.

First, $|E| = |E \cup X|$ and $\mu(E) \leq \mu(E \cup X)$, so we may assume that $E \supseteq X$. If $E_\lambda \not\subseteq E$, then either $|E_\lambda \setminus E| = 0$ or $|E_\lambda \setminus E| > 0$. In the former case, E may be replaced by $E \cup E_\lambda$; in the latter,

$$\frac{1}{2} = |E| = |E \setminus E_\lambda| + |E \cap E_\lambda| = |E \setminus E_\lambda| + |E_\lambda| - |E_\lambda \setminus E| \leq |E \setminus E_\lambda| + \frac{1}{2} - |E_\lambda \setminus E|$$

(where we have used $|E_\lambda| \le \frac{1}{2}$) and hence

$$|E \setminus E_\lambda| \ge |E_\lambda \setminus E|$$

If B denotes a Borel subset of $E \setminus E_\lambda$ for which $|B| = |E_\lambda \setminus E|$, then

$$\int_B f(x)dx \le \int_{E_\lambda \setminus E} f(x)dx$$

and it follows that

$$\mu([E \setminus B] \cup E_\lambda \cup X) = \mu_1(X) + \int_{[E\setminus B]\cup E_\lambda} f(x)dx \ge \mu_1(X) + \int_E f(x)dx \ge \mu(E)$$

so we may assume that $E_\lambda \subseteq E$.

We have reduced to the setting $E \supseteq X \cup E_\lambda$. Then $|E \setminus E_\lambda| = |E| - |E_\lambda|$ and

$$\mu([E \setminus E_\lambda] \cup X) = \mu_1(X) + \int_{E\setminus E_\lambda} f(x)dx \le \mu_1(X) + \int_A f(x)dx = \mu(F)$$

where $A \subseteq \{f = \lambda\}$ is Borel such that $|A| = |E| - |E_\lambda|$. Thus in every case $\mu(E) \le \mu(F)$, as was to be shown. $\square$

Spring 2021 Problem 2. Let μ and ν be two finite positive Borel measures on $\mathbb{R}^n$.

(a): Suppose that there exist Borel sets $A_n \subseteq X$ so that

$$\lim_{n\to\infty} \mu(A_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \nu(X \setminus A_n) = 0$$

Show that μ and ν are mutually singular.

(b): Suppose there are non-negative Borel functions $\{f_n\}_{n\ge1}$ so that $f_n(x) > 0$ for ν-a.e. x and

$$\lim_{n\to\infty} \int f_n(x)d\mu(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} \int \frac{1}{f_n(x)} d\nu(x) = 0$$

Proof. (a): Note that the conclusion is immediate if one of μ, ν are the zero measure; hence after rescaling we may as well assume that μ, ν are probability measures. Refining the sequence $\{A_n\}_n$, we may also assume

$$\mu(A_n) \le 2^{-n}, \quad \nu(X \setminus A_n) \le 2^{-n}$$

Now set

$$A = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty A_n$$

Then

$$\mu(A) = \lim_{n\to\infty} \mu\left(\bigcap_{k=n}^\infty A_n\right) \le \lim_{n\to\infty} \mu(A_n) = 0$$

and

$$\nu(X \setminus A) = \nu\left(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty [X \setminus A_n]\right) = \lim_{n\to\infty} \nu\left(\bigcup_{k=n}^\infty [X \setminus A_n]\right)$$

$$\le \lim_{n\to\infty} \sum_{k=n}^\infty \nu(X \setminus A_n) = \lim_{n\to\infty} 2^{-n+1} = 0$$

so μ, ν are mutually singular.

(b): Set

$$A_n := \{f_n \geq 1\} \quad \text{Borel}$$

Then

$$\int f_n(x) d\mu(x) \geq \int_{A_n} 1 d\mu(x) = \mu(A_n) \geq 0$$

so $\mu(A_n) \to 0$. Similarly,

$$\int \frac{1}{f_n(x)} d\nu(x) \geq \int_{X \setminus A_n} 1 d\nu(x) = \nu(X \setminus A_n) \geq 0$$

so $\nu(X \setminus A_n) \to 0$. By (a), μ, ν are mutually singular. $\square$

Spring 2021 Problem 3. Let $f \in L^2(\mathbb{R})$. For $n \geq 1$ we define

$$f_n(x) = \int_0^{2\pi} f(x+t) \cos(nt) dt$$

Prove that f_n converges to zero *both* almost everywhere in $\mathbb{R}$ *and* in the $L^2(\mathbb{R})$ topology, as $n \to \infty$.

Proof. We first note the estimate

$$\begin{aligned}
\|f_n\|_{L^2(\mathbb{R})} &= \left(\int_{\mathbb{R}} |f_n(x)|^2 dx \right)^{1/2} \\
&= \left(\int_{\mathbb{R}} \left| \int_0^{2\pi} \cos(nt) f(x+t) dt \right|^2 dx \right)^{1/2} \\
&\leq \int_0^{2\pi} |\cos(nt)| \left(\int_{\mathbb{R}} |f(x+t)|^2 dx \right)^{1/2} dt \quad \text{by Minkowski} \\
&= \|f\|_{L^2(\mathbb{R})} \int_0^{2\pi} |\cos(nt)| dt \\
&\leq 2\pi \|f\|_{L^2(\mathbb{R})}
\end{aligned}$$

Let $\varepsilon > 0$, and fix $g \in C_c^\infty(\mathbb{R})$ such that

$$\|g - f\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{4\pi}$$

We estimate

$$\begin{aligned}
\|g_n\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} g_n(x)\bar{g}_n(x)dx \\
&= \int_{\mathbb{R}} \bar{g}_n(x) \int_0^{2\pi} \cos(nt)g(x+t)dtdx \\
&= \int_0^{2\pi} \cos(nt) \int_{\mathbb{R}} g(x+t)\bar{g}_n(x)dxdt \\
&= \int_0^{2\pi} \frac{1}{n}\frac{d}{dt}[\sin(nt)] \int_{\mathbb{R}} g(x+t)\bar{g}_n(x)dxdt \\
&= -\frac{1}{n}\int_0^{2\pi} \sin(nt)\frac{d}{dt}\left[\int_{\mathbb{R}} g(x+t)\bar{g}_n(x)dx\right]dt \\
&= -\frac{1}{n}\int_0^{2\pi} \sin(nt) \int_{\mathbb{R}} g'(x+t)\bar{g}_n(x)dxdt \\
&\leq \frac{1}{n}2\pi\|g'\|_{L^2(\mathbb{R})}\|g_n\|_{L^2(\mathbb{R})} \\
&\leq \frac{1}{n}4\pi^2\|g'\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} \to 0
\end{aligned}$$

and so we may take some $N > 0$ such that for all $n \geq N$ we have

$$\|g_n\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{2}$$

Thus together we have

$$\|f_n\|_{L^2(\mathbb{R})} \leq \|g_n\|_{L^2(\mathbb{R})} + \|(g-f)_n\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq N$; we conclude that $f_n \to 0$ in $L^2(\mathbb{R})$.

We now show that the mapping $f \mapsto f_n$ is bounded from $L^2(\mathbb{R})$ to $L^\infty(\mathbb{R})$: this follows immediately from the estimate

$$|f_n(x)| \leq \int_0^{2\pi} |f(x+t)\cos(nt)|dt \leq \|f\|_{L^2(\mathbb{R})}\|\cos(n\cdot)\|_{L^2([0,2\pi])} \lesssim \|f\|_{L^2(\mathbb{R})}$$

For any $g \in C_c^\infty(\mathbb{R})$ and any $x \in \mathbb{R}$,

$$\begin{aligned}
g_n(x) &= \int_0^{2\pi} g(x+t)\cos(nt)dt \\
&= -\frac{1}{n}\int_0^{2\pi} \sin(nt)g'(x+t)dt \\
&= O\left(\frac{1}{n}\|g'\|_{L^2(\mathbb{R})}\right) \to 0
\end{aligned}$$

so $g_n(x) \to 0$ for every $x \in \mathbb{R}$, as $n \to \infty$. The final conclusion follows from the estimate

$$|f_n(x)| \leq |g_n(x)| + |(f-g)_n(x)| \lesssim |g_n(x)| + \|f-g\|_{L^2(\mathbb{R})}$$

and hence $f_n(x) \to 0$ as well.

□

Spring 2021 Problem 4. Define

$$I(f) := \int_0^1 \left(\frac{1}{2}(f'(x))^2 + \sin(f(x)) + f^4(x) \right) dx$$

for any $f \in C^1([0,1];\mathbb{R})$. Let $f_n \in C^1([0,1];\mathbb{R})$ be such that

$$I(f_n) \to \inf_{f \in C^1([0,1];\mathbb{R})} I(f)$$

Show that the sequence $\{f_n\}$ has a limit point in the space $C([0,1];\mathbb{R})$.

Proof. We first show that $\{f_n\}$ is equicontinuous. For any $x, y \in [0,1]$, and any n,

$$\begin{aligned} |f_n(y) - f_n(x)| &= \left| (y-x) \int_0^1 f_n'(x + t(y-x)) dt \right| \\ &\leq |x-y| \times \|f_n'\|_{L^1([0,1])} \\ &\leq |x-y| \times \|f_n'\|_{L^2([0,1])} \quad \text{by Hölder} \end{aligned}$$

Since $I(f_n)$ is bounded, we see that

$$\frac{1}{2}\|f_n'\|^2_{L^2([0,1])} \leq \int_0^1 \left(\frac{1}{2}(f_n'(x))^2 + [1 + \sin(f_n(x))] + f_n^4(x) \right) dx = 1 + I(f_n)$$

is uniformly bounded; hence there is a constant C independent of n, x, y such that

$$|f_n(y) - f_n(x)| \leq C|y-x|$$

from which we conclude that the family $\{f_n\}_n$ is equicontinuous. From this we can also conclude uniform boundedness: for each n and each $y \in [0,1]$, we have the upper bound

$$\begin{aligned} \|f_n\|^4_{L^4([0,1])} &= \int_0^1 |f_n(x)|^4 dx \\ &= \int_0^1 |f_n(y) + f_n(x) - f_n(y)|^4 dx \\ &\geq \int_0^1 \max(|f_n(y)| - C|x-y|, 0)^4 dx \end{aligned}$$

In particular,

$$\|f_n\|^4_{L^4([0,1])} \geq \int_0^1 \max(\|f_n\|_{L^\infty([0,1])} - C, 0)^4 dx$$

As before, we have the bound

$$\|f_n\|^4_{L^4([0,1])} \leq 1 + I(f_n)$$

uniform in n, hence $\|f_n\|_{L^\infty([0,1])}$ is uniformly bounded. Thus the family $\{f_n\}_n$ is a uniformly bounded and equicontinuous family of continuous functions on a compact domain, hence (by Arzelà-Ascoli) possess a limit point. $\square$

Spring 2021 Problem 5. Let $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ be such that the series

$$\sum_{i=1}^{\infty} x_i y_i$$

converges for all $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n y_n = 0$. Show that the series $\sum_{n=1}^{\infty} |x_n|$ converges.

Proof. We show the contrapositive: supposing $\sum_{n=1}^{\infty} |x_n|$ diverges, we wish to construct some $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n y_n = 0$ and $\sum x_i y_i$ diverges. By the divergence of $\sum |x_i|$, we may iteratively construct a sequence of natural numbers $n_1 < n_2 < n_3 < \dots$ satisfying

$$\sum_{n_j \le i < n_{j+1}} |x_i| \ge 1$$

Define

$$y_i = \frac{\operatorname{sgn}(x_i)}{j} \quad \text{where } i \text{ is such that } n_j \le i < n_{j+1}$$

Then clearly $y_i \to 0$, and

$$\sum_{i=1}^{\infty} x_i y_i = \sum_{j=1}^{\infty} \sum_{n_j \le i < n_{j+1}} \frac{|x_i|}{j} \ge \sum_{j=1}^{\infty} \frac{1}{j} = +\infty$$

as desired. □

Spring 2021 Problem 6. We say that the linear operator $T : C([0,1]) \to C([0,1])$ is *positive* if $T(f)(x) \ge 0$ for all $x \in [0,1]$, whenever $f \in C([0,1])$ satisfies $f(x) \ge 0$ for all $x \in [0,1]$. Let

$$T_n : C([0,1]) \to C([0,1])$$

be a sequence of positive linear operators such that $T_n(f) \to f$ uniformly on $[0,1]$ if f is a polynomial of degree less than or equal to 2. Show that

$$T_n(f) \to f \quad \text{uniformly on } [0,1]$$

for every $f \in C([0,1]))$.

Hint. Let $f \in C([0,1])$. Show first that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x) - f(y)| \le \varepsilon + C_\varepsilon |x-y|^2 \quad \text{for all } x, y \in [0,1]$$

Proof. Assume the estimate given in the hint. Fix $\varepsilon > 0$. Let 1 denote the constant-1 function on $[0,1]$. Since f is continuous on $[0,1]$, it is uniformly continuous, and so there is a partition $0 = x_1 < \dots < x_n = 1$ such that, whenever $x_j \le x \le x_{j+1}$,

$$|f(x_j) - f(x)| < \frac{\varepsilon}{8}$$

We compute

$$\begin{aligned}|T_n[f](x) - f(x)| &\leq |f(x) - T_n[f(x)](x)| + T_n[f](x) - T_n[f(x)](x)| \\ &\leq |f(x)||1 - T_n[1](x)| \\ &+ |T_n[f - f(x_j)](x_j)| \\ &+ |f(x_j) - f(x)||T_n[1](x_j)| \\ &+ |f(x)||T_n[1](x_j) - T_n[1](x)| \\ &= I + II + III + IV\end{aligned}$$

Since $T_n[1] \to 1$ uniformly, and $|f|$ is uniformly bounded, there is some $N_1 > 0$ such that whenever $n \geq N_1$ we have

$$I < \frac{\varepsilon}{4}$$

Similarly, there is some $N_3 > 0$ such that

$$|T_n[1](x_j)| \leq 2$$

for all $n \geq N_3$; together with the uniform continuity assumption above,

$$III < \frac{\varepsilon}{4}$$

To handle IV, note that since $T_n[1] \to 1$ uniformly we must have that for $n \gg 0$ we must have $T_n[1]$ within $\frac{\varepsilon}{8\|f\|_\infty}$ of 1; it follows that there is $N_4 > 0$ for which

$$IV \leq \|f\|_\infty(|T_n[1](x_j) - 1| + |1 - T_n[1](x)|) < \frac{\varepsilon}{4}$$

for all $n \geq N_4$.

We turn to II. Since each T_n is positive, it is order-preserving; from the inequalities

$$-\frac{\varepsilon}{8} - C_{\varepsilon/8}(x_j - y)^2 \leq f(y) - f(x_j) \leq \frac{\varepsilon}{8} + C_{\varepsilon/8}(x_j - y)^2$$

we see that

$$T_n[-\varepsilon - C_\varepsilon(x_j - \cdot)^2](x_j) \leq T_n[f - f(x_j)](x_j) \leq T_n[\varepsilon + C_\varepsilon(x_j - \cdot)^2](x_j)$$

and so, for n sufficiently large,

$$-\frac{\varepsilon}{4} \leq T_n[f - f(x_j)](x_j) \leq \frac{\varepsilon}{4}$$

Thus there is some $N_2 > 0$ such that, for all $n \geq N_2$ and $j = 1, \ldots, n$,

$$-\frac{\varepsilon}{4} \leq T_n[f - f(x_j)](x_j) \leq \frac{\varepsilon}{4}$$

from which we conclude

$$II \leq \frac{\varepsilon}{4}$$

Thus we have in total

$$|T_n[f](x) - f(x)| < \varepsilon$$

when $n \geq \max(N_1, N_2, N_3, N_4)$, if we assume the hinted estimate.

We now prove the estimate in question. Fix $\varepsilon > 0$. Since f is continuous on $[0,1]$, it is uniformly continuous, there is some $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Let $C_\varepsilon > 0$ be sufficiently large so that $C_\varepsilon \delta^2 > 2\|f\|_\infty$. Let $x, y \in [0,1]$. If $|x-y| < \delta$, then the desired estimate follows from uniformity. Otherwise,

$$\varepsilon + C_\varepsilon (x-y)^2 \geq \varepsilon + C_\varepsilon \delta^2 > \varepsilon + 2\|f\|_\infty > |f(x) - f(y)|$$

as desired; the argument is now finished.

□

Spring 2021 Problem 7. Let $\Omega = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0 \text{ and } \mathrm{Im}\, z > 0\}$. Show that there exists a *unique* bounded harmonic function $u : \Omega \to \mathbb{R}$ such that for all $x > 0$ and $y > 0$,

$$\lim_{t \to 0} u(x + it) = 0 \quad \text{and} \quad \lim_{t \to 0} u(t + iy) = 1$$

Proof. Note that $u(z) = \frac{2}{\pi}\mathrm{Arg}(z)$ is one solution, where Arg denotes the principal argument taking values in $(-\pi, \pi]$.

It remains to handle uniqueness. For u, v solutions to the given problem, we see that $w := u - v$ is a bounded harmonic real-valued function on Ω satisfying

$$\lim_{t \to 0} w(x + it) = 0 \quad \text{and} \quad \lim_{t \to 0} w(t + iy) = 0$$

Set $\eta(z) := w(\phi(z))$, for $\phi^{-1}(z) = \frac{z^2 - i}{z^2 + i}$ the conformal map $\Omega \to \mathbb{D}$ sending 0 to -1 and ∞ to 1. Then η is a bounded real-valued harmonic function on $\mathbb{D}$ such that

$$\lim_{\gamma_\theta \ni z \to e^{i\theta}} \eta(z) = 0$$

where γ_θ is the analytic arc $\phi^{-1}(s + i\mathbb{R})$ or $\phi^{-1}(\mathbb{R} + is)$ for suitable fixed s, depending on if $\theta \in (0, \pi)$ or $\in (\pi, 2\pi)$.

Let $\mu := \phi_* \lambda$ be the pushforward of the Lebesgue measure on $\partial\mathbb{D}$ onto $\partial\Omega$. For each $b \in \partial\Omega \cap i\mathbb{R}_{>0}$, define

$$\sigma(b, \varepsilon) := \sup\{r > 0 : |w(b + s)| < \varepsilon/2 \, \forall 0 < s \leq r\}$$

and similarly define $\sigma(b, \varepsilon)$ for $b \in \Omega \cap \mathbb{R}_{>0}$. If we choose $\varepsilon > 0$ and $0 < \delta < \|w\|_\infty \varepsilon$, then let ε' be sufficiently small so that

$$\mu(\{b : \sigma(b, \varepsilon') < \varepsilon\}) < \delta$$

and let $C = \{z : |z| = r\}$ for some $0 < r < 1$ sufficiently close to 1 so that

$$\lambda(\{\theta \in (0, \pi) : \mathrm{Re}\phi(re^{i\theta}) \geq \sigma(\phi(e^{i\theta}), \varepsilon')\} \cup \{\theta \in (\pi, 2\pi) : \mathrm{Im}\phi(re^{i\theta}) \geq \sigma(\phi(e^{i\theta}), \varepsilon')\}) < \delta$$

Together this implies that $\eta|_{r\partial\mathbb{D}}$ is L^1-close to 0, for all $r < 1$ large depending on $\varepsilon > 0$. We conclude by the reproducing formula for harmonic functions that η is in fact 0, so $u = v$ and we have uniqueness. □

Spring 2021 Problem 8. Show that there exists a non-zero entire function $f : \mathbb{C} \to \mathbb{C}$ and constants $b, c \in \mathbb{C}$ satisfying

$$f(0) = 0, \quad f(z+1) = e^{bz} f(z), \quad f(z+i) = e^{cz} f(z)$$

Proof. Define

$$g(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2 - \pi n + 2\pi i n z}$$

It is clear that the series converges locally uniformly in z, so defines an entire function. Clearly $g(z+1) = g(z)$ for all z. Additionally,

$$\begin{aligned} g(z+i) &= \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2 - \pi n + 2\pi i n (z+i)} \\ &= \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2 - 3\pi n + 2\pi i n z} \\ &= -e^{2\pi} e^{-2\pi i z} \sum_{n\in\mathbb{Z}} (-1)^{n+1} e^{-\pi(n+1)^2 - \pi(n+1) + 2\pi i (n+1) z} \\ &= -e^{\pi} e^{-2\pi i z} g(z) \end{aligned}$$

and

$$\begin{aligned} g(0) &= \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2 - \pi n} \\ &= \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n(n+1)} \\ &= \sum_{n\in\mathbb{Z}} (-1)^{1-n} e^{-\pi(1-n)(-n)} \\ &= -\sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n(n+1)} \\ &= -g(0) \end{aligned}$$

so that $g(0) = 0$. Using this, define

$$f(z) = e^{\pi z^2 - \pi z} g(z)$$

Then certainly $f(0) = 0$ and f is entire. We compute

$$f(z+1) = e^{\pi z^2 - \pi z} e^{2\pi z} g(z+1) = e^{2\pi z} f(z)$$

and

$$f(z+i) = e^{\pi z^2 - \pi z} e^{2\pi i z} e^{-\pi - \pi i} g(z+i) = e^{-2\pi i z} f(z)$$

so this f has the appropriate properties with $b = 2\pi, c = -2\pi i$.

□

Spring 2021 Problem 9. Let $\Omega_1 \subseteq \Omega_2$ be bounded Jordan domains in $\mathbb{C}$. We also assume that $0 \in \Omega_1$. Now suppose $f_1 : \mathbb{D} \to \Omega_1$ and $f_2 : \mathbb{D} \to \Omega_2$ are Riemann mappings, satisfying $f_1(0) = f_2(0) = 0$. Show that

$$|f_1'(0)| \le |f_2'(0)|$$

Proof. $f_2^{-1} \circ f_1$ is a holomorphic map $\mathbb{D} \to \mathbb{D}$ taking 0 to 0, hence

$$|f_2'(0)|^{-1}|f_1'(0)| = |(f_2^{-1})'(f_1(0))f_1'(0)| = |(f_2^{-1} \circ f_1)'|(0) \le 1$$

by the Schwartz lemma, which implies the desired

$$|f_1'(0)| \le |f_2'(0)|$$

$\square$

Spring 2021 Problem 10. Define

$$f(z) = \int_0^1 \frac{t^z}{e^t - 1} dt, \quad z \in \mathbb{C}, \operatorname{Re} z > 0$$

Show that f is an analytic function in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and that it admits a meromorphic continuation $\hat{f}$ to the region $\{z \in \mathbb{C} : \operatorname{Re} z > -1\}$. Compute the residue of $\hat{f}$ at $z = 0$.

Proof. Note that the integrand, as a function of t, is continuous on $(0, 1]$ and is $O(t^{\operatorname{Re}(z)-1})$, hence is absolutely integrable for all $\operatorname{Re} z > 0$. Thus $f(z)$ is well-defined in the right half-plane.

Let Δ be an arbitrary triangle in the right half-plane. Then

$$\begin{aligned}
\int_\Delta f(z)dz &= \int_\Delta \int_0^1 \frac{t^z}{e^t - 1} dtdz \\
&\overset{*}{=} \int_0^1 \int_\Delta \frac{t^z}{e^t - 1} dzdt \quad \text{by Fubini} \\
&= \int_0^1 0\, dt \quad \text{since } z \mapsto t^z \text{ is analytic for each } t > 0 \\
&= 0
\end{aligned}$$

which by Morera's implies that f is analytic in the right half-plane. To justify the use of Fubini in $(*)$, note that

$$\begin{aligned}
\int_\Delta \int_0^1 \left| \frac{t^z}{e^t - 1} \right| dt|dz| = \int_\Delta \int_0^1 O(t^{\operatorname{Re}(z)-1})dt|dz| &= \int_\Delta O\left(\frac{1}{\operatorname{Re} z}\right)|dz| \\
&= O(\text{length}(\Delta) \times \text{dist}(\Delta, i\mathbb{R})^{-1}) < \infty
\end{aligned}$$

so Fubini's applies.

Now, fix $0 < a < b \le 1$. We have the computation

$$\begin{aligned}
\int_a^b \frac{t^z}{e^t - 1} dt &= \int_a^b t^{z-1} \frac{t}{e^t - 1} dt \\
&= \frac{t^z}{z} \frac{t}{e^t - 1} \Big|_a^b - \int_a^b \frac{t^z}{z} \frac{d}{dt} \left[\frac{t}{e^t - 1} \right] dt \\
&= \frac{b^z}{z} \frac{b}{e^b - 1} - \frac{a^z}{z} \frac{a}{e^a - 1} + O\left(|z|^{-1} b^{\operatorname{Re} z + 1}\right)
\end{aligned}$$

valid for Re $z > -1, z \neq 0$. Define

$$g(a, z) = \int_a^1 \frac{t^z}{e^t - 1} dt + \frac{a^z}{z} \frac{a}{e^a - 1}$$

for $0 < a \leq 1$ and Re $z > -1, z \neq 0$. The prior estimate shows that for each fixed such z,

$$a \mapsto g(a, z)$$

is (locally uniformly in z) Cauchy as $a \to 0^+$, and hence there is some $\zeta(z) \in \mathbb{C}$ such that

$$\lim_{a \to 0^+} g(a, z) = \zeta(z)$$

Since each $g(a, z)$ is analytic in z, so is the mapping $z \mapsto \zeta(z)$. For Re$(z) > 0$, we have shown that

$$\lim_{a \to 0^+} \left[g(a, z) - \frac{a^z}{z} \frac{a}{e^a - 1} \right] = f(z)$$

from which we conclude that

$$f(z) - \zeta(z) = -\lim_{a \to 0^+} \frac{a^z}{z} \frac{a}{e^a - 1} = 0$$

Thus $f(z) = \zeta(z)$ on Re$(z) > 0$, hence f extends analytically to Re$(z) > -1, z \neq 0$ by ζ.

It remains to examine the isolated singularity at $z = 0$.

For each $a > 0$, the integral

$$\int_a^1 \frac{t^z}{e^t - 1} dt$$

is entire in z, and the function

$$\frac{a^z}{z} \frac{a}{e^a - 1}$$

is meromorphic in z with a single simple pole at $z = 0$. Thus $g(a, z)$ has a pole at $z = 0$ of residue $\frac{a}{e^a - 1}$. That is,

$$\frac{1}{2\pi i} \int_{|z| = \frac{1}{2}} g(a, z) dz = \frac{a}{e^a - 1}$$

and, taking a limit as $a \to 0^+$,

$$\frac{1}{2\pi i} \int_{|z| = \frac{1}{2}} \zeta(z) dz = 1$$

which is the residue of ζ at $z = 0$.

Lastly, we demonstrate that $z = 0$ is actually a pole of ζ: applying our initial estimate,

$$\begin{aligned} g(2^{-n}, z) - g(1, z) &= \sum_{j=0}^{n-1} g(2^{-j-1}, z) - g(2^{-j}, z) \\ &= \sum_{j=0}^{n-1} O(|z|^{-1} 2^{-j(\mathrm{Re}(z)+1)}) \\ &= O\left(|z|^{-1} \frac{1}{1 - 2^{-(\mathrm{Re}(z)+1)}} \right) \end{aligned}$$

and hence

$$\zeta(z) = g(1,z) + O\left(|z|^{-1}\frac{1}{1-2^{-(\mathrm{Re}(z)+1)}}\right) = \frac{1}{z(e-1)} + O\left(|z|^{-1}\frac{1}{1-2^{-(\mathrm{Re}(z)+1)}}\right)$$

In particular, $z = 0$ is a simple pole, so f does indeed extend meromorphically to $\mathrm{Re}(z) > -1$. $\square$

Spring 2021 Problem 11. For an entire function $f(z) = f^{(0)}(z)$, we define

$$f^{(n)}(z) = f(f^{(n-1)}(z)) \quad \text{for all } n \geq 1$$

(a): Show that if there exists an $n \geq 1$, such that $f^{(n)}$ is a polynomial, then f is a polynomial.
(b): Prove that for any $n \geq 1$ we have $f^{(n)}(z) \neq e^z$.

Proof. (a): Slightly informal. Suppose f is not a polynomial. Then ∞ is an essential singularity, so for every $R > 0$ we have that $f(\{z : |z| > R\})$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{p\}$ for some $p \in \mathbb{C}$. In either case, $f(\{z : |z| > R\})$ contains such a neighborhood of ∞, so inductively $f^{(n)}(\{z : |z| > R\})$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{p\}$ for every choice of $R > 0$.

But for any (nonconstant) polynomial P, P extends to a continuous self-map of the Riemann sphere, and so P maps a small neighborhood of ∞ to a small neighborhood of ∞; in particular, a sufficiently small neighborhood of infinity will be mapped to a non-dense subset of the sphere. We conclude that $f^{(n)}$ is not a polynomial for any $n \geq 1$, as long as f is not a polynomial.

(b): Suppose $f, n \geq 1$ are such that $f^{(n)}(z) = e^z$. Clearly f is not a polynomial. Note also that f is not surjective as a map $\mathbb{C} \to \mathbb{C}$, since otherwise $f^{(n)}(z)$ would be surjective, whereas e^z is not. Thus there is exactly one $p \in \mathbb{C}$ such that $f(\mathbb{C}) = \mathbb{C} \setminus \{p\}$. Since $f^{(n)}$ is entire (and nonconstant), $f^{(n)}(\mathbb{C})$ omits at most one (finite) value, and also omits p. Since $f^{(n)}(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, we conclude that

$$f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$$

Thus $f(z) = e^{g(z)}$ for some nonconstant entire function g. Then

$$e^z = e^{g(f^{(n-1)}(z))}$$

for all z, so there is some $k \in \mathbb{Z}$ such that

$$z + 2\pi i k \equiv g(f^{(n-1)}(z))$$

Now, the left-hand side will map a small neighborhood of ∞ to a small neighborhood of ∞. The right-hand side is of the form $g(\exp(h(z)))$, where g and $h(z)$ are nonconstant entire; so h will send a neighborhood of ∞ to a neighborhood of ∞, which under $\exp$ gets mapped to an open dense subset of $\mathbb{C}$, which under g gets mapped to an open dense subset of $\mathbb{C}$. This is a contradiction. $\square$

Spring 2021 Problem 12. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ that satisfy the following two properties:

1. $|f(z)| \leq e^{|z|^2}$ for all $z \in \mathbb{C}$,
2. $f(n^{1/3}) = n$ for all $n \in \mathbb{N}$.

Hint: $f(z) = z^3$ is one of them.

Proof. We claim that $f(z) = z^3$ is the only solution. If f is any solution, and $g(z) = f(z) - z^3$, then g is entire of order $\rho \leq 2$ by the estimate (1). If $g \neq 0$, then the zeros a_n are isolated and are $\lfloor \rho \rfloor + 1$-summable, i.e.

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lfloor \rho \rfloor + 1}} < +\infty$$

But the a_n include the numbers $n^{1/3}$, and if we take $p = \lfloor \rho \rfloor \leq 2$ then

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} \geq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p+1}{3}}} \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

a contradiction. Thus $g \equiv 0$, so $f(z) = z^3$ is the only solution. $\square$

6 Fall 2021

Fall 2021 Problem 1. Let $f : [0, 2\pi] \to \mathbb{C}$ belong to L^1 and assume that

$$\int_0^{2\pi} f(x) \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4} \right) dx = 0$$

whenever $\varphi : \mathbb{R} \to \mathbb{C}$ is smooth and (2π)-periodic. Prove that

$$f(x) = a + be^{ix} + ce^{-ix} \quad \text{a.e.}$$

for some complex scalars a, b, c.

Proof. Define the scalars a, b, c by

$$a := f(0)$$

$$b := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ix} dx$$

$$c := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{ix} dx$$

Set $g(x) = f(x) - a - be^{ix} - ce^{-ix}$. Note that, if φ is smooth and (2π)-periodic, then it possesses an L^2-convergent Fourier expansion

$$\varphi(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$$

and by Plancherel

$$\begin{aligned}
\int_0^{2\pi} (a + be^{ix} + ce^{-ix}) \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4} \right) dx &= \int_0^{2\pi} (a + be^{ix} + ce^{-ix}) \sum_{n \in \mathbb{Z}} b_n \{-n^2 + n^4\} e^{inx} dx \\
&= 2\pi (ab_0\{-0^2 + 0^4\} + bb_{-1}\{-(-1)^2 + (-1)^4\} + cb_1\{-1^2 + 1^4\}) \\
&= 0
\end{aligned}$$

Consequently, g satisfies the integral condition

$$\int_0^{2\pi} f(x) \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4} \right) dx = 0$$

for each φ as above.

Now, fix $n \notin \{-1, 0, 1\}$. Then

$$\begin{aligned}
\int_0^{2\pi} g(x) e^{inx} dx &= \frac{1}{-n^2 + n^4} \int_0^{2\pi} g(x)(-n^2 + n^4) e^{inx} dx \\
&= \frac{1}{-n^2 + n^4} \int_0^{2\pi} g(x) \left(\frac{\partial^2 [e^{inx}]}{\partial x^2} + \frac{\partial^4 [e^{inx}]}{\partial x^4} \right) dx = 0
\end{aligned}$$

while, for $k \in \{-1, 0, 1\}$,

$$\int_0^{2\pi} g(x) e^{ikx} dx = 0$$

by construction. Thus g is orthogonal to every polynomial in e^{ix}. However, by Stone-Weierstrass, polynomials in e^{ix} are L^∞-dense in $C[0,2\pi]$, so we conclude that

$$\int_0^{2\pi} g(x)h(x)dx = 0 \quad \forall h \in C[0,2\pi]$$

using $g \in L^1$. Thus $g = 0$ a.e., and hence

$$f(x) = a + be^{ix} + ce^{-ix} \quad \text{a.e.}$$

as claimed.

□

Fall 2021 Problem 2. Let $f_1, f_2, \ldots \in L^1([0,1])$ satisfy

$$\int_0^1 |f_i|^2 dx = \infty \quad \text{for every } i$$

- (a): Prove that the set

$$A_{i,M} := \left\{ g \in L^1([0,1]) : M < \int_0^1 |f_i g| dx \leq \infty \right\}$$

  is open in the norm topology of L^1 for every integer i and every $M > 0$.

- (b): Prove that some $g \in L^1$ satisfies

$$\int_0^1 |f_i g| dx = \infty \quad \text{for every } i$$

Proof. (a): Fix arbitrary i and $M > 0$. Let $g \in A_{i,M}$. It suffices to demonstrate that there is some $\varepsilon > 0$ such that, for all $\|h\|_1 < \varepsilon$, $g + h \in A_{i,M}$.

Suppose for the sake of contradiction that there is a sequence $h_j \to 0$ in L^1 for which

$$\int_0^1 |f_i(g+h_j)| dx \leq M \quad \text{for all } j$$

Then, by Fatou,

$$\int_0^1 \liminf_{j\to\infty} |f_i(g+h_j)| dx \leq \liminf_{j\to\infty} \int_0^1 |f_i(g+h_j)| dx \leq M$$

Since $h_j \to 0$ in L^1, $h_j \to 0$ in measure; in particular,

$$\liminf_{j\to\infty} |f_i(g+h_j)| = |f_i g| \quad \text{a.e.}$$

so that

$$\int_0^1 |f_i g| dx \leq M$$

which contradicts out assumption.

(b): Define
$$g(x) = \sum_{k=1}^{\infty} \frac{|f_k|(x)}{2^k \|f_k\|_1}$$
The sum converges in L^1; furthermore, by Fubini, for each i,
$$\int_0^1 |f_i g| dx = \int_0^1 |f_i| \sum_{k=1}^{\infty} \frac{|f_k|}{2^k \|f_k\|_1} dx = \sum_{k=1}^{\infty} \frac{1}{2^k \|f_k\|_1} \int_0^1 |f_i f_k| dx = +\infty$$
since one of the summands is infinite, and the others are all nonnegative. Note that, since each $\|f_i\|_2 = +\infty$, none of these f_i are 0 in L^1, so the division is fine. $\square$

Fall 2021 Problem 3. Let $\varphi : [0,1] \to [0,1]$ be Borel measurable. Prove that there is a Borel set $B \subseteq \varphi([0,1])$ such that $m(\varphi^{-1}(B)) = 1$. Here m denotes Lebesgue measure on $[0,1]$.

Proof. By Lusin's theorem, for each $n \in \mathbb{N}$ there is a compact $K_n \subseteq [0,1]$ such that $\varphi|_{K_n}$ is continuous and $m(K_n) > 1 - \frac{1}{n}$. Define
$$B = \bigcup_{n=1}^{\infty} \varphi(K_n)$$
Note that $B \subseteq \varphi([0,1])$. Note too that each $\varphi(K_n)$ is compact, hence closed, and so B is Borel. Lastly, note that for each n
$$m(\varphi^{-1}(B)) \geq m(\varphi^{-1}(\varphi(K_n))) \geq m(K_n) > 1 - \frac{1}{n}$$
whereas $m(\varphi^{-1}(B)) \leq 1$ trivially; hence $m(\varphi^{-1}(B)) = 1$ as claimed. $\square$

Fall 2021 Problem 4. Let $r_1 > r_2 > \cdots > 0$. For each positive integer n, let $\mathcal{C}_n$ be a pairwise disjoint collection of 2^n closed disks of radius r_n in $[0,1]^2$, and assume that every member of $\mathcal{C}_n$ contains exactly two members of $\mathcal{C}_{n+1}$. Let K_n be the union $\bigcup_{D \in \mathcal{C}_n} D$, and let $K = \bigcap_{n=1}^{\infty} K_n$.

- (a): Prove that there is a Borel probability measure μ such that $\mu(K) = 1$ and $\mu(D) = 2^{-n}$ for every $D \in \mathcal{C}_n$.
- (b): Prove that K is the support of μ; that is, it is the smallest closed set whose measure equals 1.

Proof. (a): For each n, set μ_n to be the measure given by restricting Lebesgue measure to K_n and rescaling it to have $\mu_n(D) = 2^{-n}$ for each $D \in \mathcal{C}_n$. Since each μ_n is supported inside of the compact set K_1, the sequence is tight, hence is sequentially precompact; that is, there is a subsequence μ_{n_j} which converges in the weak-* topology on $C(\mathbb{R}^2)^*$ to some $\mu \in C(\mathbb{R}^2)^*$; clearly μ is a probability measure.

More generally, for $D \in \mathcal{C}_n$, let χ be a bump function which is equal to 1 on D and whose support is disjoint from all other $D' \in \mathcal{C}_n$. Then
$$\langle \mu_{n_j}, \chi \rangle = 2^{-n} \quad \text{for all } n_j \geq n$$
so in particular
$$\langle \mu, \chi \rangle = 2^{-n}$$
Since this holds for any χ as described, letting $\chi \to 1_D$ we produce $\mu(D) = 2^{-n}$. Summing, this implies $\mu(K) = 1$; we have all the desired properties.

(b): We have seen that the support is contained inside K. If $x \in K$ and $x \in U$ is open, then U contains a disk of some radius $\varepsilon > 0$ centered at x. If $r_n < \varepsilon$, then U contains some $D \in \mathcal{C}_n$, so $\mu(U) \geq 2^{-n}$; since U was a generic open neighborhood of x, we conclude that x belongs to the support of μ. Since x was a generic element of K, we conclude that $\operatorname{supp} \mu = K$. □

Fall 2021 Problem 5. Let $1 \leq p \leq \infty$ and let φ and ψ be nonzero bounded linear functionals on $L^p(\mathbb{R})$. Assume that $\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$. For precisely which values of p does this imply that φ and ψ are linearly dependent? Justify your answer.

Proof. We first demonstrate that, for $p = \infty$, the implication is false. To do this, define

$$A : L^\infty(\mathbb{R}) \to \mathbb{C}, \quad f \mapsto \int_0^1 f(x)dx$$

and

$$S : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}); \quad Sf(x) := f(x+1)$$

This A is given by pairing with the L^1 function $1_{[0,1]}$, hence is a continuous linear functional on $L^\infty(\mathbb{R})$. Note that

$$|Af| = |\int_0^1 f(x)dx| \leq \|f\|_\infty$$

so that $\|A\| \leq 1$. Conversely,

$$A1 = \int_0^1 1dx = 1 = \|1\|_\infty$$

so $\|A\| = 1$. By the same token, $AS1 = 1$ and $\|AS\| \leq \|A\|\|S\| = 1$, so $\|AS\| = 1$ as well. Lastly,

$$(A + AS)(1) = A1 + A1 = 2$$

and

$$\|A + AS\| \leq \|A\| + \|A\|\|S\| = 2$$

so the equality

$$\|A + AS\| = \|A\| + \|AS\|$$

holds. To demonstrate that A and AS are linearly independent,

$$A(1_{[0,1]}) = 1, AS(1_{[0,1]}) = 0, A(1_{[1,2]}) = 0, AS(1_{[1,2]}) = 1$$

and the claim has been demonstrated.

Now consider $p = 1$. We claim the result is also false for this exponent; consider the linear functionals formed from pairing with the L^∞ functions

$$f = 2 \cdot 1_{[0,1]} + 1_{[1,2]}, \quad g = 2 \cdot 1_{[0,1]}$$

Then f, g are obviously linearly independent, whereas

$$\|f\|_\infty = \|g\|_\infty = 2, \quad \|f + g\|_\infty = 4 = \|f\|_\infty + \|g\|_\infty$$

Since L^∞ norms agree with norms as functionals as L^1, we have the desired counterexample.

Now, if $1 < p < \infty$, then all continuous linear functionals are given by pairwing with elements of $L^{p'}(\mathbb{R})$, $\frac{1}{p} + \frac{1}{p'} = 1$, and the norm of the functional is just the $L^{p'}(\mathbb{R})$ norm of the function. Hence we are investigating equations of the form

$$\|f + g\|_{p'} = \|f\|_{p'} + \|g\|_{p'} \tag{9}$$

where $1 < p' < \infty$. Suppose temporarily that we have found such f, g, and that they are both nonnegative. Suppose $E \subseteq \mathbb{R}$ is measurable with positive measure such that

If $E \subseteq \mathbb{R}$ is measurable s.t.

$$\int_E f = \lambda \int_E g$$

for suitable $\lambda \in \mathbb{R}$, and if $F \subseteq \mathbb{R}$ is measurable and disjoint from E s.t.

$$\int_F f = \mu \int_F g$$

for suitable $\mu \in \mathbb{R}$,

$\square$

Fall 2021 Problem 6. Let K be a continuous function on $\mathbb{R}^2$ that is periodic in both coordinates:

$$K(x+1, y) = K(x, y+1) = K(x, y).$$

Given any $F \in L^1([0,1] \times [0,1])$, show that

$$\int_{[0,1]^2} K(x, y+nx)F(x,y)dm(x,y) \to \int_0^1 \left(\int_0^1 K(x,s)ds\right)\left(\int_0^1 F(x,y)dy\right)dx$$

as $n \to \infty$, where m is two-dimensional Lebesgue measure.

Proof. We first present a Fourier-theoretic argument. First suppose that K and F happen to take the particular forms

$$K(x,y) = e^{2\pi i(rx+my)}, \quad F(x,y) = e^{2\pi i(lx+ky)}$$

for some m, r, l, k integers. Then we have the computations

$$\begin{aligned}
\int_{[0,1]^2} K(x, y+nx)F(x,y)dm(x,y) &= \int_0^1 \int_0^1 e^{2\pi i(rx+my+mnx)}e^{2\pi i(lx+ky)}dxdy \\
&= \left(\int_0^1 e^{2\pi ix(r+mn+l)}dx\right)\left(\int_0^1 e^{2\pi iy(m+k)}dy\right) \\
&= \begin{cases} 1 & \text{if } r+mn+l = 0 \text{ and } m+k = 0 \\ 0 & \text{otherwise} \end{cases} \\
&\overset{n\to\infty}{\to} \begin{cases} 1 & \text{if } m = 0, k = 0, r+l = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

(since, for n large, the equation $r + mn + l = 0$ fails unless $m = 0$), whereas

$$\int_0^1 \left(\int_0^1 K(x,s)ds\right)\left(\int_0^1 F(x,y)dy\right)dx = \int_0^1 \left(\int_0^1 e^{2\pi i(rx+ms)}ds\right)\left(\int_0^1 e^{2\pi i(lx+ky)}dy\right)dx$$
$$= \int_0^1 \delta_{m=0}\delta_{k=0}e^{2\pi i x(r+l)}dx$$
$$= \begin{cases} 1 & \text{if } m = 0, k = 0, r + l = 0 \\ 0 & \text{otherwise} \end{cases}$$

which agrees with the prior limit.

Now, since both sides of the desired limit are linear in K and F separately, we conclude the result whenever K and F are trigonometric polynomials.

We now take limits. Assume that the L^∞ norm of K is small. Then

$$\left|\int_{[0,1]^2} K(x,y+nx)F(x,y)dm(x,y)\right| \leq \|K\|_\infty \|F\|_1$$

is small for each n. In particular, if $K_j \to 0$ uniformly,

$$\limsup_{j\to\infty}\limsup_{n\to\infty} \left|\int_{[0,1]^2} K_j(x,y+nx)F(x,y)dm(x,y)\right| \lesssim_F \limsup_{j\to\infty} \|K_j\|_\infty = 0$$

and similarly

$$\limsup_{j\to\infty} \left|\int_0^1 \left(\int_0^1 K_j(x,s)ds\right)\left(\int_0^1 F(x,y)dy\right)dx\right| = 0$$

Since general continuous periodic K can be approximated uniformly by trigonometric polynomials, we conclude the formula

$$\int_{[0,1]^2} K(x,y+nx)F(x,y)dm(x,y) \to \int_0^1 \left(\int_0^1 K(x,s)ds\right)\left(\int_0^1 F(x,y)dy\right)dx$$

for arbitrary K as in the setup, where F is still taken to be a trigonometric polynomial.

Lastly, since we also have the estimates

$$\left|\int_{[0,1]^2} K(x,y+nx)F(x,y)dm(x,y)\right| \leq \|F\|_\infty \|K\|_1$$

and

$$\left|\int_0^1 \left(\int_0^1 K(x,s)ds\right)\left(\int_0^1 F(x,y)dy\right)dx\right| \leq \|F\|_\infty \|K\|_1$$

we conclude the general statement by approximating F uniformly by trigonometric polynomials. □

Fall 2021 Problem 7. Let f and g be functions that are continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$. Suppose that $\text{Re}(f)$ and $\text{Re}(g)$ agree on $\partial\mathbb{D}$. Prove that $f - g$ is an imaginary constant on $\mathbb{D}$.

Proof. By assumption, $i(f-g)$ is a continuous function on $\overline{\mathbb{D}}$ that is holomorphic on $\mathbb{D}$ and maps $\partial\mathbb{D}$ to $\mathbb{R}$. By the Schwarz reflection principle, $i(f-g)$ extends to an entire function h satisfying the functional equation

$$h(1/\bar{z}) = \overline{h(z)}$$

for all $z \neq 0$. But since h is continuous on $\overline{\mathbb{D}}$, it is bounded on $\overline{\mathbb{D}} \setminus \{0\}$, while

$$h(\overline{\mathbb{D}} \setminus \{0\}) = \overline{\overline{[h(\overline{\mathbb{D}} \setminus \{0\})]}} = \overline{[\bar{h}(\overline{\mathbb{D}} \setminus \{0\})]} = \overline{[h(\phi(\overline{\mathbb{D}} \setminus \{0\}))]} = \overline{[h(\mathbb{C} \setminus \mathbb{D}]}$$

so the latter is a bounded set; consequently, $h(\mathbb{C})$ is a bounded set and we conclude that h is a bounded entire function, hence constant. By the functional equation, $h = \bar{h}$ everywhere, so h is real-valued. Thus $f - g = -ih$ is a purely imaginary constant, as claimed. □

Fall 2021 Problem 8. Throughout this question, U, V, and W are proper nonempty subsets of $\mathbb{C}$ that are open and simply connected, and u and v are fixed points in U and V respectively. We say that a sequence of functions converges *normally* if it converges uniformly on compact sets.

- (a): Prove that, for any compact set $K \subseteq U$, there is a compact set $L \subseteq V$ such that $f(K) \subseteq L$ for any holomorphic map $f : U \to V$ that satisfies $f(u) = v$.
- (b): Let $f_1, f_2, \ldots$ be a sequence of holomorphic maps $U \to V$ that all satsify $f_n(u) = v$ and that converge normally to another holomorphic map $f : U \to V$. Let $g : W \to U$ and $h : V \to W$ be conformal equivalences. Prove that $f_n \circ g$ converges normally to $f \circ g$ and $h \circ f_n$ converges normally to $h \circ f$.

Proof. (a) is false if "simply connected" does not necessarily entail "connected," so for the rest of the problem we assume that U, V, and W are connected.

(a): Assume first $U = V = \mathbb{D}$ and $u = v = 0$. Then, by the Schwarz lemma, any $f : U \to V$ satisfying $f(u) = v$ also satisfies $|f(z)| \leq |z|$ for all $z \in U$. If $K \subseteq U$ is compact, then it is contained in $\{|z| \leq r\}$ for some $r < 1$. Set $L = \{|z| \leq r\}$. By the Schwarz lemma, $f(K) \subseteq L$ for any $f : U \to V$ holomorphic with $f(u) = v$, as desired.

More generally, let $\phi_1 : \mathbb{D} \to U$ and $\phi_2 : \mathbb{D} \to V$ be conformal maps satisfying $\phi_1(0) = u, \phi_2(0) = v$ by the Riemann mapping theorem. If $K \subseteq U$ is compact, then $\phi_1^{-1}(K)$ is compact in $\mathbb{D}$, so we may find an associated compact L' in $\mathbb{D}$ by the above. Set $L = \phi_2(L')$ compact. Then, for any $f : U \to V$ with $f(u) = v$, the associated function

$$g = \phi_2^{-1} \circ f \circ \phi_1$$

is holomorphic, maps from $\mathbb{D}$ to $\mathbb{D}$, and has $g(0) = 0$; by the above, $g(\phi_1^{-1}(K)) \subseteq \phi_2^{-1}(L)$, and hence $f(K) \subseteq L$ as desired.

(b): First fix $K \subseteq W$ compact. Then $g(K)$ is compact, so $(f_n \circ g)|_K = (f_n|_{g(K)}) \circ (g|_K)$ converges uniformly to $(f|_{g(K)}) \circ (g|_K) = (f \circ g)|_K$, as desired.

Similarly, fixing $K \subseteq U$ compact, $(h \circ f_n)|_K = h \circ (f_n|_K)$ converges uniformly to $h \circ (f|_K) = (h \circ f)|_K$, so we have the desired result. □

Fall 2021 Problem 9. Compute the number of solutions, including multiplicity, of the equation

$$z^5 \cos z + 5iz^4 + 2 = 0$$

inside the unit disk $|z| < 1$.

Proof. Note first that, for $|z| = 1$,

$$|z^5 \cos z| = |\cos z| = \left|\frac{e^{iz} + e^{-iz}}{2}\right| \leq \frac{e^y + e^{-y}}{2} \leq \frac{e + e^{-1}}{2} < 2$$

by a few easy calculator-free estimates. Therefore

$$|z^5 \cos z + 2| < 4 < 5 = |5iz^4|$$

for any $|z| = 1$. By Rouché's theorem, the number of zeroes of the given function in the unit disk is equal to the number of zeroes of $5iz^4$ in the same domain, which is 4. $\square$

Fall 2021 Problem 10. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ that satisfy $|f'(z)| \leq 2|f(z)|$ for all $z \in \mathbb{C}$.

Proof. Clearly constants satisfy that inequality, so suppose f is not a constant. Then the log-derivative

$$\frac{f'}{f}$$

has a simple pole at every zero of f, hence is unbounded; consequently, the inequality $|f'(z)| \leq 2|f(z)|$ cannot hold for z sufficiently close to any zero of f, and so we can assume f has no zeroes.

Any zero-free f may be written as e^g (e.g. by integrating the entire function $\frac{f'}{f}$) for some entire g; the given inequality then takes the form

$$|g'(z)e^g| \leq 2|e^g|$$

or

$$|g'(z)| \leq 2$$

Since g is entire, g' is as well, so by Liouville we conclude that g' is constant and so $g = az + b$ for some complex constants a, b with $|a| \leq 2$. Thus the possible entire functions f satisfying the given inequality are all of the form

$$f(z) = e^{az+b}, \quad |a| \leq 2$$

and it is immediate to verify that such functions are sufficient as well. $\square$

Fall 2021 Problem 11. For each $p \in (-1, 1)$, compute the improper Riemann integral

$$\int_0^\infty \frac{x^p}{x^2 + 1} dx.$$

Proof. For each $p \in (-1, 1)$, there is a unique holomorphic branch of the function $z \mapsto z^p$ on $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$ which maps $\mathbb{R}_{>0}$ to $\mathbb{R}_{>0}$. For each $0 < \varepsilon < R < \infty$ define the curve $\Gamma = \Gamma_{\varepsilon,R}$ to be composed of the four curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ where

- γ_1 is the straight line segment connecting ε to R;
- γ_2 is the half-circle connecting R to $-R$ through the upper half-plane;
- γ_3 is the straight line segment connecting $-R$ to $-\varepsilon$;
- γ_4 is the half-circle connecting $-\varepsilon$ to ε through the upper half-plane.

By the residue theorem,

$$\begin{aligned}
\int_\Gamma \frac{z^p}{z^2+1}dz &= 2\pi i \text{Res}\left[\frac{z^p}{z^2+1}dz, i\right] \\
&= 2\pi i \left.\frac{z^p}{z-i}\right|_{z=i} \\
&= \pi e^{\frac{\pi i p}{2}}
\end{aligned}$$

We now consider each of the γ_j separately. Clearly

$$\lim_{\varepsilon\to 0^+, R\to+\infty} \int_{\gamma_1} \frac{z^p}{z^2+1}dz = \int_0^\infty \frac{x^p}{x^2+1}dx$$

Here we have used $|p| < 1$, so that the integrand is in $L^1(0,\infty)$. Similarly, by the triangle inequality,

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{z^p}{z^2+1}dz\right| &\leq \int_0^\pi \frac{R^p}{R^2-1}Rd\theta \\
&= \pi\frac{R^{p+1}}{R^2-1} \overset{R\to\infty}{\to} 0 \quad \text{since } p < 1
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{\gamma_4} \frac{z^p}{z^2+1}dz\right| &\leq \int_0^\pi \frac{\varepsilon^p}{1-\varepsilon^2}\varepsilon d\theta \\
&= \pi\frac{\varepsilon^{1+p}}{1-\varepsilon^2} \overset{\varepsilon\to 0}{\to} 0 \quad \text{since } p > -1
\end{aligned}$$

Lastly, along γ_3, $z = re^{i\pi}$ with r varying from R to ε; our branch of z^p then evaluates to $r^p e^{i\pi p}$, so

$$\begin{aligned}
\int_{\gamma_3} \frac{z^p}{z^2+1}dz &= \int_R^\varepsilon \frac{r^p e^{i\pi p}}{r^2+1}e^{\pi i}dr \\
&= e^{i\pi p}\int_\varepsilon^R \frac{x^p}{x^2+1}dx \\
&\overset{\varepsilon\to 0, R\to\infty}{\to} e^{i\pi p}\int_0^\infty \frac{x^p}{x^2+1}dx
\end{aligned}$$

Thus in total we conclude

$$\pi e^{\frac{\pi i p}{2}} = \left(1+e^{i\pi p}\right)\int_0^\infty \frac{x^p}{x^2+1}dx$$

so that

$$\int_0^\infty \frac{x^p}{x^2+1}dx = \frac{\pi}{e^{-\frac{\pi i p}{2}} + e^{\frac{\pi i p}{2}}} = \frac{\pi}{2}\cos(\pi p/2)$$

$\square$

Fall 2021 Problem 12. Let $f(z)$ be a holomorphic function on the set $\mathcal{B} = \{z : |z| < 2\}$ that satisfies $|f(z)| < 1$ for all $z \in \mathcal{B}$. Assume also that

$$f(1) = f(-1) = f(i) = f(-i) = 0$$

- (a): Show that $|f(0)| \leq 1/16$.
- (b): Show that there is such a function $f : \mathcal{B} \to \mathbb{D}$ with $|f(0)| = 1/16$.

Proof. We modify f in several steps. First set

$$f_1(z) = f(2z), \quad z \in \mathbb{D}$$

Then f_1 vanishes on $\pm 1/2, \pm i/2$. By standard Blashke factor theory, f_1 may be written as

$$f_1(z) = \frac{z - 1/2}{1 - \frac{1}{2}z} \frac{z + 1/2}{1 + \frac{1}{2}z} \frac{z - i/2}{1 + \frac{i}{2}z} \frac{z + i/2}{1 - \frac{i}{2}z} g(z)$$

for some $g : \mathbb{D} \to \overline{\mathbb{D}}$ holomorphic (this follows from Schwarz reflection over $\partial\mathbb{D}$, say). Consequently,

$$|f(0)| = |f_1(0)| = \frac{1}{16}|g(0)| \leq \frac{1}{16}$$

Setting $g(z) = 1$, we obtain an equality. $\square$

7 Spring 2022

Spring 2022 Problem 1.

- (a): Given a finite Borel measure μ on $\mathbb{R}$, its *support* is the set

$$S = \{x \in \mathbb{R} : \mu((x-\varepsilon, x+\varepsilon)) > 0 \text{ for every } \varepsilon > 0\}$$

  Prove that S is closed, that $\mu(\mathbb{R} \setminus S) = 0$, and that any other set with these last two properties must contain S.

- (b): Prove that there is a finite Borel measure μ on $\mathbb{R}$ such that
  - μ has support equal to $\mathbb{R}$
  - μ and Lebesgue measure are mutually singular.

Proof. (a): Denotes $S^c = \mathbb{R} \setminus S$. If $x \in S^c$, then there is some $\varepsilon > 0$ such that $\mu((x-\varepsilon, x+\varepsilon)) = 0$; but then $(x-\varepsilon, x+\varepsilon) \subseteq S^c$, so S^c is open, hence S is closed.

Notice that, since μ is finite Borel, μ is inner regular, so $\mu(S^c) = \sup\{\mu(K) : K^{\text{compact}} \subseteq S^c\}$. Fix one such K; we will verify that $\mu(K) = 0$. For each $x \in K$ there is $\varepsilon_x > 0$ such that $\mu((x-\varepsilon_x, x+\varepsilon_x)) = 0$. By compactness, we may find $x_1, \ldots, x_n \in K$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\{(x_j - \varepsilon_j, x_j + \varepsilon_j)\}_{j=1}^n$ cover K and each interval has μ-measure 0. But then

$$\mu(K) \leq \sum_{j=1}^n \mu(x_j - \varepsilon_j, x_j + \varepsilon_j) = 0$$

so $\mu(K) = 0$. By inner regularity, $\mu(S^c) = 0$.

Finally, let $T \subseteq \mathbb{R}$ be closed with $\mu(\mathbb{R} \setminus T) = 0$. If $x \in \mathbb{R} \setminus T$ is arbitrary, then there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R} \setminus T$ because T is closed, so $\mu((x-\varepsilon, x+\varepsilon)) \leq \mu(\mathbb{R} \setminus T) = 0$. Thus $x \in \mathbb{R} \setminus S$ by the definition of S. Consequently, $(\mathbb{R} \setminus T) \subseteq (\mathbb{R} \setminus S)$, so $S \subseteq T$ as was to be shown.

(b): Let $\{q_n\}_{n=1}^\infty$ be an enumeration of $\mathbb{Q}$. Let μ_0 be Cantor measure, supported on the usual middle-thirds Cantor set C. Let $\tau_n : \mathbb{R} \to \mathbb{R}$ be translation by q_n. Then we define

$$\mu := \sum_{n=1}^\infty 2^{-n} (\tau_n)_* \mu_0$$

Then μ is Borel, nonnegative, and has total mass $\|\mu_0\| < \infty$. For any $x \in \mathbb{R}$ and $\varepsilon > 0$ fix $q_n \in (x, x+\varepsilon)$ rational; then

$$\mu((x-\varepsilon, x+\varepsilon)) \geq \mu_n((x, x+\varepsilon)) > 0$$

hence the support of μ is all of $\mathbb{R}$. Lastly, note that, for $A = \bigcup_{n=1}^\infty \tau_n(C)$, we have

$$\mu(\mathbb{R} \setminus A) \leq \sum_{n=1}^\infty 2^{-n} \mu_n(\bigcap_{m=1}^\infty \mathbb{R} \setminus \tau_n(C)) \leq \sum_{n=1}^\infty 2^{-n} \mu_n(\mathbb{R} \setminus \tau_n(C)) = 0$$

whereas, since Lebesgue measure m is translation-invariant, and $m(C) = 0$,

$$m(A) \leq \sum_{n=1}^\infty m(C) = 0$$

hence m and μ are mutually singular, as claimed. $\square$

Spring 2022 Problem 2. Let $\mathcal{F}$ be an arbitrary collection of 1-Lipschitz functions, and for each $f \in \mathcal{F}$ let

$$L_f := \{(x,y) \in \mathbb{R}^2 : y \leq f(x)\}.$$

Let $L := \bigcup_{f \in \mathcal{F}} L_f$.

- (a): Prove that L is a Lebesgue measurable subset of the plane.
- (b): Prove that L is not necessarily Borel measurable. You may quote without proof the fact that *there exists a non-Borel subset of* $\mathbb{R}$.

Proof. (a): First, note that $(x,y) \in L$ if and only if $f(x) \geq y$ for some $f \in \mathcal{F}$. If, for some $x \in \mathbb{R}$ there are arbitrarily large $y \in \mathbb{R}$ such that $(x,y) \in L$, then by the 1-Lipschitz condition we have $L = \mathbb{R}^2$. For the remainder of the problem, we assume that for each $x \in \mathbb{R}$, the collection $\{y \in \mathbb{R} : (x,y) \in L\}$ is bounded above.

Let $h(x) = \sup_{f \in \mathcal{F}} f(x)$; then $L = \text{int}(L) \uplus (L \cap \partial L)$, where $\partial L = \text{graph}(h)$. It suffices to argue that graph(h) is a nullset in $\mathbb{R}^2$. Since h is 1-Lipschitz (being the supremum of a family of 1-Lipschitz functions), it suffices to show that the graph of a 1-Lipschitz function is null.

It suffices to show that, if $u : [0,1] \to \mathbb{R}$ is 1-Lipschitz, then graph(u) is null. For each n, let A_n be the set

$$A_n = \bigcup_{j=0}^{n-1} A_{n,j}$$

where $A_{n,j}$ is the triangle with vertices $(\frac{j}{n}, u(\frac{j}{n})), (\frac{j+1}{n}, u(\frac{j}{n}) - \frac{1}{n}), (\frac{j+1}{n}, u(\frac{j}{n}) + \frac{1}{n})$. Then A_n contains graph(u), using the 1-Lipschitz condition.

Then

$$|\text{graph}(u)| \leq \sum_{j=0}^{n-1} |A_{n,j}| = \sum_{j=0}^{n-1} \frac{1}{n^2} = \frac{1}{n}$$

which implies $|\text{graph}(u)| = 0$, as claimed.

Thus L is Lebesgue measurable, being the union of a Borel set and a nullset.

(b): Let A be a non-Borel subset of $\mathbb{R}$. Let $\mathcal{F}$ be the union $\mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{x \mapsto 1 - |x - a| : a \in A\}$$

and

$$\mathcal{F}_2 = \{x \mapsto c : c \in (-\infty, 1)\}$$

Then $L = \mathbb{R} \times (-\infty, 1) \cup A \times \{1\}$, which is not Borel. □

Spring 2022 Problem 3. Let X be a real Banach space and let X^* be its dual. If $Y \subseteq X$, then let

$$Y^\perp := \{\ell \in X^* : \ell(y) = 0 \, \forall y \in Y\}$$

On the other hand, if $Z \subseteq X^*$, then let

$$^\perp Z := \{x \in X : \ell(x) = 0 \, \forall \ell \in Z\}.$$

- (a): Prove that $^\perp(Y^\perp)$ is the closed linear span of Y in X for any $Y \subseteq X$.

- (b): Provide an example of a real Banach space X and a subset $Z \subseteq X^*$ for which $({}^\perp Z)^\perp$ is not the closed linear span of Z in X^*.

Proof. (a): Let $Y \subseteq X$ be an arbitrary subset. We first notice $\overline{\operatorname{span} Y} \subseteq {}^\perp(Y^\perp)$: clearly $Y \subseteq {}^\perp(Y^\perp)$ and ${}^\perp(Y^\perp)$ is a linear subspace of X, so automatically $\operatorname{span} Y \subseteq^\perp (Y^\perp)$. Note that ${}^\perp(Y^\perp)$ is closed: it is the intersection of closed subsets of X. Thus $\overline{\operatorname{span} Y} \subseteq {}^\perp(Y^\perp)$ as claimed.

We now show the reverse. Suppose $y \in X \setminus \overline{\operatorname{span} Y}$. By Hahn-Banach, we may find $\ell \in X^*$ such that $\ell(\overline{\operatorname{span} Y}) = \{0\}$ and $\ell(y) \neq 0$. Then $\ell \in Y^\perp$, and so $y \notin {}^\perp(Y^\perp)$. Thus we have shown the inclusion $X \setminus \overline{\operatorname{span} Y} \subseteq X \setminus^\perp (Y^\perp)$, which is to say ${}^\perp(Y^\perp) \subseteq \overline{\operatorname{span} Y}$, as was to be shown.

(b): Let $X = L^1([0,1], m)$, so that $X^* = L^\infty([0,1], m)$. Let $Z \subseteq L^\infty([0,1], m)$ be the collection of continuous mean-zero functions, i.e.

$$f \in Z \iff \text{f continuous and } \int_0^1 f(x)dx = 0$$

Then Z is its own closed linear span in $L^\infty([0,1], m)$. We claim ${}^\perp Z$ is the collection of constant functions in X: if $g \in X$ is nonconstant, then we may find a pair of separated intervals $I_1, I_2 \subseteq [0,1]$ (i.e. intervals of positive distance apart) with $m(I_1) = m(I_2)$ and such that

$$\int_{I_1} g(x)dx < \int_{I_2} g(x)dx$$

But then we may let $f = 1$ on I_1, $f = -1$ on I_2, and zero outside of small neighborhoods of I_1, I_2 (interpolated to give a continuous function), which may be made mean zero and satisfying

$$\int_0^1 f(x)g(x)dx < 0$$

so that $g \not\perp Z$. Thus ${}^\perp Z$ is the collection of constant functions, whereas $({}^\perp Z)^\perp$ is the collection of mean-zero functions in $L^\infty([0,1], m)$, which need not be continuous. This concludes the counterexample upon recalling $Z = \overline{\operatorname{span} Z}$. □

Spring 2022 Problem 4. Let $f : [0,\infty) \to [0,\infty)$. Assume that $f(0) = 0$ and that f is convex. Prove that

$$f(x) = \int_0^x g(y)dy$$

for some increasing function $g : [0,\infty) \to [0,\infty)$.

Proof. For $x \in [0,\infty)$, define $g(x)$ to be the supremal λ such that

$$f(t) \geq f(x) + \lambda(t - x), \quad \forall t \in [0,\infty)$$

Note that $f(x) \geq 0$ everywhere, so $g(x) \geq 0$ as well. By convexity, and elementary geometric considerations, g is increasing as well.

Fix $x \in [0,\infty)$. To show the integral equality, we show the inequalities

$$f(x) \geq \int_0^x g(y)dy$$

and

$$f(x) \leq \int_0^x g(y)dy$$

Since g is increasing, it is Riemann integrable. Thus

$$\int_0^x g(y)dy = \sup_n \sum_{j=0}^{n-1} \frac{x}{n} g(\frac{jx}{n})$$

Since $f(t) \geq f(\frac{jx}{n}) + g(\frac{jx}{n})(t - \frac{jx}{n})$ for all t, we in particular have

$$f(\frac{(j+1)x}{n}) - f(\frac{jx}{n}) \geq g(\frac{jx}{n})\frac{x}{n}$$

Thus

$$\sum_{j=0}^{n-1} \frac{x}{n} g(\frac{jx}{n}) \leq \sum_{j=0}^{n-1} f(\frac{(j+1)x}{n}) - f(\frac{jx}{n}) = f(x) - f(0) = f(x)$$

so, taking a supremum over n,

$$f(x) \geq \int_0^x g(y)dy$$

For the reverse inequality, we consider the right Riemann sum, instead of the left: note that

$$f(\frac{(j+1)x}{n}) - f(\frac{jx}{n}) \leq g(\frac{(j+1)x}{n}\frac{x}{n}$$

so

$$f(x) = \sum_{j=0}^{n-1} f(\frac{(j+1)x}{n}) - f(\frac{jx}{n}) \leq \sum_{j=0}^{n-1} \frac{x}{n} g(\frac{(j+1)x}{n})$$

so certainly

$$f(x) \leq \int_0^x g(y)dy$$

as claimed. □

Spring 2022 Problem 5. Let μ be a Borel measure on $\mathbb{R}^2$, and assume it has the following property: for every fixed $r > 0$, the quantity $\mu(B(x,r))$ is finite and independent of x, where $B(x,r)$ is the open ball of radius r around x.

- (a): Prove that there is a finite constant c such that $\mu(B(x,r)) \leq cr^2$ whenever $0 < r \leq 1$.
- (b): Prove that μ is a constant multiple of Lebesgue measure.

Proof. (a): We argue by geometric considerations. For each $0 < r \leq 1$, let c_r be the unique constant such that $\mu(B(x,r)) = c_r r^2$ for each $x \in \mathbb{R}^2$. We note that, for $\lambda \gg 1$, there are $\Omega(\lambda^2)$ disjoint disks of radius r/λ that fit in any disk of radius r; this may be seen by considering the grid of points with separation r/λ, centered at the center of the large disk, and considering a square inscribed in the large disk; by adding a small disk at every other point in the grid in the square, we get $\Omega(\lambda^2)$ small disks in the large disk, as claimed.

Consequently,

$$c_r r^2 = \mu(B(x,r)) \geq \Omega(\lambda^2) c_{r/\lambda} (\frac{r}{\lambda})^2 = c_{r/\lambda} \Omega(1) r^2$$

so

$$c_r \gtrsim c_{r/\lambda}$$

independent of r, λ, as long as λ is sufficiently large. To finish, note that $B(x,r)$ can be covered by $O(\lambda^2)$ disks $B(x', r/\lambda)$ for *all* $\lambda \geq 1$, so

$$c_r r^2 \lesssim \lambda^2 c_{r/\lambda} (\frac{r}{\lambda})^2$$

and

$$c_r \lesssim c_{r/\lambda}$$

for all $\lambda \geq 1$. Thus the constants c_r are all comparable, so we may find some c such that $\mu(B(x,r)) \leq cr^2$ independently of $0 < r \leq 1$.

(b): We omit some details. By part (a), μ is absolutely continuous with respect to Lebesgue measure m: if N is a (Borel) nullset in $\mathbb{R}^2$, then for each $\varepsilon > 0$ we may find a cover of N by balls B_i such that $\sum_i m(B_i) < \varepsilon$; the latter implies $\mu(\bigcup_i B_i) \lesssim \varepsilon$ for a constant depending on the constant from (a), so indeed $\mu(N) = 0$.

Thus we may write $d\mu = f dm$ for some nonnegative locally integrable Borel function f. By the assumption, the average of f on $B(x,r)$ is independent of x. If f is nonconstant, then there is some $\varepsilon > 0$ and positive measure sets A, B such that $\sup_{x \in A} f(x) + \varepsilon < \inf_{x \in B} f(x)$. By Lebesgue differentiation, a.e. point of A (resp. B) is a Lebesgue point for A (resp. for B). Consequently, we may find some $x \in A, y \in B$ and $r > 0$ such that $\mu(B(x,r)) < \mu(B(y,r))$, contradicting our assumption. Thus f is constant a.e., so μ is a constant multiple of Lebesgue measure. $\square$

Spring 2022 Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and (2π)-periodic. Prove that

$$\int_0^{2\pi} |f'(t)|^2 dt + \int_0^{2\pi} |f''(t)|^2 dt \leq \int_0^{2\pi} |f(t)|^2 dt + \int_0^{2\pi} |f'''(t)|^2 dt$$

Proof. Consider the Fourier expansion

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$$

Then, by Plancherel,

$$\int_0^{2\pi} |f(t)|^2 dt = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2$$

$$\int_0^{2\pi} |f'(t)|^2 dt = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 n^2$$

$$\int_0^{2\pi} |f''(t)|^2 dt = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 n^4$$

$$\int_0^{2\pi} |f'''(t)|^2 dt = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 n^6$$

so the inequality to be demonstrated is just

$$\sum_{n=-\infty}^{\infty} |a_n|^2(n^2 + n^4) \leq \sum_{n=-\infty}^{\infty} |a_n|^2(1 + n^6) \tag{10}$$

But note that $n^6 - n^4 - n^2 + 1 = (n^2 - 1)(n^4 - 1) \geq 0$ for all $n \in \mathbb{Z}$; thus $n^2 + n^4 \leq 1 + n^6$ for all n, and (10) follows immediately. $\square$

Spring 2022 Problem 7. Let $f : D(0,1) \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function in the unit disk $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ that extends continuously to the boundary $\{z \in \mathbb{C} : |z| = 1\}$. Suppose also that $|f(z)| = 1$ whenever $|z| = 1$. Show that f is a rational function, in the sense that there exist polynomials $P, Q : \mathbb{C} \to \mathbb{C}$ with Q not identically zero such that $f(z) = P(z)/Q(z)$ whenever $|z| \leq 1$ and $Q(z) \neq 0$.

Proof. Set S be the collection of poles of f in $D(0,1)$. Let $u(z) = \frac{1}{\bar{z}}$ for all $z \in \mathbb{C} \setminus \{\infty\}$; we adopt the usual convention that u exchanges 0 and ∞. Then $u \circ f \circ u$ is holomorphic on $\mathbb{C} \setminus (\overline{D(0,1)} \cup u(S))$, and has a pole at each element of $u(S)$. Furthermore, $u \circ f \circ u$ is equal to f on $\{|z| = 1\}$, since u fixes that set pointwise. Consequently, $f \cup (u \circ f \circ u)$ is a continuous function $\mathbb{C} \to \mathbb{C} \cup \{\infty\}$ that is meromorphic on $\mathbb{C} \setminus \{|z| = 1\}$. If we assume that by "extends continuously to the boundary" we mean $\mathbb{C}$-valued, then $f \cup (u \circ f \circ u)$ extends continuously to the boundary, in the sense that poles do not accumulate to the boundary and the extension to $\{|z| = 1\}$ is $\mathbb{C}$-valued, then by Morera's theorem we conclude that the extension is holomorphic on a neighborhood of $\{|z| = 1\}$.

Call this extension g. Then g is meromorphic on the plane and has finite order at ∞, hence is meromorphic on the Riemann sphere. If we set Q to be a polynomial which vanishes to sufficiently large order at each pole of g, then Qg extends to be holomorphic on the plane and has finite order at ∞; by considering the power series of Qg, we may only have finitely-many terms by the order at ∞, so Qg is a polynomial P. Thus $g = P/Q$, and g agrees with f on $D(0,1)$; this concludes the argument. $\square$

Spring 2022 Problem 8. Let U be a connected open subset of $\mathbb{C}$, let V be a nonempty open subset of U, and let K be a compact subset of U. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $f : U \to \mathbb{C}$ is a holomorphic function that obeys the bounds $|f(z)| \leq \delta$ for all $z \in V$ and $|f(z)| \leq 1$ for all $z \in U$, then $|f(z)| \leq \varepsilon$ for all $z \in K$.

Proof. Suppose the statement is not true. Then there exists a sequence $f_1, f_2, \ldots$ of holomorphic functions on U, and an $\varepsilon > 0$, such that $|f_j(z)| \leq 1$ on U, $|f_j(z)| \to 0$ uniformly on V, and $|f_j(z_j)| > \varepsilon$ for some $z_j \in K$. Since the family $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded, it is normal (by Montel's theorem), so there is some subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ which is uniformly convergent on compact sets, say to f.

Since $|f_{n_j}(z)| \to 0$ on V, f vanishes identically on V. Since f is the local uniform limit of holomorphic functions, it is also holomorphic, so $f \equiv 0$ on U, using the fact that U is connected. On the other hand, the sequence z_{n_j} has a limit point z' in K, and by the condition $|f_{n_j}(z_{n_j})| > \varepsilon$ we have $|f(z')| \geq \varepsilon$, a contradiction. $\square$

Spring 2022 Problem 9. Establish the identity

$$\int_0^1 \log|1 - re^{2\pi i\theta}| d\theta = \max(\log r, 0)$$

for all $0 < r < \infty$.

Proof. We first assume $r \neq 1$. In this case, from the identity

$$\log|1 - re^{2\pi i\theta}| = \log(r|1 - r^{-1}e^{2\pi i\theta}|) = \log r + \log|1 - r^{-1}e^{2\pi i\theta}|$$

it suffices to prove the identity in the case $0 < r < 1$.

In this case, $1 - re^{2\pi i\theta}$ is in the right half-plane for all θ. Thus $\log|1 - re^{2\pi i\theta}| = \mathrm{Re}(\mathrm{Log}(1 - z))$, where Log is the principal logarithm and $z = re^{2\pi i\theta}$. Consequently,

$$\int_0^1 \log|1 - re^{2\pi i\theta}| d\theta = \mathrm{Avg}_{|z-1|=r}\mathrm{Re}(\mathrm{Log}(1 - z)) = \mathrm{Re}(\mathrm{Log}(1)) = 0$$

where we have used the mean value principle for harmonic functions.

We now argue via DCT that the result extends to $r = 1$. We will only study the half of the integral $0 < \theta < \frac{1}{2}$ for brevity, but one may notice that the analysis is entirely symmetric. The RHS of the claimed inequality varies continuously in r, so we need to show that the LHS has the natural limiting value. Observe that

$$1 - re^{2\pi i\theta} = (1 - e^{2\pi i\theta}) + (1 - r)e^{2\pi i\theta}$$

Thus

$$1 - re^{2\pi i\theta} = 1 - e^{2\pi i\theta} + O(|1 - r|)$$

Consequently,

$$\big|\log|1 - re^{2\pi i\theta}|\big| \leq \big|\log|1 - e^{2\pi i\theta}|\big| + O(|1 - r|)$$

Observe that

$$1 - e^{2\pi i\theta} = -(2i)e^{\pi i\theta}\sin(\pi\theta)$$

so that

$$|1 - e^{2\pi i\theta}| = 2|\sin(\pi\theta)|$$

and

$$\log|1 - e^{2\pi i\theta}| = \log 2 + \log|\sin(\pi\theta)|, \quad (0 < \theta < 1)$$

Since $\sin(\pi\theta) \sim \pi\theta$ for $\theta \sim 0$, we have that

$$\big|\log|1 - e^{2\pi i\theta}|\big| \lesssim |\log\theta|$$

for $0 < \theta < \frac{1}{2}$. All together, we have the inequality

$$\big|\log|1 - re^{2\pi i\theta}|\big| \lesssim 1 + |\log\theta| + |r - 1|$$

Over the interval $r \in [1, 2]$, the preceding is uniformly bounded by a function integrable over $\theta \in [0, \frac{1}{2}]$. Also, as $r \to 1^+$, we have the pointwise limit

$$\log|1 - re^{2\pi i\theta}| \to \log|1 - e^{2\pi i\theta}|$$

Thus, by dominated convergence,

$$0 = \lim_{r\to 1^+} \log r = \lim_{r\to 1^+} \int_0^1 \log|1 - re^{2\pi i\theta}| d\theta = \int_0^1 \log|1 - e^{2\pi i\theta}| d\theta$$

as was to be shown. $\square$

Spring 2022 Problem 10. Let n be a natural number, and let α be a complex number with $|\alpha| < 1$. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) := e^z(1-z)^n - \alpha$.

- (a): Show that f has exactly n roots (counting multiplicity) in the right half-plane $\{z : \text{Re}\, z > 0\}$.
- (b): If $\alpha \neq 0$, show that the n roots in (a) are all simple.

Proof. (a): For $R > 0$, let Γ_R be the boundary of a half-disk, defined as the union of the diameter connecting iR to $-iR$ and the half-circle $\{e^{i\theta}R : \theta \in [-\pi/2, \pi/2]$. Since $|f(z)| \geq |z-1|^n - |\alpha|$ in the right half-plane, there are only finitely many such roots; additionally, note that f has no roots along the imaginary axis, since $|\alpha| < 1$. Thus, for R sufficiently large, the roots of f in the right half-plane are enclosed by Γ_R.

Let $g(z) = e^z(1-z)^n$. By the argument principle, the number of roots of f in the right half-plane is equal to the winding number of the curve $g(\Gamma_R)$ around α. Along the diameter $iR \to -iR$, $|e^z| = 1$ and $|1-z|^n \geq 1^n = 1$; similarly, for $z = e^{i\theta}R, \theta \in [-\pi/2, \pi/2]\}$, $|e^z| \geq 1$ and $|1-z|^n \geq (R-1)^n \geq 1$ (again, assuming R is sufficiently large). Thus $\{|z| < 1\} \subseteq \mathbb{C} \setminus g(\Gamma_R)$, so the winding number of $g(\Gamma_R)$ about α equals the winding number about 0. Thus, the number in question is the number of zeroes, counting multiplicity, of $g(z) = e^z(1-z)^n$ in the right half-plane, which is clearly n.

(b): If a root of f has multiplicity greater than 1, then f' vanishes at that root. However,

$$f'(z) = e^z(z-1)^{n-1}(z-1+n)$$

so the only possible roots of high multiplicity are at $z = 1$ and $z = 1-n$. Since $f(1) = -\alpha \neq 0$, there is not a root at $z = 1$; if $n > 1$ then $1-n$ is not in the right half-plane, and for $n = 1$ we have already argued that f has no zero at $z = 0 \in i\mathbb{R}$. Thus f' never vanishes at any of the zeroes considered in part (a), assuming $\alpha \neq 0$, so the n roots are all simple. $\square$

Spring 2022 Problem 11. Let $u : D(0,1) \to \mathbb{R}^+$ be a non-negative harmonic function on the unit disk $D(0,1) := \{z \in \mathbb{C} : |z| < 1\}$ with $u(0) = 1$. Show that

$$\frac{1-|z|}{1+|z|} \leq u(z) \leq \frac{1+|z|}{1-|z|}$$

for all $z \in D(0,1)$.

Proof. Let v be the unique harmonic conjugate to u on $D(0,1)$ such that $v(0) = 0$. Then $f := u + iv$ is a holomorphic function $D(0,1) \to \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$ with $f(0) = 1$. Let $\phi(z) = \frac{z-1}{z+1}$. Then ϕ maps $\{z \in \mathbb{C} : \text{Re}(z) \geq 0\} \cup \{\infty\}$ to $\overline{D(0,1)}$ and maps 1 to 0; thus $g := \phi \circ f$ is a holomorphic function $D(0,1) \to \overline{D(0,1)}$ with $g(0) = 0$.

If g is constant, then f is constant, and u is the constant 1 function; the inequality is trivial in this case. Assume instead that g is not constant. By the open mapping theorem, g in fact takes values in $D(0,1)$. Then, by the Schwarz lemma,

$$|g(z)| \leq |z|$$

for all $|z| < 1$, i.e.

$$\frac{|f(z)-1|}{|f(z)+1|} \leq |z|$$

Note that

$$\frac{|f(z)-1|}{|f(z)+1|} = \sqrt{\frac{(u(z)-1)^2+v^2}{(u(z)+1)^2+v^2}} \geq \frac{|u(z)-1|}{|u(z)+1|}$$

so

$$\frac{|u-1|}{u+1} \leq |z|$$

Since u is nonnegative, case analysis ($u \leq 1$ versus $u > 1$) and elementary estimates ($\frac{x-1}{x+1}$ is increasing on $(-1,\infty)$) give the result in question. $\square$

Spring 2022 Problem 12. Do the following:

- (a): Establish the identity

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)}$$

  for any $z \in \mathbb{C} \setminus \mathbb{Z}$.

- (b): Establish the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof. (a):[8] We start with the Weierstrass factorization of $\mathrm{sinc}(z)$: note that

$$\frac{\sin(\pi z)}{\pi z}$$

is entire and has only simple zeroes, which are precisely at the nonzero integers. Thus it has a Weierstrass product expansion

$$\frac{\sin(\pi z)}{\pi z} = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

(where we have invoked the fact that, at least for $|z| < 1$, the quantities $\frac{z^2}{n^2}$ are absolutely summable), for some entire function g. We skip over demonstrating that $g = 0$ in this case.

Thus

$$\frac{\pi^2}{\sin^2(\pi z)} = \frac{1}{z^2} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^{-2} = \frac{1}{z^2} \prod_{n=1}^{\infty} \frac{n^4}{(z^2-n^2)^2} = \frac{1}{z^2} \prod_{n\neq 0} \frac{n^2}{(z-n)^2}$$

[8] An alternate approach to this sort of problem is presented in the solution to Fall 2022 problem 8 below, which does not depend on a pre-known factorization formula.

This gives us the desired expansion, by a partial fraction decomposition:

$$\begin{aligned}
\left(\prod_{|n|\le N}(z-n)^2\right)\left(\sum_{|n|\le N}\frac{1}{(z-n)^2}\right) &= \sum_{|n|\le N}\prod_{|j|\le N; j\neq n}(z-j)^2 \\
&= \prod_{n=1}^{N}(z-n)^2(z+n)^2 + z^2\sum_{1\le|n|\le N}\prod_{1\le|j|\le N; j\neq n}(z-j)^2 \\
&= \left(\prod_{1\le|n|\le N}n^2\right)\prod_{n=1}^{N}(1-(\frac{z}{n})^2)^2 \\
&+ \left(\prod_{1\le|n|\le N}n^2\right)z^2\sum_{1\le|n|\le N}\frac{1}{n^2}\prod_{1\le|j|\le N; j\neq n}(1-\frac{z}{j})^2
\end{aligned}$$

which rearranges to

$$\begin{aligned}
\left(z^2\prod_{1\le|n|\le N}\frac{(z-n)^2}{n^2}\right)\left(\sum_{|n|\le N}\frac{1}{(z-n)^2}\right) &= \prod_{n=1}^{N}(1-(\frac{z}{n})^2)^2 \\
&+ z^2\sum_{1\le|n|\le N}\frac{1}{n^2}\prod_{1\le|j|\le N; j\neq n}(1-\frac{z}{j})^2 \\
&= 1 + o_{N\to\infty}(1)
\end{aligned}$$

using a few elementary estimates from the theory of infinite products. Taking a limit as $N \to \infty$ of both sides produces

$$\frac{\sin^2(\pi z)}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2} = 1$$

which is the desired result.

(b): For each $N \in \mathbb{N}$, let Γ_N be the rectangle contour with vertices $\pm(N+\frac{1}{2}) \pm iN$. Then

$$\frac{1}{2\pi i}\int_{\Gamma_N}\frac{\pi^2}{z\sin^2(\pi z)}dz = \sum_{1\le|n|\le N}\text{Res}\left[\frac{1}{z(z-n)^2}, n\right] + \text{Res}\left[\frac{\pi^2}{z\sin^2(\pi z)}, 0\right]$$

Note that

$$\begin{aligned}
\frac{\pi^2}{z\sin^2(\pi z)} &= \frac{\pi^2}{z}\frac{1}{\pi^2 z^2}\frac{1}{(1-\frac{\pi^2 z^2}{6}+O(z^4))^2} \\
&= \frac{1}{z^3}\frac{1}{1-\frac{\pi^2 z^2}{3}+O(z^4)} \\
&= \frac{1}{z^3}(1+\frac{\pi^2 z^2}{3}+O(z^4)) \\
&= \frac{1}{z^3}+\frac{\pi^2}{3z}+O(1)
\end{aligned}$$

for z near 0; consequently,

$$\operatorname{Res}\left[\frac{\pi^2}{z\sin^2(\pi z)}, 0\right] = \frac{\pi^2}{3}$$

Additionally,

$$\operatorname{Res}\left[\frac{1}{z(z-n)^2}, n\right] = -\frac{1}{n^2}$$

so we have

$$\frac{1}{2\pi i}\int_{\Gamma_N} \frac{\pi^2}{z\sin^2(\pi z)} dz = -\sum_{n=1}^{N} \frac{2}{n^2} + \frac{\pi^2}{3}$$

On the other hand, along the top/bottom edges of Γ_N,

$$|\frac{\pi^2}{z\sin^2(\pi z)}| \leq \frac{\pi^2}{N}\frac{2}{e^N - e^{-N}}$$

so by an ML estimate the contribution from the top edge of the contour goes to 0 as $N \to \infty$. On the other hand, along the left and right edges,

$$|\frac{\pi^2}{z\sin^2(\pi z)}| \leq \frac{\pi^2}{N}\frac{2}{e^{\pi y} + e^{-\pi y}} \leq \frac{2\pi^2}{N}\frac{1}{1+e^{-2\pi|y|}} e^{-\pi|y|} \quad \text{with } z = x + iy$$

and so the contribution along each of the edges is no more than

$$\frac{4\pi^2}{N}\int_0^N e^{-\pi y} dy = O(N^{-1})$$

which tends to 0 as $N \to \infty$. Thus

$$0 = \frac{\pi^2}{3} - \lim_{N\to\infty}\sum_{n=1}^{N}\frac{2}{n^2} = \frac{\pi^2}{3} - \sum_{n=1}^{\infty}\frac{2}{n^2}$$

so that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

as claimed. □

8 Fall 2022

Fall 2022 Problem 1. Let $f \in L^1(\mathbb{R}^d)$ and let $a \in \mathbb{R}^d \setminus \{0\}$. For each $k \in \mathbb{N}$, determine the limit

$$\lim_{t\to\infty} \int \left|\sum_{j=1}^{k} f(jx+ta)\right| dx.$$

Proof. We claim that

$$\lim_{t\to\infty} \int |\sum_{j=1}^{k} f(jx+ta)|dx = (\sum_{j=1}^{k} j^{-d})\|f\|_1$$

By the triangle inequality and change-of-variable, it will suffice to show that

$$\liminf_{t\to\infty} \int |\sum_{j=1}^{k} f(jx+ta)|dx > (\sum_{j=1}^{k} j^{-d})\|f\|_1 - \varepsilon$$

for each $\varepsilon > 0$.

Fix $\varepsilon > 0$. Let $R > 0$ be sufficiently large so that

$$\int_{|x|>R} |f(x)|dx < \varepsilon$$

Let t be sufficiently large so that the R-balls centered at $-ta, -\frac{t}{2}a, -\frac{t}{3}a, \ldots, -\frac{t}{k}a$ (which we write as $B_R(\frac{t}{j}a)$) are all disjoint. We may also take t sufficiently large so that $|ta| > k(k+1)R$. Then

$$\begin{aligned}\int |\sum_{j=1}^{k} f(jx+ta)|dx &= \sum_{j=1}^{k}\int_{B_R(-\frac{t}{j}a)} |\sum_{j=1}^{k} f(jx+ta)|dx \\ &\quad + \int_{\mathbb{R}^d\setminus\bigcup_{j=1}^{k} B_R(-\frac{t}{j}a)} |\sum_{j=1}^{k} f(jx+ta)|dx \\ &= \sum_{j=1}^{k} I_j + II\end{aligned}$$

We analyze each contribution separately. First, considering II,

$$\int_{\mathbb{R}^d\setminus\bigcup_{j=1}^{k} B_R(\frac{t}{j}a)} |\sum_{j=1}^{k} f(jx+ta)|dx \geq 0$$

trivially. Next, considering I_j,

$$\int_{B_R(\frac{t}{j}a)} |\sum_{j=1}^{k} f(jx+ta)|dx \geq \int_{B_R(-\frac{t}{j}a)} |f(jx+ta)|dx - \sum_{j'\neq j}\int_{B_R(-\frac{t}{j}a)} |f(j'x+ta)|dx$$

By a change-of-variable,

$$\int_{B_R(-\frac{t}{j}a)} |f(jx+ta)|dx = j^{-d}\int_{|y|<jR} |f(y)|dy \geq j^{-d}\|f\| - j^{-d}\varepsilon$$

and, if $j' \neq j$,

$$\int_{B_R(-\frac{t}{j}a)} |f(j'x+ta)|dx = (j')^{-d} \int_{B_{j'R}((1-\frac{j'}{j})ta)} |f(y)|dy$$

Observe the containment

$$B_{j'R}((1-\frac{j'}{j})ta) \subseteq \{x : |x| > \frac{1}{k}ta - j'R\} \subseteq \mathbb{R}^d \setminus B_R$$

where we have used the second assumption on t. Thus

$$\int_{B_R(-\frac{t}{j}a)} |f(j'x+ta)|dx < (j')^{-d}\varepsilon$$

We conclude that

$$\int_{B_R(\frac{t}{j}a)} |\sum_{j'=1}^{k} f(j'x+ta)|dx \geq j^{-d}\|f\|_1 - \sum_{j'=1}^{k}(j')^{-d}\varepsilon$$

and so

$$\liminf_{t\to\infty} \int |\sum_{j=1}^{k} f(jx+ta)|dx \geq \|f\|_1(\sum_{j=1}^{k} j^{-d}) - \varepsilon k \sum_{j=1}^{k} j^{-d}$$

Since $\varepsilon > 0$ was arbitrary, we must have

$$\liminf_{t\to\infty} \int |\sum_{j=1}^{k} f(jx+ta)|dx \geq (\sum_{j=1}^{k} j^{-d})\|f\|_1$$

and so we have concluded that

$$\lim_{t\to\infty} \int |\sum_{j=1}^{k} f(jx+ta)|dx = (\sum_{j=1}^{k} j^{-d})\|f\|_1$$

□

Fall 2022 Problem 2. Let $f \in L^p(\mathbb{R}^d)$, for some $1 \leq p < 2$. Show that the series

$$\sum_{n=1}^{\infty} \frac{f(x+n)}{\sqrt{n}}$$

converges absolutely for almost all $x \in \mathbb{R}$. For each $2 \leq p \leq \infty$, give an example of a function $f \in L^p(\mathbb{R})$, give an example of a function $f \in L^p(\mathbb{R})$ for which the series diverges for every $x \in \mathbb{R}$.

Proof. Let $f \in L^p(\mathbb{R})$ with $1 \leq p < 2$. Let $k \in \mathbb{Z}$ be arbitrary. Then, writing p' for the dual exponent, we have

$$\begin{aligned}\int_k^{k+1} (\sum_{n=1}^{\infty} \frac{f(x+n)}{\sqrt{n}})^p dx &\leq \int_k^{k+1} (\sum_{n=1}^{\infty} |f(x+n)|^p)(\sum_{n=1}^{\infty} n^{-\frac{p'}{2}})^{p/p'} dx \quad \text{by Hölder} \\ &\lesssim_p \int_k^{k+1} \sum_{n=1}^{\infty} |f(x+n)|^p dx \quad \text{since } p' > 2 \\ &= \int_{\mathbb{R}} |f(x)|^p dx < \infty\end{aligned}$$

so certainly $\sum_{n=1}^{\infty} \frac{f(x+n)}{\sqrt{n}}$ converges for a.e. $x \in [k, k+1]$. Since $k \in \mathbb{Z}$ was arbitrary and the countable union of null sets is null, we conclude that the series converges for a.e. $x \in \mathbb{R}$, as claimed.

For the next part, consider the function

$$f(x) = \frac{1}{\log x \sqrt{x}} 1_{x > e+1}$$

It is clear that $f \in L^\infty(\mathbb{R})$; on the other hand, by a change-of-variable, one can verify that $f \in L^2(\mathbb{R})$ as well, so by Hölder we see that $f \in L^p(\mathbb{R})$ for each $2 \leq p \leq \infty$. It will suffice to show that the series in question diverges for this particular f.

Indeed, for $x > e+1$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\log(x+n)\sqrt{x+n}} \frac{1}{\sqrt{n}} &\geq \sum_{n > x} \frac{1}{\log(x+n)\sqrt{x+n}} \frac{1}{\sqrt{n}} \\ &\geq \sum_{n > x} \frac{1}{\log(2n) n \sqrt{2}}\end{aligned}$$

which diverges by an integral comparison.

□

Fall 2022 Problem 3. Fix $K > 0$ and let $\mathcal{M}_K(\mathbb{R}^d)$ denote the space of finite positive Borel measures μ on $\mathbb{R}^d$ with $\mu(\mathbb{R}^d) \leq K$. When $\mu_1, \mu_2 \in \mathcal{M}_K(\mathbb{R}^d)$, write $\mu_1 \leq \mu_2$ if $\mu_1(U) \leq \mu_2(U)$ for each Borel set $U \subseteq \mathbb{R}^d$. Show that the set

$$\{(\mu_1, \mu_2) \in \mathcal{M}_K(\mathbb{R}^d) \times \mathcal{M}_K(\mathbb{R}^d) : \mu_1 \leq \mu_2\}$$

is compact for the weak-$*$ topology. Here $\mathcal{M}_K(\mathbb{R}^d)$ is viewed as a subset of the space of finite Borel measures on $\mathbb{R}^d$, equipped with the weak-$*$ topology.

Proof. Write the set under consideration as A. Denote also $\mathcal{M}(\mathbb{R}^d)$ the space of finite positive Borel measures on $\mathbb{R}^d$. We argue first that A is pre-compact, and second that A is closed.

Step 1: Observe that $A \subseteq (K\mathcal{M}_1(\mathbb{R}^d)) \times (K\mathcal{M}_1(\mathbb{R}^d))$. By Prokhorov's theorem, $\mathcal{M}_1(\mathbb{R}^d)$ is weak-$*$ pre-compact in $\mathcal{M}(\mathbb{R}^d)$; since scalar multiplication is weak-$*$ continuous, we see that $K\mathcal{M}_1(\mathbb{R}^d)$ is weak-$*$ pre-compact. Finally, abstractly we have that the product of pre-compact sets is pre-compact in the product topology, so $(K\mathcal{M}_1(\mathbb{R}^d)) \times (K\mathcal{K}_1(\mathbb{R}^d))$ is pre-compact in $\mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d)$. A subset of a pre-compact set is pre-compact, so A is pre-compact.

Step 2: Suppose $\{(\mu_1^{(n)}, \mu_2^{(n)})\}_{n=1}^{\infty} \subseteq A$ converge weak-$*$'ly to $(\nu_1, \nu_2) \in \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d)$. Then, for each $f \in C_c(\mathbb{R}^d)$ nonnegative,

$$\begin{aligned}\int f d\nu_2 &= \lim_{n\to\infty} \int f d\mu_2^{(n)} \\ &\geq \limsup_{n\to\infty} \int f d\mu_1^{(n)} \\ &= \int f d\nu_1\end{aligned}$$

so that $\int f d\nu_1 \leq \int f d\nu_2$ for all $f \in C_c(\mathbb{R}^d)$ nonnegative. If $U \subseteq \mathbb{R}^d$ is Borel, then by Urysohn and Borel regularity there is a sequence $k \mapsto f_k \in C_c(\mathbb{R}^d)$ such that $f_k \geq 1_U$ for each k, and

$$\lim_{k\to\infty} \int f_k d\nu_1 = \nu_1(U), \quad \lim_{k\to\infty} \int f_k d\nu_2 = \nu_2(U)$$

Since each $\int f_k d\nu_1 \leq \int f_k d\nu_2$, it follows that $\nu_1(U) \leq \nu_2(U)$. Thus $\nu_1 \leq \nu_2$. Finally, observe that, for a function $f : \mathbb{R}^d \to \mathbb{R}$ with $f \in C_c(\mathbb{R}^d)$, $0 \leq f \leq 1$, and $f \equiv 1$ on $B(0,1)$,

$$\nu_i(\mathbb{R}^d) = \lim_{R\to\infty} \int f(R^{-1}x) d\nu_i(x) = \lim_{R\to\infty} \lim_{n\to\infty} \int f(R^{-1}x) d\mu_i^{(n)}(x) \leq K, \quad i = 1, 2$$

so indeed $(\nu_1, \nu_2) \in A$, as was to be shown.

$\square$

Fall 2022 Problem 4. Suppose $2 \leq p < \infty$. If μ, ν are positive measures on $\mathbb{R}^d$ and f, f_j are finitely many functions in $L^p(\mathbb{R}^d)$ satisfying $f = \sum_j f_j$, the inequality

$$\|f\|_{L^p(\mu)} \leq M \left(\sum_j \|f_j\|^2_{L^p(\nu)} \right)^{1/2} \tag{11}$$

is called an $\ell^2 L^p$ decoupling inequality and $M > 1$ is called the decoupling constant. Show that if $\mu = \sum_k \mu_k$ and $\nu = \sum_k \nu_k$, where the sums are finite, and the $\ell^2 L^p$ decoupling inequalities hold with decoupling constant M,

$$\|f\|_{L^p(\mu_k)} \leq M \left(\sum_j \|f_j\|^2_{L^p(\nu_k)} \right)^{1/2}$$

for all k, then 11 holds with the same decoupling constant M.

Proof. By Minkowski,

$$\begin{aligned}
\|f\|^p_{L^p(\mu)} &= \sum_k \|f\|^p_{L^p(\mu_k)} \\
&\leq M^p \sum_k (\sum_j \|f_j\|^2_{L^p(\nu_k)})^{p/2} \\
&\leq M^p (\sum_j (\sum_k \|f_j\|^p_{L^p(\nu_k)})^{2/p})^{p/2} \\
&= M^p (\sum_j \|f_j\|^2_{L^p(\nu)})^{p/2}
\end{aligned}$$

i.e.

$$\|f\|_{L^p(\mu)} \leq M (\sum_j \|f_j\|^2_{L^p(\nu)})^{1/2}$$

$\square$

Fall 2022 Problem 5. Let us define $Tf(x) = \int_0^x f(t)dt$, $x \in [0,1]$, for $f \in L^2([0,1])$.

(a) Prove that $T : L^2([0,1]) \to L^2([0,1])$ is a linear continuous map which is *compact*, in the sense that for any bounded sequence $f_n \in L^2([0,1])$, the sequence Tf_n has a convergent subsequence in $L^2([0,1])$.

(b) Prove that T has no eigenvalues, i.e. prove that there is no $\lambda \in \mathbb{C}$ such that $Tf = \lambda f$ for some nonzero $f \in L^2([0,1])$.

(c) Show that the spectrum of T is $\{0\}$, i.e. show that the map $f \mapsto Tf - \lambda f$ is an isomorphism of $L^2([0,1])$ for each $0 \neq \lambda \in \mathbb{C}$, and that it is not an isomorphism of $L^2([0,1])$ for $\lambda = 0$.

Proof. (a): Fix any sequence $\{f_n\}_{n=1}^\infty$ which are in the closed unit ball in $L^2[0,1]$, say. Then, for each $x \in [0,1]$, by Hölder we have

$$|Tf_n(x)| \leq \|f_n\|_2 \leq 1$$

so $\{Tf_n\}_{n=1}^\infty$ is uniformly bounded. For each ε, if we set $\delta = \varepsilon^2$ then we see that, for any $|x-y| < \delta$ with $x < y$, then

$$|\int_x^y f_n(t)dt| \leq |x-y|^{1/2}\|f_n\|_2 < \delta^2 = \varepsilon$$

i.e. $|Tf_n(x) - Tf_n(y)| < \varepsilon$. Thus $\{Tf_n\}_{n=1}^\infty$ is equicontinuous. By Arzelà-Ascoli, we may find a subsequence $\{f_{n_k}\}_{n=1}^\infty$ for which $\{Tf_{n_k}\}_{k=1}^\infty$ converges uniformly to some continuous f. But then

$$\|Tf_{n_k} - f\|_2 \leq \|Tf_{n_k} - f\|_\infty \overset{k\to\infty}{\longrightarrow} 0$$

so $\{Tf_{n_k}\}_{k=1}^\infty$ converges in $L^2[0,1]$. Thus $T(B_1)$ is a compact subset of $L^2[0,1]$, so T is a compact operator, as claimed.

(b): Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T with eigenfunction f. Then, for any $n \in \mathbb{N}$ and any $x \in [0,1]$,

$$\begin{aligned}|\lambda^n f(x)| = |T^n f(x)| &= |\int_{0 \leq x_1 \leq \cdots \leq x_n \leq x} f(x_1)dx_1 \cdots dx_n| \\ &\leq \|f\|_2 (\frac{x}{n!})^{1/2}\end{aligned}$$

by Hölder. Thus, if we take $\|f\|_2 = 1$, we have

$$|\lambda| \leq \frac{1}{(n!)^{1/2n}} \to 0$$

by trivial estimates. Therefore the only potential eigenvalue is $\lambda = 0$. On the other hand, if $f \neq 0$ a.e., then either $\mathrm{Re}(f), \mathrm{Im}(f) \neq 0$ a.e., so we may assume f is real and nontrivial; by multiplying by -1 we may also assume that $f > 0$ for a positive-measure subset of $[0,1]$. Say $A_n := \{x : f(x) > \frac{1}{n}\}$ has positive measure. Then by Lebesgue differentiation we may find $x \in A_n$ such that

$$\int_{x-\delta}^{x+\delta} f(x)dx > 0$$

But then $Tf(x-\delta) \neq Tf(x+\delta)$. Since Tf is continuous, we conclude that Tf is not a.e. 0. Since f was arbitrary nonzero, we conclude that 0 is not an eigenvalue. Thus T has no eigenvalues, as claimed.

(c): By (b), $T - \lambda$ is injective for every λ. By (a), T is not surjective (since T is compact, hence bounded, if it were surjective then by the open mapping theorem we would have $T(B_1^\circ)$ is open and, by (a), precompact, which cannot happen since $L^2[0,1]$ is infinite dimensional). Thus it remains to show that $T - \lambda$ is surjective for each $\lambda \neq 0$.

We note that range$(T-\lambda)$ contains all polynomials. First, it contains constants, since

$$(T-\lambda)[e^{\lambda^{-1}x}] = \lambda e^{\lambda^{-1}x} - 1 - \lambda e^{\lambda^{-1}x} = -1$$

Next, by induction it contains the monomials, as for each $n \geq 1$ we have

$$T(x^{n-1}) = \frac{1}{n}x^n - \lambda x^{n-1}$$

and the base case is covered by the argument for the constants above. Thus indeed $P[x] \subseteq$ range$(T-\lambda)$. We claim that the range of $T-\lambda$ is closed.

To see this, note that for each g we have

$$\langle Tg, g\rangle = \int_{0\leq y\leq x\leq 1} g(x)g(y)dydx = \frac{1}{2}\int_{[0,1]^2} g(x)g(y)dxdy = \frac{1}{2}(\int g(x))^2 \geq 0$$

so that, for each f,

$$\|Tf\|_2^2 + \lambda^2\|f\|_2^2 - 2\mathrm{Re}\lambda\langle Tf, f\rangle = \langle Tf - \lambda f, Tf - \lambda f\rangle \geq 0$$

implies

$$\|Tf\|_2^2 + \lambda^2\|f\|_2^2 \geq \frac{1}{2}(\int f)^2$$

□

Fall 2022 Problem 6. Let $E = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \in \mathbb{Q}\}$. Show that E does not contain a set of the form $A_1 \times A_2$, where $A_1, A_2 \subseteq \mathbb{R}$ are measurable, both of positive Lebesgue measure.

Proof. Let A_1, A_2 be Lebesgue measurable with finite positive measure. It suffices to show that E does not contain $A_1 \times A_2$. Set $f(x) = 1_{A_1} * 1_{-A_2}(x)$. Then

$$\int_{\mathbb{R}} f(x)dx = \int 1_{-A_2}(y)\int 1_{A_1}(x-y)dxdy = m(A_1)m(A_2) > 0$$

In particular, there is some point $x \in \mathbb{R}$ for which $f(x) > 0$. On the other hand, f is continuous (which we don't prove here), so $f(z) > 0$ on an interval $z \in (x-\delta, x+\delta)$, which by the definition of f implies

$$(x-\delta, x+\delta) \subseteq A_1 - A_2$$

and certainly $(x-\delta, x+\delta)$ contains rational numbers. Thus E does not contain $A_1 \times A_2$, as claimed. □

Fall 2022 Problem 7. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(a) Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic injective and $g : \mathbb{D} \to \mathbb{C}$ be holomorphic such that $g(0) = f(0)$ and $g(\mathbb{D}) \subseteq f(\mathbb{D})$. Show that $g(\mathbb{D}_r) \subseteq f(\mathbb{D}_r)$, for each $0 < r < 1$. Here $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$.

(b) Let $g : \mathbb{D} \to \mathbb{C}$ be holomorphic such that $g(0) = 0$ and $|\mathrm{Re}\, g(z)| < 1$ for all $z \in \mathbb{D}$. Show that

$$|\mathrm{Im}\, g(z)| \leq \frac{2}{\pi}\log\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}$$

Proof. (a): Write $h = f^{-1} \circ g$. Then h maps $\mathbb{D}$ to $\mathbb{D}$ holomorphically and $h(0) = 0$. By the Schwartz lemma,

$$|h(se^{i\theta})| \leq s \quad \forall se^{i\theta} \in \mathbb{D}$$

i.e.

$$f^{-1}(g(\overline{\mathbb{D}_s})) \subseteq \overline{\mathbb{D}_s}$$

Taking a union over all $s < r$, we see

$$f^{-1}(g(\mathbb{D}_r)) \subseteq \mathbb{D}_r$$

and so

$$g(\mathbb{D}_r) \subseteq f(\mathbb{D}_r)$$

as claimed.

(b): Write $f(z) = \frac{2i}{\pi}\mathrm{Log}(\frac{1+z}{1-z})$, where Log denotes the principal logarithm. Note that for each $|z| < 1$ one has that $\frac{1+z}{1-z}$ is in the right half-plane, so this is well-defined. The map $z \mapsto \frac{1+z}{1-z}$ is a biholomorphic map from $\mathbb{D}$ to the right half-plane, so the image of f is the set $\{w \in \mathbb{C} : |\mathrm{Re}(w)| < 1\}$. Since $f(0) = 0$, by the previous part we have

$$g(\mathbb{D}_r) \subseteq f(\mathbb{D}_r)$$

for all $0 < r \leq 1$. In particular, for each $|z| = r$,

$$|\mathrm{Im}(g(z))| \leq \sup_{|w| \leq r} |\mathrm{Im}(f(z))|$$

On the other hand,

$$\mathrm{Im}(f(w)) = \frac{2}{\pi} \log |\frac{1+w}{1-w}|$$

By the triangle and reverse triangle inequalities,

$$|\frac{1+w}{1-w}| \leq \frac{1+|w|}{1-|w|}$$

and so, for any $|w| \leq r$,

$$|\mathrm{Im}(f(w))| \leq \frac{2}{\pi} \log \frac{1+r}{1-r}$$

and hence

$$|g(z)| \leq \frac{2}{\pi} \log \frac{1+|z|}{1-|z|}$$

as claimed. □

Fall 2022 Problem 8. Show that

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2},$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$, with the series in the right hand side converging uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

Proof. We start by computing the residues of $\frac{\pi}{\sin \pi z}$ at $z \in \mathbb{Z}$. If $n \in \mathbb{Z}$ is even, then

$$\frac{\pi}{\sin \pi(n+z)} = \frac{\pi}{\sin \pi z}$$

so it remains only to compute the residues at $\{0, 1\}$. To handle the former case,

$$\operatorname{Res}\left[\frac{\pi}{\sin \pi z}, 0\right] = \left.\frac{\pi}{\pi \cos \pi z}\right|_{z=0} = 1$$

and, in the latter case,

$$\operatorname{Res}\left[\frac{\pi}{\sin \pi z}, 1\right] = \left.\frac{\pi}{\pi \cos \pi z}\right|_{z=1} = -1$$

Consequently,

$$\operatorname{Res}\left[\frac{\pi}{\sin \pi z}, n\right] = (-1)^n \quad \forall n \in \mathbb{Z}$$

and we may observe that $\frac{\pi}{\sin \pi z}$ has no other singularities in $\mathbb{C}$.

We apply this to a Cauchy integral for the function under consideration. Let $N \in \mathbb{N}$ and define the paths as follows: $\gamma_N^{(1)}$ is the straight-line path from $-(N+\frac{1}{2}) - iN$ to $(N+\frac{1}{2}) - iN$; $\gamma_N^{(2)}$ is the straight-line path from $(N+\frac{1}{2}) + iN$; $\gamma_N^{(3)}$ is the straight-line path from $(N+\frac{1}{2}) + iN$ to $-(N+\frac{1}{2}) + iN$; $\gamma_N^{(4)}$ is the straight-line path from $-(N+\frac{1}{2}) - iN$. Lastly, use Γ_N to denote the concatenated path formed by $\gamma_N^{(1)} \to \gamma_N^{(2)} \to \gamma_N^{(3)} \to \gamma_N^{(4)}$.

If $z \in \mathbb{C} \setminus \mathbb{Z}$ is enclosed by Γ_N, then

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\pi}{(w-z)\sin \pi w} dw = \frac{\pi}{\sin \pi z} + \sum_{|n| \leq N} \frac{(-1)^n}{n-z}$$

We claim that the left-hand side has limit 0 as $N \to \infty$; we may assume that $|z| \leq \frac{1}{2}N$. We control the contributions along the $\gamma_N^{(i)}$. First we study $\gamma_N^{(1)}$. For any w along $\gamma_N^{(1)}$, we may write $w = x - iN$ with $-(N+\frac{1}{2}) \leq x \leq N + \frac{1}{2}$. Then

$$\frac{1}{\sin \pi w} = \frac{2i}{e^{\pi i(x-iN)} - e^{-\pi i(x-iN)}} = e^{-\pi N} \frac{2i}{e^{\pi i x} - e^{-\pi i x} e^{-2\pi N}}$$

so that

$$\left|\frac{\pi}{(w-z)\sin \pi w}\right| \leq \frac{2\pi}{N} e^{-\pi N} \frac{1}{1 - e^{-2\pi N}} \leq \frac{2\pi}{N} e^{-\pi N}(1 + 2e^{-2\pi N}) \leq \frac{4\pi}{N} e^{-\pi N}$$

by elementary estimates. Similarly, along $\gamma_N^{(3)}$ we have

$$\left|\frac{\pi}{(w-z)\sin \pi w}\right| \leq \frac{4\pi e^{-\pi N}}{N}$$

so that

$$\int_{\gamma_N^{(1)} \cup \gamma_N^{(3)}} \left|\frac{\pi}{(w-z)\sin \pi w}\right| |dw| \leq 16\pi e^{-\pi N}$$

Along $\gamma_N^{(2)}$, we have $w = (N+\frac{1}{2}) + iy$ with $-N \leq y \leq N$. Then

$$\frac{1}{\sin \pi w} = \frac{2i}{e^{\pi i((N+\frac{1}{2})+iy)} - e^{-\pi i((N+\frac{1}{2})+iy)}} = \frac{2(-1)^N}{e^{-\pi y} + e^{\pi y}}$$

so that, by elementary estimates,

$$\left|\frac{\pi}{(w-z)\sin\pi w}\right| \leq \frac{4\pi}{N}e^{-\pi|y|}$$

and

$$\int_{\gamma_N^{(2)}} \left|\frac{\pi}{(w-z)\sin\pi w}\right| |dw| \leq 2\int_0^{N^{1/2}} \frac{4\pi}{N}e^{-\pi t}dt + 2\int_{N^{1/2}}^{N} \frac{4\pi}{N}e^{-\pi t}dt$$

which may be bounded as

$$\int_0^{N^{1/2}} \frac{1}{N}e^{-\pi t}dt \leq N^{-1/2}$$

$$\int_{N^{1/2}}^{N} \frac{1}{N}e^{-\pi t}dt \leq e^{-\pi N^{1/2}}$$

so that

$$\int_{\gamma_N^{(2)}} \left|\frac{\pi}{(w-z)\sin\pi w}\right| |dw| \leq 8\pi\left(N^{-1/2}+e^{-\pi N^{1/2}}\right)$$

By the same token (e.g. directly using the symmetry on $\sin$),

$$\int_{\gamma_N^{(4)}} \left|\frac{\pi}{(w-z)\sin\pi w}\right| |dw| \leq 8\pi\left(N^{-1/2}+e^{-\pi N^{1/2}}\right)$$

so we conclude

$$\int_{\Gamma_N} \left|\frac{\pi}{(w-z)\sin\pi w}\right| |dw| \leq 16\pi\left(e^{-\pi N}+N^{-1/2}+e^{-\pi N^{1/2}}\right)$$

which implies

$$\lim_{N\to\infty}\int_{\Gamma_N} \frac{\pi}{(w-z)\sin\pi w}dw = 0$$

and hence

$$\frac{\pi}{\sin\pi z} = -\lim_{N\to\infty}\sum_{|n|\leq N}\frac{(-1)^n}{n-z}$$

Finally, to compute the latter limit, notice that

$$\frac{(-1)^n}{n-z}+\frac{(-1)^{-n}}{-n-z} = (-1)^n\frac{n-z+(-n-z)}{(n-z)(-n-z)} = \frac{2(-1)^{n+1}z}{z^2-n^2}$$

so that

$$\frac{\pi}{\sin\pi z} = \frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2-n^2}$$

for each particular $z\in\mathbb{C}\setminus\mathbb{Z}$. Finally, observe that for any $z\in\mathbb{C}\setminus\mathbb{Z}$ with $\mathrm{dist}(z,\mathbb{Z})>\delta$ for some $\delta\in(0,1)$, if we set $N\in\mathbb{N}$ such that $N>2|z|+1$, then for any $|w-z|\leq\delta$ we have, for $n\geq N$,

$$\left|\frac{2(-1)^n w}{w^2-n^2}\right| \leq 2|w|\frac{1}{n^2}\frac{1}{1-n^{-1}(|z|+\delta)} \leq 4|w|\frac{1}{n^2}$$

The latter is summable in n, so by the Weierstrass M-test we conclude that the series $2z\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2-n^2}$ converges uniformly on $\{w : |w-z| \leq \delta\}$. Since the series converges uniformly on a neighborhood of each point $z \in \mathbb{C}\setminus\mathbb{Z}$, we conclude that the series converges locally uniformly on $\mathbb{C}\setminus\mathbb{Z}$, as was to be shown.

□

Fall 2022 Problem 9. Let $\Omega \subseteq \mathbb{C}$ be open connected and $f_j : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions. Suppose that $f_j(a)$ converges as $j \to \infty$, for some $a \in \Omega$, and that the sequence Re f_j converges as $j \to \infty$, uniformly on compact subsets of Ω. Show that f_j converges as $j \to \infty$, uniformly on compact subsets of Ω.

Proof. Consider arbitrary compact set $K \subseteq \Omega$. Then the functions $\{\text{Re}(f_j)\}_j$ are uniformly bounded on K, and so there is some $R > 0$ such that $\text{Re}(f_j)(z) \in [-R, R]$ for each $z \in K$. Then f_j omits each value in $\mathbb{C}\setminus([-R,R]+i\mathbb{R})$ on K, and so $\{f_j\}_j$ is normal in K° (the interior of K) by Montel's theorem. Writing Ω as a union of compact sets K_n such that $K_n \subseteq K_{n+1}^\circ$, we conclude that the family $\{f_j\}_j$ is normal.

We wish to show that f_j converges uniformly on compact sets. Fix any holomorphic function f which is a limit point of some subsequence of the f_j. Suppose for the sake of contradiction that there is some compact set $K \subseteq \Omega$ and $\varepsilon > 0$ and a subsequence f_{j_k} such that $\|f - f_{j_k}\|_\infty \geq \varepsilon$ for each k. By normality, there is a further subsequence that is locally uniformly convergent; to abbreviate notation, write this subsequence as $\{g_k\}_k$. □

Fall 2022 Problem 10. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and set

$$m(r) = \frac{1}{2\pi}\int_0^{2\pi} \log_+ |f(re^{i\varphi})|d\varphi$$

Here $\log_+ t = \max(\log t, 0)$. Suppose that

$$\limsup_{r\to\infty}\frac{m(r)}{\log r} < \infty$$

Show that f is a polynomial.

Proof. Under the finiteness assumption, there are $R > 0$ and $C > 0$ such that

$$m(r) \leq C\log r, \quad \forall r \geq R$$

Fix any $a \in \mathbb{C}, a \neq -f(0)$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})+a|d\theta \leq \log(1+|a|) + \frac{1}{2\pi}\int_0^{2\pi}\log_+|f(re^{i\theta})|d\theta \leq \log(1+|a|) + C\log r$$

for each $r \geq R$ such that f does not take the value $-a$ on the circle of radius r. On the other hand, by Jensen's formula, writing n_r for the number of zeroes of $f(z)+a$ in the disk of radius r,

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})+a|d\theta = \log|f(0)+a| - \sum_{k=1}^{n_r}\log(\frac{|z_k|}{r})$$

where $\{z_k\}_{k=1}^{\infty}$ is an enumeration of the zeroes of $f(z)+a$, counting multiplicity, with increasing moduli. Then

$$n_r + \frac{\log|f(0)+a|}{\log r} - \frac{1}{\log r}\sum_{k=1}^{n_r}\log|z_k| \leq \frac{\log(1+|a|)}{\log r} + C$$

On the other hand, for each $k \leq n_r$ we have $|z_k| \leq r$, so we must have that n_r is bounded in r. Thus f attains the value a at most finitely times. Since this is true for all $a \neq -f(0)$, we see by the second Picard theorem that f is not transcendental. Thus f is a polynomial, as claimed. $\square$

Fall 2022 Problem 11. Let $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and define

$$u(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt, \quad \text{Im}\, z \neq 0$$

(a) Prove that u is a holomorphic function on $\mathbb{C}\setminus\mathbb{R}$ such that $u(z) \to 0$ as $|\text{Im}\, z| \to \infty$.

(b) Show that the limit

$$\lim_{y\to 0^+}(u(z) - u(\bar{z})), \quad z = x+iy,$$

exists for each $x \in \mathbb{R}$ and compute it.

Proof. (a): To prove the first statement, fix any $z_0 \in \mathbb{C}\setminus\mathbb{R}$; for simplicity, we assume *Im*$(z_0) > 0$. Let T be any triangle contained strictly in the upper half-plane whose interior contains z_0. Then

$$\int_{\mathbb{R}}\int_T \left|\frac{f(t)}{t-z}\right||dz|dt \leq \text{length}(T)\max_{x+iy\in T}\frac{1}{y}\int_{\mathbb{R}}|f(t)|dt < \infty$$

(where we have used compactness of T). Thus we may appeal to Fubini to obtain

$$\int_T u(z)dz = \int_{\mathbb{R}}\int_T \frac{f(t)}{t-z}dzdt = \int_{\mathbb{R}} 0dt = 0$$

Thus, by Morera's theorem, u is analytic in the upper half-plane. By an analogous argument, u is analytic in the lower half-plane.

To prove the second statement, note that

$$|u(x+iy)| \leq \frac{1}{2\pi y}\|f\|_{L^1(\mathbb{R})}$$

so certainly $u(z) \to 0$ as *Im*$(z) \to \infty$.

(b): For each z,

$$u(z) - u(\bar{z}) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(t)\left(\frac{1}{t-z} - \frac{1}{t-\bar{z}}\right)dt = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\frac{y}{(t-x)^2+y^2}dt$$

The latter integral may be divided for each $\delta > 0$ as

$$\begin{aligned}\int_{\mathbb{R}} f(t)\frac{y}{(t-x)^2+y^2}dt &= \int_{x-\delta}^{x+\delta}[f(t)-f(x)]\frac{y}{(t-x)^2+y^2}dt \\ &+ \int_{(-\infty,x-\delta]\cup[x+\delta,\infty)}[f(t)-f(x)]\frac{y}{(t-x)^2+y^2}dt \\ &+ f(x)\int_{\mathbb{R}}\frac{y}{(t-x)^2+y^2}dt \\ &= I + II + III\end{aligned}$$

We evaluate each summand separately. First, considering III,

$$\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2} dt = \frac{1}{y} \int_{\mathbb{R}} \frac{1}{(t/y)^2+1} dt = \int_{\mathbb{R}} \frac{1}{1+u^2} du = \pi$$

Next, for arbitrary $\varepsilon > 0$, pick $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ if $|t - x| < \delta$. Then, considering I,

$$|\int_{x-\delta}^{x+\delta} [f(t) - f(x)] \frac{y}{(t-x)^2+y^2} dt| \le \varepsilon \int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2} dt = \varepsilon\pi$$

and, considering II,

$$\int_{(-\infty,x-\delta]\cup[x+\delta,\infty)} [f(t) - f(x)] \frac{y}{(t-x)^2+y^2} dt \le 2\frac{y}{\delta^2+y^2} \|f\|_1$$

Thus, for each ε,

$$\limsup_{y\to 0^+} |u(z) - u(\bar{z}) - f(x)| \le \varepsilon$$

and hence

$$\lim_{y\to 0^+} u(z) - u(\bar{z}) = f(x)$$

□

Fall 2022 Problem 12. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and assume that f is *not* of the form $z \mapsto z + a$ for some $a \in \mathbb{C}$.

(a) Show that the composition $f \circ f$ has a fixed point.

(b) Find an entire $f : \mathbb{C} \to \mathbb{C}$ (not of the form $z \mapsto z + a$) with no fixed points.

Hint: one approach centers on the function $z \mapsto [f(f(z)) - z]/[f(z) - z]$.

Proof. (a): Write $g(z) = \frac{f(f(z))-z}{f(z)-z}$. Assuming f is entire and $f \circ f$ has no fixed points, we must have that f has no fixed points as well, so g is entire and never takes the value 0. On the other hand, if $g(z) = 1$, then $f(f(z)) - z = f(z) - z$, i.e. $f(f(z)) = f(z)$, i.e. $f(z)$ is a fixed point for f. Thus g cannot take value 1. By Picard, g is constant, so there is some $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that

$$f(f(z)) - z = \lambda(f(z) - z) \quad \forall z \in \mathbb{C}$$

i.e.

$$f(f(z)) = \lambda f(z) + (1 - \lambda)z$$

Differentiating this once we obtain

$$f'(f(z))f'(z) = \lambda f'(z) + 1 - \lambda$$

which immediately implies f' is nowhere vanishing, since $\lambda \neq 1$. But then the left-hand side of the preceding also never vanishes, so that $\lambda f'(z) + 1 - \lambda$ never takes the value 0; that is,

$$f'(z) \neq \frac{\lambda - 1}{\lambda} \quad \forall z$$

Since $\lambda \notin \{0, 1\}$, the right-hand side is some element of $\mathbb{C} \setminus \{0\}$. In particular, f' also omits this value in addition to 0, so by Picard once again we see that f' is constant. Thus f is affine, as was to be shown.

(b): The function $g(z) = e^z$ has no zeroes on $\mathbb{C}$, so the entire function $f(z) = e^z + z$ has no fixed points. □

9 Spring 2023

Spring 2023 Problem 1. Let P be the set of all Borel probability measures on $[0,1]$ and let m be Lebesgue measure.

(a) Prove that $\mu \in P$ satisfies $\mu \ll m$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int f d\mu < \varepsilon \quad \text{whenever} \quad f \in C([0,1]),\ 0 \leq f \leq 1 \text{ and } \int f dm < \delta$$

[Hint: $\mu + m$ is a measure, so there are results about approximating one kind of function with another in $L^1(\mu + m)$.]

(b) Give P the weak-$*$ topology via the Riesz representation theorem identifying $C([0,1])^*$, and let

$$A = \{\mu \in P : \mu \ll m\}.$$

Prove that A is a Borel subset of P for the weak-$*$ topology.

Proof. (a): First assume that $\mu \in P$ satisfies the condition described above w.r.t. continuous functions. We verify a (well-known) equivalent criterion for absolute continuity, namely

$$\forall \varepsilon > 0\, \exists \delta > 0 \quad \text{s.t.} \quad \forall A \subseteq [0,1] \text{ Borel}, m(A) < \delta \implies \mu(A) < \varepsilon \tag{12}$$

Let $\varepsilon > 0$ be as in 12, and define $\delta = \frac{1}{2}\delta'$, where δ' is the quantity in the problem assumption associated to $\frac{1}{2}\varepsilon$. Suppose $A \subseteq [0,1]$ has $m(A) < \delta$. By regularity of Lebesgue measure, there exists $U \supseteq A$ open such that $m(U) < \delta'$.

For each $n \in \mathbb{N}$, let

$$U_{1/n} = \{x \in U : \operatorname{dist}(x, \mathbb{R} \setminus U) \geq \frac{1}{n}\}$$

Since $x \mapsto \operatorname{dist}(x, \mathbb{R} \setminus U)$ is 1-Lipschitz (by the triangle inequality), the set $U_{1/n}$ is closed and disjoint from $\mathbb{R} \setminus U$. By Urysohn, we may find $f_n : [0,1] \to \mathbb{R}$ continuous such that $f_n|_{U_{1/n}} \equiv 1$ and $f_n|_{\mathbb{R}\setminus U} \equiv 0$. This $\lim_n f_n \equiv 1_U$ pointwise. By the dominated convergence theorem,

$$\lim_{n\to\infty} \int f_n d\mu = \mu(U)$$

whereas

$$\int f_n dm \leq \int_U dm < \delta'$$

so that

$$\int f_n d\mu < \frac{1}{2}\varepsilon'$$

for every n. Putting the last three together, we see

$$\mu(U) \leq \frac{1}{2}\varepsilon' < \varepsilon$$

and hence $\mu \ll m$, as was to be shown.

Next assume that $\mu \in P$ is such that $\mu \ll m$. It follows that $\mu + m \ll m$, so it follows that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \quad \text{s.t.} \, \forall A \subseteq [0,1], m(A) < \delta \implies \mu(A) < \varepsilon \tag{13}$$

We claim that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \quad \text{s.t.} \, \forall f : [0,1] \to [0,1] \text{ simple}, \int f dm < \delta \implies \int f d\mu < \varepsilon \tag{14}$$

To prove the claim, let $\phi : (0,1] \to (0,1]$ be a strictly increasing continuous function such that $\lim_{\varepsilon \to 0^+} \phi(\varepsilon) = 0$ and

$$A \subseteq [0,1] \text{ Borel}, m(A) < \phi(\varepsilon) \implies \mu(A) < \varepsilon$$

We may assume that $\phi(1) = 1$, so that ϕ is invertible. Take $f : [0,1] \to [0,1]$ simple. Write $f = \sum_{i=1}^{n} c_i 1_{A_i}$ for pairwise disjoint Borel sets $A_i \subseteq [0,1]$ and distinct constants $c_i \in (0,1]$. For $k \geq 1$, let

$$\mathcal{C}_k = \{1 \leq i \leq n : 2^{-k} < c_i \leq 2^{-k+1}\}$$

and

$$A_{\sim k} = \bigcup_{i \in \mathcal{C}_k} A_i$$

Suppose that $\int f dm < \delta$. In particular, for each $k \geq 0$, we have $2^{-k} m(A_{\sim k}) \leq \delta$. Thus, if $k < \log_2(\delta^{-1})$,

$$m(A_{\sim k}) \leq 2^k \delta < 1$$

and so

$$\mu(A_{\sim k}) \leq \phi^{-1}(2^k \delta)$$

from which it follows

$$\begin{aligned}
\int f d\mu &\lesssim \sum_{k \geq 1} 2^{-k} \mu(A_{\sim k}) \\
&= \sum_{k=0}^{\log_2(\delta^{-1})} 2^{-k} \mu(A_{\sim k}) + \sum_{k > \log_2(\delta^{-1})} 2^{-k} \mu(A_{\sim k}) \\
&< \sum_{k=0}^{\log_2(\delta^{-1})} 2^{-k} \phi^{-1}(2^k \delta) + \delta \\
&\lesssim \int_{\delta}^{1} \phi^{-1}(\delta t^{-1}) dt + \delta \\
&= \int_{\delta}^{\delta^{1/2}} \phi^{-1}(\delta t^{-1}) dt + \int_{\delta^{1/2}}^{1} \phi^{-1}(\delta t^{-1}) dt + \delta \\
&\leq \delta^{1/2} + \phi^{-1}(\delta^{1/2}) + \delta
\end{aligned}$$

Consequently, if we take $C > 1$ sufficiently large and define $\psi : (0,1] \to (0,1]$ such that, for any $\varepsilon > 0$, the number $\delta := \psi(\varepsilon)$ satisfies

$$\delta^{1/2} + \phi(\delta^{1/2}) + \delta \leq C^{-1} \varepsilon$$

then we conclude

$$f : [0,1] \to [0,1] \text{ simple}, \int f dm < \psi(\varepsilon) \implies \int f d\mu < \varepsilon$$

which is the desired claim 14. Finally, approximating continuous functions by simple functions, the desired result follows.

(b): For each $k \in \mathbb{N}$, define A_k to be the set

$$A_k = \{f \in C([0,1]) : \|f\|_\infty \leq 1, f \geq 0, \int f dm \leq \frac{1}{k}\}$$

Observe that $C([0,1])$ is a separable metric space, so A_k is separable with the subspace topology. Let $\{f_m^{(k)}\}_{m=1}^\infty$ be a countable dense subset of A_k, for each k.

Suppose $\mu \in P$ is such that $\int f_m^{(k)} d\mu \leq \frac{1}{n}$ for all n. We claim that $\int f d\mu \leq \frac{1}{n}$ whenever $f \in A_k$. Indeed, for $f \in A_k$, write $\{f_{m_j}^{(k)}\}_{j=1}^\infty$ for a subsequence such that $f_{m_j}^{(k)} \xrightarrow{j \to \infty} f$ in $C([0,1])$. Then

$$\limsup_{j \to \infty} \int |f - f_{m_j}^{(k)}| d\mu \leq \limsup_{j \to \infty} \|f - f_{m_j}^{(k)}\|_\infty = 0$$

so that $\int f d\mu \leq \limsup_{j \to \infty} \int f_{m_j}^{(k)} \leq \frac{1}{n}$, as was to be shown.

Trivially, if $\int f d\mu \leq \frac{1}{n}$ for all $f \in A_k$, then $\int f_m^{(k)} d\mu \leq \frac{1}{n}$ for all m. As a consequence, if we define a set B as

$$B = \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \{\mu \in P : \forall f \in A_k, \int f d\mu \leq \frac{1}{n}\}$$

then we have

$$B = \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \bigcap_{m=1}^\infty \{\mu \in P : \int f_m^{(k)} d\mu \leq \frac{1}{n}\}$$

Write the set inside the latter display as $B_{m,k,n} \subseteq P$. Then $B_{m,k,n}$ is of the form $\mathrm{ev}_f^{-1}(-\infty, \frac{1}{n}]$, where $\mathrm{ev}_f : P \to \mathbb{R}$ is defined as $\mathrm{ev}_f(\mu) = \int f d\mu$. This is clearly a weak-$*$ Borel subset of P, so we conclude that B is a weak-$*$ Borel subset of P.

Finally, by (a) we see that $B = \{\mu \in P : \mu \ll m\}$, so we have verified that the latter set is weak-$*$ Borel in P, as was to be shown.

$\square$

Spring 2023 Problem 2. Here are two Banach spaces of real-valued functions on $[0,1]$:

- Let C be the space of continuous functions with norm

$$\|f\|_C = \sup\{|f(x)| : 0 \leq x \leq 1\};$$

- Let L be the space of functions f for which the quantity

$$\|f\|_L = |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : 0 \leq x < y \leq 1\right\}$$

  is finite. These are called the **Lipschitz** functions. You may assume without proof that L is a vector space and $\|\cdot\|_L$ is a complete norm on L, and that $L \subseteq C$. Let B_C and B_L be the closed unit balls of C and L, respectively.

(a) Prove that B_L is a closed subset of C for the norm $\|\cdot\|_C$.

(b) For any $f \in C$, define
$$\Phi(f) = \int_0^1 f(x)^4 dx - \left(\int_0^1 f(x)dx\right)^4.$$
Prove that Φ attains its maximum on B_L.

(c) Prove that the functional Φ from part (b) does not attain its maximum on B_C. [Hint: we certainly have $\Phi(f) \leq 1$ for $f \in B_C$. How close can it get?]

Proof. (a): Suppose $\{f_n\}_{n\in\mathbb{N}} \subseteq B_L$ and $f_n \to f \in C$ in the norm $\|\cdot\|_C$. Then, for any $0 \leq x < y \leq 1$,
$$|f(0)| + \frac{|f(x)-f(y)|}{|x-y|} = \lim_n |f_n(0)| + \lim_n \frac{|f_n(x)-f_n(y)|}{|x-y|} \leq \limsup_n \|f_n\|_L \leq 1$$
hence
$$|f(0)| + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} : 0 \leq x < y \leq 1\right\} \leq 1$$
so that $f \in B_L$. Thus B_L is closed in C for the uniform norm, as was to be shown.

(b): By (a), B_L is closed in the uniform norm on C; we demonstrate as well that it is compact. It suffices to demonstrate that it is pre-compact. By the definition of the norm $\|\cdot\|_L$, B_L is bounded and uniformly equicontinuous. By Arzelà-Ascoli, we conclude that B_L is pre-compact, hence compact by (a).

we demonstrate that Φ is continuous with respect to the norm $\|\cdot\|_C$. Clearly
$$C \ni f \mapsto \int_0^1 f(x)dx \in \mathbb{R}$$
is continuous with respect to the uniform norm, hence
$$C \ni f \mapsto -\left(\int_0^1 f(x)dx\right)^4 \in \mathbb{R}$$
is also continuous with respect to the uniform norm. Similarly, since by Hölder we have
$$\left(\int_0^1 f(x)^4 dx\right)^{1/4} \leq \|f\|_C$$
the map $f \mapsto \|f\|_4$ is continuous, so
$$C \ni f \mapsto \int_0^1 f(x)^4 dx \in \mathbb{R}$$
is continuous as well. Thus Φ is a continuous function on C. Restricting Φ to the compact set B_L, we conclude that Φ achieves a maximum on B_L, as was to be shown.

(c): Clearly, for each $f \in B_C$,
$$\Phi(f) \leq \int_0^1 f(x)^4 dx \leq \|f\|_C^4$$

and in particular $\Phi \le 1$ on B_C. If we write f_n for the function ($n \ge 4$)

$$f_n(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ -n(x - \frac{1}{2} + \frac{1}{n}) + 1 & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ -1 & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$

then $\int_0^1 f_n(x)dx = 0$ and

$$\int f_n(x)^4 dx = (1 - \frac{2}{n}) + \frac{2}{5n}$$

Thus $\sup_{f \in B_C} \Phi(f) = 1$. It remains to demonstrate that $\Phi(f) \neq 1$ for all $f \in B_C$. Observe that, if $f \in B_C$ is such that $\int f(x)dx \neq 0$, then

$$\Phi(f) < \int_0^1 f(x)^4 dx \le 1$$

so in particular $\Phi(f) \neq 1$. Additionally, since $\Phi(f) \le \|f\|_C^4$, we need only to consider the case $\|f\|_C = 1$ and $\int_0^1 f(x)dx = 0$.

Fix such an f. In particular, $f(x) \in \{-1, 1\}$ for some $0 \le x \le 1$; up to possibly exchanging f with $-f$, we may assume $f(x) = 1$. Since f is continuous and $\int f = 0$, we must have $f < 0$ somewhere on $[0, 1]$, so by the intermediate value theorem we may find $t \in [0, 1]$ such that $f(t) = 0$. By continuity, there is $\frac{1}{10} > \delta > 0$ such that $|f| \le \frac{1}{2}$ on $(t - \delta, t + \delta)$. But then

$$\Phi(f) \le \int f(x)^4 dx = \int_{[0,1] \cap (t-\delta, t+\delta)} f(x)^4 dx + \int_{[0,1] \setminus (t-\delta, t+\delta)} f(x)^4 dx \le \frac{1}{8}\delta + (1 - \delta) < 1$$

so in particular $\Phi(f) \neq 1$. Thus Φ fails to achieve its maximum on B_C, as was to be shown. □

Spring 2023 Problem 3. Let $(X, \mathcal{M})$ be a measurable space and let V be a separable real Banach space. (Being 'separable' means that V has a countable dense subset for the norm topology.)

(a) Prove that there are dual vectors $L_1, L_2, \dots \in V^*$ such that $\|L_n\| = 1$ for every n and

$$\|v\| = \sup_n |L_n(v)| \quad \text{for all } v \in V.$$

[Warning: the separability of V does *not* imply the separability of V^* in general.]

(b) Now let $\phi : X \to V$, and assume that $L \circ \phi$ is measurable from $\mathcal{M}$ to $\mathcal{B}(\mathbb{R})$ for every $L \in V^*$. Prove that ϕ is measurable from $\mathcal{M}$ to the Borel σ-algebra generated by the norm topology of V. [Hint: start by observing that every open subset of V is a countable union of open balls.]

Proof. (a): By the Banach-Alaoglu theorem, the closed unit ball B of V^* is weak-$*$'ly metrizable compact. In particular, it is totally bounded, so by taking finite $1/n$-nets for every n we obtain a countable subset of B which is weak-$*$'ly dense in B, say $K_1, K_2, \ldots \in V^*$.

Removing 0 from the sequence $\{K_n\}_{n=1}^\infty$ if necessary, define $L_n = \frac{K_n}{\|K_n\|}$ for every n. This sequence satisfies $\|L_n\| = 1$ for every n; we claim in addition that

$$\|v\| = \sup_n |L_n(v)| \quad \text{for all } v \in V.$$

To prove this, fix any $v \in V$. By Hahn-Banach we may find $L \in V^*$ such that $\|L\| = 1$ and $|L(v)| = \|v\|$. Since $\|L\| = 1$, we have that $L \in V^*$ and thus L is a limit point of the sequence $\{K_n\}_{n=1}^\infty$, say

$$L = \lim_{k\to\infty} K_{n_k} \quad \text{weak-}*\text{'ly}$$

In particular,

$$\|v\| = |L(v)| = \lim_{k\to\infty} |K_{n_k}(v)|$$

Since $|K_{n_k}(v)| \le \|K_{n_k}\|\|v\| \le \|v\|$, we get by squeezing that $\lim_k \|K_{n_k}\| = 1$. Thus

$$|L_{n_k}(v)| = \frac{|K_{n_k}(v)|}{\|K_{n_k}\|} \to \frac{|L(v)|}{1} = \|v\|$$

and thus $\|v\|$ is a limit point of the $|L_n(v)|$. Since $|L_n(v)| \le \|L_n\|\|v\| = \|v\|$ for all n, we conclude that $\|v\| = \sup_n |L_n(v)|$, as was to be shown.

(b): As suggested by the hint, we start by noting that every open subset of V is a countable union of open balls. Indeed, if $U \subseteq V$ is open, then, writing D for a countable dense subset of V, let $\mathcal{U} = \{(x,r) \in D \times \mathbb{Q}_{>0} : B(x,r) \subseteq U\}$. Then $\mathcal{U}$ is a countable family of open balls, $\bigcup_{B\in\mathcal{U}} B \subseteq U$, and for each $y \in U$ we may find $n \in \mathbb{N}$ such that $B(y, \frac{2}{n}) \subseteq U$; since D is dense, there exists $z \in B(y, \frac{1}{n}) \cap D$, so $y \in B(z, \frac{1}{n}) \subseteq U$ by the triangle inequality and $B(z, \frac{1}{n}) \in \mathcal{U}$, so we conclude that $\bigcup_{B\in\mathcal{U}} B = U$.

In particular, if $\phi^{-1}(B) \in \mathcal{M}$ for every open metric ball $B \subseteq V$, then $\phi^{-1}(U) \in \mathcal{M}$ for every U open subset of V, so ϕ is measurable in the desired sense. Thus it suffices to verify measurability on open metric balls.

We start by verifying measurability on closed metric balls centered at 0. Given $r > 0$, we observe that $\phi^{-1}(\overline{B(0,r)}) \subseteq \bigcap_{n=1}^\infty (L_n \circ \phi)^{-1}[-r,r]$, where $\{L_n\}_{n=1}^\infty$ are the dual vectors identified in part (a). Indeed, if $\|\phi(x)\| \le r$, then $|L_n(\phi(x))| \le \|L_n\|\|\phi(x)\| \le r$ for each n.

On the other hand, if $\|\phi(x)\| > r$, then $\sup_n |L_n(\phi(x))| > r$, so there is some n such that $|L_n(\phi(x))| > r$, so $x \notin \bigcap_{n=1}^\infty (L_n \circ \phi)^{-1}[-r,r]$. Thus we have the equality

$$\phi^{-1}(\overline{B(0,r)}) = \bigcap_{n=1}^\infty (L_n \circ \phi)^{-1}[-r,r]$$

so by the hypothesis that all $L_n \circ \phi$ are measurable entails that $\phi^{-1}(\overline{B(0,r)}) \in \mathcal{M}$ for each $r > 0$. By taking countable increasing unions, we see that $\phi^{-1}(B(0,s)) \in \mathcal{M}$ for every $s > 0$.

Finally, observe for every $v \in V$ that the shift map $y \mapsto y - v$ preserves the family of open balls. Additionally, writing $\psi : X \to V$ for the map $x \mapsto \phi(x) - v$, we see that $L(\psi(x)) = L(\phi(x)) - L(v)$ is a constant shift of $L(\phi(\cdot))$, hence each $L \circ \psi$ is measurable, so in the preceding arguments we have verified that $\psi^{-1}(B(0,r)) \in \mathcal{M}$ for each $r > 0$. On the other hand, $\psi^{-1}(B(0,r)) = \phi^{-1}(B(v,r))$, so we conclude that ϕ pulls *all* metric open balls to elements of $\mathcal{M}$. By the discussion at the start of this portion, we conclude that ϕ is measurable from $\mathcal{M}$ to the Borel σ-algebra generated by the norm topology of V, as was to be shown.

□

Spring 2023 Problem 4. Let $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ be sequences in the unit ball of $L^2([0,1])$, and assume that

(i) $f_n \to f$ and $g_n \to g$ Lebesgue almost everywhere,

(ii) all these functions also lie in the unit ball of $L^p([0,1])$ for some $p \in [1,\infty]$.

For which values of p do (i) and (ii) imply that $\langle f_n, g_n\rangle \to \langle f, g\rangle$? Justify your answers with proofs or counterexamples.

Proof. We claim that the implication holds precisely when $p > 2$. We first show the negative direction, namely that if $p \le 2$ then we may find sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ such that $f_n \to f$ and $g_n \to g$ a.e., such that all these functions belong to the unit balls of L^2 and L^p. Observe that, by Hölder,

$$\|h\|_{L^p([0,1])} \le \|h\|_{L^2([0,1])}$$

so it suffices to treat the case $p = 2$. Let $f_n = g_n = \sqrt{n}1_{[0,1/n]}$ and $f = g = 0$. Then $\|f_n\|_2 = 1$ and $f_n \to f$ (similarly for the g's). On the other hand,

$$\langle f_n, g_n\rangle = 1 \not\to 0 = \langle f, g\rangle$$

so the implication indeed fails for $p = 2$, hence for all $1 \le p \le 2$.

We now assume $p > 2$ and aim to demonstrate that (i) and (ii) imply that $\langle f_n, g_n\rangle \to \langle f, g\rangle$. We first demonstrate the following claim:

Claim: For any $C > 0$ and any $\{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty$ in the balls of radius C of L^2 and L^p such that $f_n \to 0$ a.e., we have $\langle f_n, g_n\rangle \to 0$.

Proof of claim: By Egorov's theorem, for each $\varepsilon > 0$ we may find $A_\varepsilon \subseteq [0,1]$ such that $f_n \to 0$ uniformly on A_ε and $m([0,1]\setminus A_\varepsilon) < \varepsilon$. Then

$$|\int_{A_\varepsilon} f_n g_n| \le \left(\int_{A_\varepsilon} |f_n g_n|^{p/2}\right)^{2/p} m(A_\varepsilon)^{1-\frac{2}{p}} \le C\varepsilon^{1-\frac{2}{p}}$$

Thus, for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} |\int f_n g_n| \le C\varepsilon^{1-\frac{2}{p}}$$

so indeed $\langle f_n, g_n\rangle \to 0$, and the claim is proven.

To complete the problem, observe that $f_n - f \to 0$ and $g_n - g \to 0$ a.e.. By the claim, $\langle f_n - f, g_n - g\rangle \to 0$. Expanding the inner product, the cross terms $\langle f, g_n\rangle$ and $\langle f_n, g\rangle$ limit to zero as well (applying the claim again). We are done. $\square$

Spring 2023 Problem 5. Let e_1, e_2, e_3 be the usual basis of $\mathbb{R}^3$, and let $w \in \mathbb{R}^3$.

(a) Prove that there does not exist any $f \in L^2(\mathbb{R}^3)$ such that

$$f(x) = \frac{1}{6}\sum_{j=1}^{3}(f(x+e_j) + f(x-e_j)) + e^{iw\cdot x - |x|^2/2} \quad \text{for a.e. } x.$$

(b) Prove that, for any $\varepsilon > 0$, there exists $f \in L^2(\mathbb{R}^3)$ such that

$$\int \left| f(x) - \frac{1}{6}\sum_{j=1}^{3}(f(x+e_j) + f(x-e_j)) - e^{iw\cdot x - |x|^2/2}\right|^2 dx < \varepsilon$$

Proof. (a): Suppose for the sake of contradiction that such an f exists. Then the Fourier transform

$$\hat{f}(\xi) = \int e^{-2\pi i \xi \cdot x} f(x) dx$$

belongs to $L^2(\mathbb{R}^3)$ and satisfies

$$\hat{f}(\xi) = \frac{1}{6}\sum_{j=1}^{3}(e^{2\pi i \xi \cdot e_j} + e^{-2\pi i \xi \cdot e_j})\hat{f}(\xi) + e^{-|\xi - w|^2/2} \quad \text{for a.e. } \xi$$

i.e.

$$\hat{f}(\xi) = \frac{1}{3}\sum_{j=1}^{3}\cos(2\pi \xi \cdot e_j)\hat{f}(\xi) + e^{-|\xi - w|^2/2} \quad \text{for a.e. } \xi$$

This we may write as

$$\hat{f}(\xi) = \frac{e^{-|\xi-w|^2/2}}{1 - \frac{1}{3}\sum_{j=1}^{3}\cos(2\pi\xi \cdot e_j)} \quad \text{for a.e. } \xi \text{ such that the denominator is nonvanishing}$$

In particular, if $0 < |\xi| < \frac{1}{10}$, we have $\cos(2\pi\xi \cdot e_j) \geq 1 - 10|\xi \cdot e_j|^2$; thus in this regime we have

$$1 - \frac{1}{3}\sum_{j=1}^{3}\cos(2\pi\xi \cdot e_j) \leq 10|\xi|^2$$

so that

$$|\hat{f}(\xi)| \geq \frac{1}{10}e^{-|\xi-w|^2/2}|\xi|^{-2}$$

for a.e. ξ satisfying $0 < |\xi| < \frac{1}{10}$. But then

$$\int_{0<|\xi|<\frac{1}{10}} |\hat{f}(\xi)|^2 \geq \int_{0<|\xi|<\frac{1}{10}} \frac{1}{100}e^{-|\xi-w|^2}|\xi|^{-4} = +\infty$$

which contradicts our previous assertion that $\hat{f} \in L^2(\mathbb{R}^3)$. Thus we have a contradiction, as was to be shown.

(b): For $\delta > 0$, define $g = g_\delta \in L^2(\mathbb{R}^3)$ by

$$g(\xi) = \frac{e^{-|\xi-w|^2/2}}{1 - \frac{1}{3}\sum_{j=1}^{3}\cos(2\pi\xi \cdot e_j)} 1_{\mathbb{R}^3 \setminus \bigcup_{\mathbf{n}\in\mathbb{Z}^3} B(\mathbf{n},\delta)}(\xi)$$

Then

$$g(\xi) - \frac{1}{3}\sum_{j=1}^{3}\cos(2\pi\xi \cdot e_j)g(\xi) = e^{-|\xi-w|^2/2} 1_{\mathbb{R}^3 \setminus \bigcup_{\mathbf{n}\in\mathbb{Z}^3} B(\mathbf{n},\delta)}(\xi)$$

so that, writing $f = g^\vee$,

$$f(x) - \frac{1}{6}\sum_{j=1}^{3}(f(x+e_j) + f(x-e_j)) = e^{ix\cdot w - |x|^2/2} - \int e^{2\pi i x \cdot \xi} e^{-|\xi-w|^2/2} \sum_{\mathbf{n}\in\mathbb{Z}^3} 1_{B(\mathbf{n},\delta)}(\xi) d\xi$$

Then

$$\int \left| f(x) - \frac{1}{6}\sum_{j=1}^{3}(f(x+e_j)+f(x-e_j)) - e^{ix\cdot w-|x|^2/2}\right|^2 dx = \int_{\bigcup_{\mathbf{n}\in\mathbb{Z}^3} B(\mathbf{n},\delta)} e^{-|\xi-w|^2} d\xi$$

Since $e^{-|\xi-w|^2}$ is integrable in ξ, the dominated convergence theorem implies

$$\lim_{\delta\downarrow 0}\int_{\bigcup_{\mathbf{n}\in\mathbb{Z}^3} B(\mathbf{n},\delta)} e^{-|\xi-w|^2} d\xi = 0$$

Thus, for each $\varepsilon > 0$ we may find $\delta > 0$ such that $f = g_\delta^\vee$ satisfies

$$\int \left| f(x) - \frac{1}{6}\sum_{j=1}^{3}(f(x+e_j)+f(x-e_j)) - e^{ix\cdot w-|x|^2/2}\right|^2 dx < \varepsilon$$

as was to be shown.

□

Spring 2023 Problem 6. Let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous, let $f \in L^1(\mathbb{R}^2)$, and define

$$A_r f(x,y) = \frac{1}{2r}\int_{-r}^{r} f(x+s, y+\phi(x+s)-\phi(x))ds \quad ((x,y)\in\mathbb{R}^2, r>0)$$

whenever the integrand lies in $L^1(-r,r)$ as a function of s. With this definition, prove that $A_r f(x,y)$ exists for Lebesgue almost every (x,y), that it satisfies

$$\int_{\mathbb{R}^2} A_r f = \int_{\mathbb{R}^2} f \quad \text{for every } r > 0,$$

and that $A_r f \to f$ pointwise a.e. as $r \downarrow 0$.

[Hint: start by understanding the case $\phi = 0$, and then draw a picture to help you see a reduction of the general case to that one.]

Proof. [Incomplete; the last claim is only established for $\phi \equiv 0$.]

We first observe that, by Tonelli, for arbitrary $r > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^2} A_r|f| &= \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x+s,y+\phi(x+s)-\phi(x))| dydxds \\
&= \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x+s,y)| dydxds \quad \text{by translation-invariance of Lebesgue measure} \\
&= \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)| dydxds \quad \text{by translation-invariance of Lebesgue measure} \\
&= \int_{\mathbb{R}^2} |f|
\end{aligned}$$

Since the latter expression is finite, we note in particular that $A_r|f|(x,y)$ is finite for a.e. $(x,y) \in \mathbb{R}^2$; thus $A_r f(x,y)$ exists for Lebesgue a.e. (x,y). Furthermore, since the integrals in the preceding display are all finite, Fubini-Tonelli implies that the calculation remains valid when $|f|$ is replaced by f in each instance. In particular,

$$\int_{\mathbb{R}^2} A_r f = \int_{\mathbb{R}^2} f$$

for every $r > 0$.

Suppose $\phi = 0$. Let $Y \subseteq \mathbb{R}$ be the set of y such that $x \mapsto f(x, y)$ is measurable and L^1. Define the function

$$g : \mathbb{R}^2 \times (0, \infty) \to \mathbb{R}, \quad g(x, y, r) = f(x, y) - \frac{1}{2r} \int_{-r}^{r} f(x + s, y) ds$$

when $y \in Y$, and 0 otherwise. Then, for each $(x, y) \in \mathbb{R}^2$, we see that $r \mapsto g(x, y, r)$ is continuous. Consequently, we have an equality between the two sets

$$\begin{aligned} A &:= \left\{ (x, y) \in \mathbb{R}^2 : \forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall r \in \mathbb{R}_{>0} \left(r \leq \frac{1}{k} \implies |g(x, y, r)| \leq \frac{1}{n} \right) \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 : \forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall r \in \mathbb{Q}_{>0} \left(r \leq \frac{1}{k} \implies |g(x, y, r)| \leq \frac{1}{n} \right) \right\} \end{aligned}$$

Observe that A is the set of (x, y) such that $A_r f(x, y) \to f(x, y)$ as $r \to 0$. The rightmost set in the last display equals

$$\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{r \in \mathbb{Q}; 0 < r \leq \frac{1}{k}} g(\cdot, \cdot\cdot, r)^{-1} ([-1/n, 1/n])$$

Note that Fubini implies that Y is conull (hence Lebesgue measurable), so $g(\cdot, \cdot\cdot, r)$ is measurable for every $r > 0$. Consequently, A is Lebesgue measurable. By Fubini,

$$\begin{aligned} m^2(\mathbb{R}^2 \setminus A) &= \int_{\mathbb{R}^2} 1_{\mathbb{R}^2 \setminus A}(x, y) dm^2(x, y) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{\mathbb{R}^2 \setminus A}(x, y) dx dy \end{aligned}$$

For each $y \in Y$, Lebesgue differentiation implies that the set $\{x \in \mathbb{R} : (x, y) \in A\}$ is conull. Thus the last display implies

$$m^2(\mathbb{R}^2 \setminus A) = 0$$

as was to be shown.

□

Spring 2023 Problem 7. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic such that $\sup_{|z|<1} |f(z)| \leq r$, for some $r < 1$.

(a) Show that f has a fixed point $a \in \mathbb{D}$.

(b) Let

$$f^{(n)} = f \circ f \circ \cdots \circ f$$

be the n-fold iterate of f. Show that $f^{(n)} \to a$ uniformly on compact subsets of $\mathbb{D}$.

Proof. (a): By the fact $\sup_{|z|<1}|f(z)| \le r$, we have in particular that the restriction of f to $\overline{\frac{1+r}{2}\mathbb{D}}$ has range contained in $\overline{\frac{1+r}{2}\mathbb{D}}$. By Brouwer's fixed point theorem, f has a fixed point $a \in \overline{\frac{1+r}{2}\mathbb{D}} \subseteq \mathbb{D}$.

(b): We start by assuming that $a = 0$; we will remove the assumption later. Write $g(z) = \frac{1}{r-\varepsilon}f(z)$ for some $0 < \varepsilon < r$. Then $g : \mathbb{D} \to \mathbb{D}$ is holomorphic. By the Schwarz-Pick theorem we have

$$\frac{|g'(z)|}{1-|g(z)|^2} \le \frac{1}{1-|z|^2}$$

which we in particular write as

$$|g'(z)| \le \frac{1}{1-|z|^2}$$

Thus

$$|f'(z)| \le \frac{r-\varepsilon}{1-|z|^2}$$

for arbitrarily small $\varepsilon > 0$, so we have

$$|f'(z)| \le \frac{r}{1-|z|^2}$$

for all $z \in \mathbb{D}$. Integrating, we see that, for each $z \neq 0$,

$$|f(z)| \le \int_0^{|z|} |f'(tz/|z|)|dt \le \int_0^{|z|} \frac{r}{1-t^2}dt$$

In particular, if $|z| \le s$ for some $s \in (0,1)$,

$$|f(z)| \le \frac{r}{1-s^2}|z|$$

Thus $|f(z)| \le \frac{1}{1-s^2}|z|$ for all $|z| \le s$. Taking iterates,

$$|f^{(n)}(z)| \le r|f^{(n-1)}(z)| \le \cdots \le (\frac{r}{1-s^2})^n|z|$$

on the disk $|z| \le s$. Suppose s is sufficiently small so that $\frac{r}{1-s^2} < 1$. Then $f^{(n)} \to 0$ uniformly on $s\mathbb{D}$.

On the other hand, the family $A = \{f^{(n)}\}_{n=1}^{\infty}$ is a normal family, by Montel's theorem (e.g. $\{i, 1\}$ are omitted by every iterate of f). If g is a limit point of A in the topology of local uniform convergence, then $g \equiv 0$ on $s\mathbb{D}$, so g is the zero function. Thus 0 is the only limit point of the pre-compact set A, so $f^{(n)} \to 0$ locally uniformly.

Next, suppose $a \in \mathbb{D}$ is general. Write $\phi(z) = \frac{z-a}{1+\bar{a}z}$. Then, writing $h := \phi \circ f \circ \phi^{-1}$, we observe that $h(0) = 0$ and $h(\mathbb{D}) \subseteq \phi(f(\mathbb{D})) \subseteq \phi(\overline{r\mathbb{D}})$, which is a compact subset of $\mathbb{D}$, hence is contained in a closed disk of the form $\overline{\rho\mathbb{D}}$, $\rho < 1$. Consequently, h satisfies the hypotheses of f discussed previously, but with $a = 0$; we have then verified that $h^{(n)} \to 0$ locally uniformly. But

$$f^{(n)} = \phi^{-1} \circ h^{(n)} \circ \phi$$

so that $f \to a$ locally uniformly, as was to be shown. $\square$

Spring 2023 Problem 8. Let
$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$
be a conformal map from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto the domain $\{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1, |\mathrm{Im}\, z| < 1\}$. Show that $a_n = 0$ for all $n \neq 4k+1, k = 0, 1, 2 \ldots$.

Proof. Write $g(z)$ for the function $z \mapsto -if^{-1}(if(z))$. Observe that, since multiplication by i is an automorphism of $f(\mathbb{D})$, $g(z)$ is well-defined on $\mathbb{D}$ and defines an conformal self-map of $\mathbb{D}$. Furthermore, since $f(0) = 0$, we have $g(0) = 0$. Additionally,
$$g'(0) = -i(f^{-1})'(if(0))if'(0) = (f^{-1})'(0)f'(0) = 1$$
By the Schwarz lemma, $g(z) = z$; thus we have the identity
$$f(iz) = if(z)$$
In particular, plugging in to the power series for f,
$$\sum_{n=1}^{\infty} a_n i^n z^n = \sum_{n=1}^{\infty} a_n i z^n \quad \forall z \in \mathbb{D}$$
i.e.
$$\sum_{n=1}^{\infty} a_n z^n (i^n - i) \equiv 0 \quad (z \in \mathbb{D})$$
By the uniqueness of power series, we must have $a_n(i^n - i) = 0$ for all $n \geq 1$. Thus $a_n = 0$ whenever n is not of the form $4k+1, k = 0, 1, 2, \ldots$, as was to be shown.

$\square$

Spring 2023 Problem 9. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and set
$$u(z) = \log(1 + |f(z)|^2).$$
Suppose that
$$\limsup_{r\to\infty} \frac{1}{r^2} \iint_{|z|<r} u(z) dm(z) < \infty,$$
where $dm(z)$ denotes the Lebesgue measure on $\mathbb{C}$. Show that f is constant.

Proof. Write $v(z) = \log(|f(z)|^2)$. We verify that v is harmonic: indeed, writing $\partial, \overline{\partial}$ for the Wirtinger derivatives,
$$\overline{\partial}\partial v = \overline{\partial}[\frac{f'\overline{f}}{|f|^2}] = \overline{\partial}[\frac{f'}{f}] = 0$$
Then
$$\limsup_{r\to\infty} \frac{1}{r^2} \iint_{|z|<r} v(z) dm(z) \leq \limsup_{r\to\infty} \frac{1}{r^2} \iint_{|z|<r} u(z) dm(z) < \infty$$

Write D for a real number that is greater than the above limsup. By the mean value property for harmonic functions,

$$v(w) = \frac{1}{\pi r^2} \iint_{|z|<r} v(w+z)dm(z)$$

for each $w \in \mathbb{C}$. Then

$$v(w) = \frac{1}{\pi} \limsup_{r\to\infty} \frac{1}{r^2} \iint_{|z|<r} v(w+z)dm(z) \leq \frac{1}{\pi} \limsup_{r\to\infty} \frac{1}{r^2} \iint_{|z|<|w|+r} u(z)dm(z) \leq \frac{D}{\pi}$$

so that v is bounded from above. By a standard fact, this implies (since v is harmonic) that v is constant, so f is constant, as was to be shown. $\square$

Spring 2023 Problem 10. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $n \geq 1$ be an integer. A function of the form

$$B(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}, \quad z \in \mathbb{D},$$

where $a_j \in \mathbb{D}, 1 \leq j \leq n$, and $|\lambda| = 1$ is called a Blaschke product of degree n. Let $\alpha \in \mathbb{D}$. Show that the function

$$z \mapsto \frac{B(z) + \alpha}{1 + \bar{\alpha}B(z)}$$

is a Blaschke product of degree n.

Proof. Write the latter function as f. Observe that $|f(z)| < 1$ for all $z \in \mathbb{D}$, that $|f(z)| = 1$ when $|z| = 1$, and that f has poles precisely when $B(z) = 1/\bar{\alpha} \in \mathbb{C} \setminus \overline{\mathbb{D}}$. In particular, f is analytic in a neighborhood of $\overline{\mathbb{D}}$, and so has only finitely many zeroes in $\overline{\mathbb{D}}$. Since $|f(z)| = 1$ on $|z| = 1$, those zeroes all belong to $\mathbb{D}$. Consequently, the integral

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)} dz$$

is well-defined, and counts the zeroes of f. This integral takes the form

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{\frac{1-|\alpha|^2}{(1+\bar{\alpha}B(z))^2}}{\frac{B(z)+\alpha}{1+\bar{\alpha}B(z)}} B'(z)dz = \frac{1-|\alpha|^2}{2\pi i} \int_{|z|=1} \frac{B'(z)}{(1+\bar{\alpha}B(z))(B(z)+\alpha)} dz$$

which is clearly a continuous function of $\alpha \in \mathbb{D}$. Since it is equal to the number of zeroes of f, it is constant on $\mathbb{D}$, so we conclude (by moving α to 0)

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{B'(z)}{B(z)} dz = n$$

so f has n zeroes in $\mathbb{D}$, counting multiplicity. Write these as $b_1, \ldots, b_n$. Then

$$g(z) := f(z) \prod_{j=1}^{n} \frac{1 + \bar{b}_j z}{z - b_j}$$

is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, has $|g(z)| = 1$ on $|z| = 1$, and has no zeroes in $\mathbb{D}$. Thus $1/g$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ and has $|1/g(z)| = 1$ on $|z| = 1$, so $|1/g(z)| \leq 1$ on $\overline{\mathbb{D}}$, so $|g| \geq 1$

on $\overline{\mathbb{D}}$ by the maximum modulus principle. On the other hand, from $|g| \leq 1$ on $|z| = 1$ we see also $|g| \leq 1$ on $\overline{\mathbb{D}}$, so $g(\mathbb{D}) \subseteq \{|z| = 1\}$. By the open mapping theorem, g is constant, i.e. there is $|\gamma| = 1$ so that $g \equiv \gamma$. Thus

$$f(z) = \gamma \prod_{j=1}^{n} \frac{z - b_j}{1 + \bar{b}_j z}$$

so indeed f is a Blaschke product of degree n, as was to be shown.

$\square$

Spring 2023 Problem 11. Show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i a t}}{\cosh(\pi t)} dt = \frac{1}{\cosh \pi a},$$

for all $a \in \mathbb{R}$. Here $\cosh(y) = (e^y + e^{-y})/2$. Justify all manipulations.

Proof. We define the following curves, depending on an integer parameter $N \geq 1$: $\gamma_N^{(1)}$ from $-N$ to N; $\gamma_N^{(2)}$ from N to $N + Ni$; $\gamma_N^{(3)}$ from $N + Ni$ to $-N + Ni$; $\gamma_N^{(4)}$ from $-N + Ni$ to $-N$. Write Γ_N for the concatenation of the $\gamma_N^{(j)}$ in the obvious way. Write $f = f_a$ for the function

$$f(z) = \frac{e^{2\pi i a z}}{\cosh(\pi z)}, \quad (z \in \mathbb{C} \setminus (i\mathbb{Z} + \frac{i}{2}))$$

We have modified the sign in the exponential, for the sake of convenience later. We consider first the case $a > 0$.

Observe that f has a singularity at every point $(n + \frac{1}{2})i$, $n \in \mathbb{Z}$. Since $\cosh$ has simple zeroes, we may compute the residue as

$$\text{Res}\left[f(z), (n + \frac{1}{2})i\right] = 2\frac{e^{-2\pi a(n+\frac{1}{2})}}{e^{\pi i(n+\frac{1}{2})} - e^{-\pi i(n+\frac{1}{2})}} = -2i(-1)^n e^{-2\pi a(n+\frac{1}{2})}$$

so that

$$\int_{\Gamma_N} f(z)dz = 4\pi \sum_{n=0}^{N-1} (-1)^n e^{-2\pi a(n+\frac{1}{2})}$$

Thus (evaluating the geometric series)

$$\lim_{N\to\infty} \int_{\Gamma_N} f(z)dz = \frac{4\pi e^{-\pi a}}{1 + e^{-2\pi a}} = \frac{2\pi}{\cosh(\pi a)}$$

We control the integrals along $\gamma_N^{(j)}$, $2 \leq j \leq 4$. We first handle $\gamma_N^{(3)}$. For z on $\gamma_N^{(3)}$, we have $z = x + iN$ for $-N \leq x \leq N$, and

$$|f(x + i(N + \frac{1}{2}))| = 2\frac{e^{-2\pi a N}}{e^{\pi x} + e^{-\pi x}} \leq 2e^{-2\pi a N}$$

so that

$$|\int_{\gamma_N^{(3)}} f(z)dz| \leq 4Ne^{-2\pi a N} \to 0 \quad (N \to \infty)$$

We handle the integral along $\gamma_N^{(2)}$. Points z along here may be written as $z = N + iy, 0 \leq y \leq N$. Then

$$|f(z)| = 2\frac{e^{-2\pi a y}}{|e^{\pi(N+iy)} + e^{-\pi(N+iy)}|} \leq 4e^{-\pi N - 2\pi a y}$$

and consequently

$$|\int_{\gamma_N^{(2)}} f(z)dz| \leq 4Ne^{-\pi N} \to 0$$

Precisely the same analysis holds for $\gamma_N^{(4)}$. Consequently,

$$\lim_{N\to\infty} \int_{\Gamma_N} f(z)dz = \int_{-\infty}^{\infty} \frac{e^{2\pi i a t}}{\cosh(\pi t)}dt = \frac{1}{\cosh \pi a} \quad (a > 0)$$

Applying complex conjugation to both sides results in the desired formula, for $a > 0$. Replacing a by $-a$, and using the fact that $\cosh$ is even, we have the result for all $a \neq 0$.

Finally, since the left-hand side is the Fourier transform of the integrable function $t \mapsto \frac{1}{\cosh(\pi t)}$, it is continuous in a, so we obtain the result for all a by sending $a \to 0$.

$\square$

Spring 2023 Problem 12. (a) Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be entire functions that satisfy

$$f^2 + g^2 = 1, \quad \text{or equivalently,} \quad (f + ig)(f - ig) = 1 \tag{15}$$

throughout the complex plane. Show that there exists $h : \mathbb{C} \to \mathbb{C}$ entire so that

$$f(z) = \cos(h(z)) \quad \text{and} \quad g(z) = \sin(h(z)) \tag{16}$$

for all $z \in \mathbb{C}$.

(b) Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ and $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be holomorphic functions satisfying 15 in $\mathbb{C} \setminus \{0\}$. Show that there need not exist $h : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that the representation 16 holds for all $z \in \mathbb{C} \setminus \{0\}$.

Proof. (a): Observe that $2ff' + 2gg' = 0$ on $\mathbb{C}$; we write this as

$$-\frac{f'}{g} = \frac{g'}{f} \quad \text{when } f \neq 0 \neq g$$

Write η for the function in the above display. We claim that η has only removable singularities. Since $f^2 + g^2 = 1$, we see that f and g do not vanish simultaneously; consequently, if f'/g has a singularity at z_0, then g'/f does not, so all singularities of η are removable.

Write h for a global antiderivative of η; we will specify the choice of constant later. Fix $z_0 \in \mathbb{C}$ such that $h'(z_0) \neq 0$, and write $w_0 = h(z_0)$. Then there is a holomorphic inverse of h near w_0, which we write as h^{-1}. Define f_1, g_1 by

$$f_1 = f \circ h^{-1}, \quad g_1 = g \circ h^{-1} \quad \text{near } w_0$$

Then

$$\frac{f_1'(w)}{g_1(w)} = \frac{f'(h^{-1}(w))}{g(h^{-1}(w))}(h^{-1})'(w) = -h'(h^{-1}(w))(h^{-1})'(w) = -1$$

and

$$\frac{g_1'(w)}{f_1(w)} = \frac{g'(h^{-1}(w))}{f(h^{-1}(w))}(h^{-1})'(w) = h'(h^{-1}(w))(h^{-1})'(w) = 1$$

so that f_1, g_1 satisfy the system

$$\begin{cases} f_1' = -g_1 \\ g_1' = f_1 \end{cases}$$

In particular, there are $A, B \in \mathbb{C}$ such that

$$f_1(w) = A\cos w + B\sin w, \quad g_1(w) = -B\cos w + A\sin w$$

for all w near w_0. Thus, for all z near z_0, there is an identity

$$f(z) = A\cos(h(z)) + B\sin(h(z)), \quad g(z) = -B\cos(h(z)) + A\sin(h(z))$$

Since f, g are entire, it follows that the previous display holds on all of $\mathbb{C}$. Then

$$1 = f(z)^2 + g(z)^2 = A^2 + B^2$$

We omit details in the final steps. As is well-known, there is a constant φ depending on A, B such that

$$A\cos(w) + B\sin(w) = \sqrt{A^2 + B^2}\cos(w + \varphi) = \cos(w + \varphi), \quad (w \in \mathbb{C})$$

so by replacing h by a constant shift we may write $f(z) = \cos(h(z))$; it follows that $g(z) = \pm\sin(h(z))$, as was to be shown.

(b): Write

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad g(z) = \frac{i}{2}\left(z - \frac{1}{z}\right).$$

Then

$$f(z)^2 + g(z)^2 = \frac{1}{4}\left[z^2 + 2 + \frac{1}{z^2} - z^2 + 2 - \frac{1}{z^2}\right] = 1$$

on all of $\mathbb{C} \setminus \{0\}$. To conclude, it suffices to show that there does not exist $h : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ holomorphic such that $f(z) = \cos(h(z)$ for all $z \in \mathbb{C} \setminus \{0\}$.

Suppose otherwise. We consider the possible singularities of h at 0. If 0 is removable, then $h(z) \to a$ as $z \to 0$ for some $a \in \mathbb{C}$, and so $\cos(h(z)) \to \cos(a)$ as $z \to 0$, which violates the fact that $f = \cos \circ h$ has a pole at 0.

Suppose that h has a pole at 0. Then $\cos \circ h$ has an essential singularity at 0 by straightforward arguments, which violates $f = \cos \circ h$.

Suppose h has an essential singularity at 0. Then h fails to be injective on any punctured neighborhood at 0, so $\cos \circ h$ fails to be injective on any punctured neighborhood at 0, whereas f is clearly locally injective near 0. Finally, by the usual classification of singularities theorem, the last three options are all there are, so no such h can exist.

$\square$

10 Fall 2023

Fall 2023 Problem 1. Prove that finite linear combinations of functions from the family

$$\left\{x \mapsto \frac{b}{(x-a)^2+b^2} : a \in \mathbb{R} \text{ and } b \in \mathbb{R}\right\}$$

are dense in $L^1(\mathbb{R})$.

Proof. Let A be the set of finite linear combinations of the preceding family. We claim that the following holds:

$$\exists \varepsilon > 0 \forall f \in L^\infty(\mathbb{R}) \exists g \in A \quad \text{s.t.} \int fg \geq \varepsilon \|f\|_\infty \|g\|_1$$

Suppose for a moment that the claim is true. If $\overline{A} \neq L^1(\mathbb{R})$, then by Hahn-Banach there is $f \in L^\infty(\mathbb{R}) \simeq L^1(\mathbb{R})^*$ such that $\|f\|_\infty = 1$ and $\int fg = 0$ for all $g \in \overline{A}$, which in particular violates the claim.

We now demonstrate the claim. Let $f \in L^\infty$ have unit norm. By replacing f by $-f$ if necessary, we may assume that $f_+ := \max(f, 0)$ has unit norm. Write also $f_- = \max(-f, 0)$. Then there is a measurable $F \subseteq \mathbb{R}$ of positive measure such that $f \geq \frac{1}{2} 1_F$. Let $a \in F$ be a Lebesgue point of F. For $n \in \mathbb{N}$, consider the function $g_n \in A$ defined as

$$g_n(x) = \frac{1}{\pi} n^{3/2} \frac{n^{-2}}{(x-a)^2 + n^{-1}}$$

Note then that

$$\int |g_n(x)| dx = \frac{1}{\pi} n^{1/2} \int \frac{1}{(n^{1/2}x - n^{1/2}a)^2 + 1} dx = \frac{1}{\pi} \int \frac{1}{x^2+1} dx = 1$$

Since a is a Lebesgue point of F,

$$\lim_{n \to \infty} \frac{1}{2} n^{1/2} \int 1_{[a - \frac{1}{n^{1/2}}, a + \frac{1}{n^{1/2}}]} 1_F = 1$$

On the other hand, notice that $g_n(a + n^{-1/2}) = \frac{1}{2\pi} n^{1/2}$, so that $g_n \geq \frac{n^{1/2}}{2\pi} 1_{[a - n^{-1/2}, a + n^{-1/2}]}$. Thus

$$\int f_+ g_n \geq \frac{n^{1/2}}{4\pi} \int 1_F 1_{[a - n^{-1/2}, a + n^{-1/2}]}$$

so that, for n sufficiently large,

$$\int f_+ g_n \geq \frac{1}{4\pi} = \frac{1}{4\pi} \|f\|_\infty \|g_n\|_1$$

We now consider f_-. For $\theta > 0$, write $h_{n,\theta} = \frac{n^{1/2}}{2\theta} 1_{[a-\theta n^{-1/2}, a+\theta n^{-1/2}]}$. Then, for each n and θ,

$$\begin{aligned}
\int f_- h_{n,\theta} &= \int (f_+ - f) h_{n,\theta} \\
&= \int_F (f_+ - f) h_{n,\theta} + \int_{F^c} (f_+ - f) h_{n,\theta} \\
&\leq \int 1_{F^c} (f_+ - \frac{1}{2} 1_F) h_{n,\theta} \\
&= \int 1_{F^c} f_+ h_{n,\theta} \\
&= \int f_+ h_{n,\theta} - \int 1_F f_+ h_{n,\theta} \\
&\leq 1 - \int 1_F h_{n,\theta}
\end{aligned}$$

by various trivial manipulations. The last integral has limit 1, since a is a Lebesgue point, so we find

$$\limsup_n \int f_- h_{n,\theta} \leq 1 - 1 = 0$$

On the other hand, the integrand is nonnegative, so $\int f_- h_{n,\theta} \to 0$ as $n \to \infty$. We compare the pairings against $h_{n,\theta}$ to those against g_n. From the form

$$g_n(x) = \frac{1}{\pi} n^{3/2} \frac{n^{-2}}{(x-a)^2 + n^{-1}} = \frac{n^{1/2}}{\pi} \frac{1}{(n^{1/2}x - n^{1/2}a)^2 + 1}$$

we see that, for each R, there is a collection $\theta_1, \ldots, \theta_N$ and scalars $\alpha_1, \ldots, \alpha_N > 0$ such that

$$g_n(x) \leq \sum_{j=1}^N \alpha_j h_{n,\theta}(x) \quad \text{for all } n \text{ and all } |x-a| \leq n^{-1/2} R$$

Note too that

$$\int_{|x-a| > n^{-1/2}R} g_n(x) dx = \int_{|x-a|>R} g_1(x) dx \overset{R \to \infty}{\longrightarrow} 0$$

Consequently, for each $R > 0$, we may produce the estimate

$$\begin{aligned}
\limsup_n \int f_- g_n &\leq \limsup_n \int_{|x-a| \geq n^{-1/2}R} f_-(x) g_n + (x) \limsup_n \int_{|x-a| \leq n^{-1/2}R} f_-(x) g_n(x) \\
&\leq \limsup_n \int_{|x-a| \geq n^{-1/2}R} g_n(x) + \sum_{j=1}^N \alpha_j \limsup_n \int_{|x-a| \leq n^{-1/2}R} f_-(x) h_{n,\theta_j}(x) \\
&\leq \int_{|x-a| \geq R} g_1(x)
\end{aligned}$$

Sending $R \to \infty$, we see that

$$\lim_n \int f_- g_n = 0$$

and we reach the conclusion that, for n sufficiently large and $\varepsilon > 0$ sufficiently small,

$$\int fg_n = \int f_+ g_n - \int f_- g_n \geq \varepsilon \|f\|_\infty \|g_n\|_1$$

as was to be demonstrated in the claim.

$\square$

Fall 2023 Problem 2. Let E denote the set of real numbers in $[0,1]$ without the digit 9 in their decimal expansion, that is, $x \in E$ if it admits the representation

$$x = \sum_{n \geq 0} \frac{a_n}{10^n} \quad \text{with} \quad a_n \in \{0,1,2,3,4,5,6,7,8\}.$$

(a) Show that E is a Borel set.

(b) Show that E has Lebesgue measure zero.

Proof. If $n \geq 0$ and $0 \leq m_0, \ldots, m_n \leq 9$, write $m = (m_1, \ldots, m_n)$ and

$$I_{m,n} = \left[\sum_{k=0}^{n} \frac{m_k}{10^k}, 10^{-n} + \sum_{k=0}^{n} \frac{m_k}{10^k} \right)$$

We claim that

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\substack{m_0,\ldots,m_n \\ 0 \leq m_k \leq 8}} I_{m,n}$$

For brevity we write E' for the right-hand side. To establish the claim, note first that, if $x \in E$, then we may find $\{m_k\}_{k \geq 0}$ with $0 \leq m_k \leq 8$ for all k such that

$$x = \sum_{k \geq 0} \frac{m_k}{10^k}$$

In particular, for each n we have

$$\sum_{k=0}^{n} \frac{m_k}{10^k} \leq x \leq 9 \cdot 10^{-n-1} + \sum_{k=0}^{n} \frac{m_k}{10^k} < 10^{-n} + \sum_{k=0}^{n} \frac{m_k}{10^k}$$

so that $x \in I_{m,n}$. Thus, for each n there is a tuple m for which $x \in I_{m,n}$, which is to say that $x \in E'$. Thus we have shown that

$$E \subseteq E'$$

We now consider the reverse inclusion. To this end, note first that $I_{m,n} \cap I_{m',n} = \emptyset$ unless $m = m'$, and for $n' > n$ we have

$$\forall m \in \{0, \ldots, 8\}^n \, \forall m' \in \{0, \ldots, 8\}^{n'} \quad (I_{m',n'} \subseteq I_{m,n} \quad \text{or} \quad I_{m',n'} \cap I_{m,n} = \emptyset)$$

This is trivial to demonstrate, so we omit the proof. Suppose now that $x \in E'$. By the previous fact, we may find a sequence $m^{(1)}, m^{(2)}, \ldots$ for which

$$I_{m^{(1)},1} \supseteq I_{m^{(2)},2} \supseteq \cdots, x \in \bigcap_{k=1}^{\infty} I_{m^{(k)},k}$$

In particular, $m^{(k+1)}$ extends $m^{(k)}$. Thus there is a single sequence $m_1, m_2, \ldots$ such that $x \in I_{(m_1,\ldots,m_n),n}$ for all n. Since the diameter of any $I_{m,n}$ is 10^{-n}, it follows that

$$x = \sum_{n\geq 0} \frac{m_n}{10^n}$$

and each $m_k \in \{0, \ldots, 8\}$, so $x \in E$. Thus $E = E'$.

Finally, note that each $I_{m,n}$ is Borel, so E' is clearly Borel, so E is Borel.

(b): Write μ for Lebesgue measure. Observe that, for each n,

$$\mu\left(\bigcup_{\substack{m_0,\ldots,m_n \\ 0\leq m_k\leq 8}} I_{m,n}\right) \leq 9^{n+1}10^{-n}$$

which limits to 0 as $n \to \infty$. From $E = E'$ above, it follows that $\mu(E) = 0$, as was to be shown. $\square$

Fall 2023 Problem 3. Let U and V be closed subspaces of a Hilbert space $\mathcal{H}$ over $\mathbb{R}$ so that

$$\sup\{\langle u, v\rangle : u \in U \text{ and } v \in V \text{ are unit vectors}\} < 1$$

Define

$$W = \{u + v : u \in U \text{ and } v \in V\}$$

(a) Show that each $w \in W$ admits a unique decomposition $w = u + v$ with $u \in U$ and $v \in V$.

(b) Show that the set W is closed in $\mathcal{H}$.

(c) Show that there is a bounded linear map $T : W \to U$ so that

$$w - T(w) \in V \quad \text{for all} \quad w \in W.$$

Proof. (a): Suppose $w = u_0 + v_0 = u_1 + v_1$ with $u_0, u_1 \in U$, $v_0, v_1 \in V$. For $t \in \mathbb{R}$, write

$$u_t = tu_1 + (1-t)u_0, \quad v_t = tv_1 + (1-t)v_0$$

Then $u_t + v_t = w$ for all $t \in \mathbb{R}$, and

$$\|w\|^2 = \|u_t\|^2 + \|v_t\|^2 + 2\langle u_t, v_t\rangle$$

If $u_t \neq 0 \neq v_t$,

$$\|w\|^2 > \|u_t\|^2 + \|v_t\|^2 - 2\|u_t\|\|v_t\| = (\|u_t\| - \|v_t\|)^2$$

Differentiating,

$$0 = \langle u_1 - u_0, u_t\rangle + \langle v_1 - v_0, v_t\rangle + \langle u_1 - u_0, v_t\rangle + \langle u_t, v_1 - v_0\rangle$$

and again,

$$0 = \|u_1 - u_0\|^2 + \|v_1 - v_0\|^2 + 2\langle u_1 - u_0, v_1 - v_0\rangle$$

But $u_1 - u_0 \in U$ and $v_1 - v_0 \in V$, so by the assumed inequality

$$\|u_1 - u_0\|^2 + \|v_1 - v_0\|^2 + 2\langle u_1 - u_0, v_1 - v_0\rangle \geq (\|u_1 - u_0\| - \|v_1 - v_0\|)^2$$

with equality iff $u_1 = u_0$ or $v_1 = v_0$. Since the left-hand side is 0, we conclude that this is the case; it clearly follows from $u_0 + v_0 = u_1 + v_1$ that the other equality holds as well, as was to be shown.

(b): We first show a lower bound on the map $(u, v) \mapsto u + v$. Note

$$\|u + v\|^2 \geq \|u\|^2 + \|v\|^2 - 2c\langle u, v\rangle \geq \|u\|^2 + \|v\|^2 - 2c\|u\|\|v\| \geq (1 - c)(\|u\|^2 + \|v\|^2)$$

so that, if $\{u_n + v_n\}_n$ converges to w, it is Cauchy, and from

$$\|(u_n - u_m) + (v_n - v_m)\| \geq (1 - c)^{1/2}(\|u_n - u_m\|^2 + \|v_n - v_m\|^2)^{1/2}$$

which implies that $\{u_n\}_n, \{v_n\}_n$ are Cauchy, hence convergent to u, v, respectively. By the boundedness of $(u, v) \mapsto u + v$, we conclude that $\{u_n + v_n\}_n$ is convergent to $u + v$, so indeed $w \in U + V = W$.

(c): Define $T : W \to U$ by $T(u + v) = u$. By (a), this is well defined as a function. Clearly T is linear, and has the property that $w - T(w) \in V$ for all $w \in W$.

We claim that T is bounded. Indeed, for each $x \geq 0$, the function

$$y \mapsto x^2 + y^2 - 2cxy$$

has a minimum at the root

$$2y - 2cx = 0, \quad \text{i.e. } y = cx$$

from which

$$x^2 + y^2 - 2cxy \geq x^2 + c^2x^2 - 2c^2x^2 = (1 - c^2)x^2$$

so that

$$(1 - c^2)\|u\|^2 \leq (\|u\| - \|v\|)^2 + 2(1 - c)\|u\|\|v\| \leq \|u\|^2 + \|v\|^2 - 2c|\langle u, v\rangle| \leq \|u + v\|^2$$

and hence

$$\|u\| \leq (1 - c^2)^{-1/2}\|u + v\|$$

as was to be shown. □

Fall 2023 Problem 4. For $f \in C^1([0, 1]; \mathbb{R})$, we define

$$E(f) := \int_0^1 (|f'(x)|^2 + |f(x)|^6 - |f(x)|^4)dx$$

(i) Show that

$$E_{\min} = \inf_{f \in C^1([0,1];\mathbb{R})} E(f) > -\infty.$$

(ii) Show that if $f_n \in C^1([0, 1]; \mathbb{R})$ is a minimizing sequence, that is, $E(f_n) \to E_{\min}$ as $n \to \infty$, then the sequence $\{f_n\}$ admits a subsequence that converges in the space $C([0, 1]; \mathbb{R})$.

Proof. (i): The polynomial $t \mapsto t^6 - t^4$ is even with positive leading coefficient, so there is some $c \in \mathbb{R}$ such that $t^6 - t^4 \geq c$ for all $t \in \mathbb{R}$. Thus for any $f \in C^1([0, 1]; \mathbb{R})$

$$E(f) \geq \int_0^1 cdx = c$$

and thus $E_{\min} \geq c > -\infty$, as was to be shown.

(ii): We show that any such family $\{f_n\}_n$ is uniformly bounded and equicontinuous. We start with the latter. For any $0 \leq x < y \leq 1$,

$$|f_n(y) - f_n(x)| = \left|\int_x^y f_n'(t)dt\right| \leq |x-y|^{1/2} \left(\int_0^1 |f_n'(t)|^2 dt\right)^{1/2}$$

Let $A > 0$ be such that $E(f_n) < A$ for all n. Recall the number c from (i). Then

$$\int_0^1 |f_n'(t)|^2 dt \leq -c + E(f_n) < A - c$$

so that

$$|f_n(y) - f_n(x)| \leq |x-y|^{1/2}|A-c|^{1/2}$$

for all n and all $0 \leq x < y \leq 1$. It follows that $\{f_n\}_n$ is equicontinuous.

We now show that $\{f_n\}_n$ is uniformly bounded. If $x \in [0,1]$ and $|f(x)| = r$, then taking small $\delta > 0$ such that $|f_n(x) - f_n(y)| \leq 1$ for all n and x, y with $|x-y| \leq \delta$,

$$E(f_n) \geq c + \int_{(x-\delta,x+\delta)\cap[0,1]} |f'(t)|^2 + |f(t)|^6 - |f(t)|^4 dt \geq c + 2\delta(r-1)$$

If $r > 0$ is so large that $c + 2\delta(r-1) > A$, then we violate the hypothesis that $E(f_n) < A$. Thus the f_n are uniformly bounded.

Finally, observe that Arzelà-Ascoli now implies the desired result. $\square$

Fall 2023 Problem 5. Let $\omega : \mathbb{R} \to (0,\infty)$ be a locally integrable function to which we associate a Borel measure via

$$\omega(E) = \int_E \omega(x)dx.$$

Let M denote the Hardy-Littlewood maximal function

$$(Mf)(x) = \sup_{r>0} \frac{1}{2r}\int_{x-r}^{x+r} |f(y)|dy.$$

Assume that the function $\frac{1}{\omega}$ is locally integrable and that there exists $C > 0$ so that

$$\omega(\{x \in \mathbb{R} : |(Mf)(x)| > \lambda\}) \leq \frac{C}{\lambda^2}\int_{\mathbb{R}} |f(x)|^2\omega(x)dx$$

uniformly in $\lambda > 0$ and functions $f : \mathbb{R} \to \mathbb{R}$ for which the right-hand side above is finite. Prove that

$$\sup_{x\in\mathbb{R}, r>0} \left(\frac{1}{2r}\int_{x-r}^{x+r} \omega(y)dy\right)\left(\frac{1}{2r}\int_{x-r}^{x+r} \frac{1}{\omega(y)}dy\right) < \infty.$$

Hint: Apply the hypothesis to a well chosen function f and constant λ.

Proof. We exhibit a particular upper-bound to the expression in the supremum. Fix $x \in \mathbb{R}$ and $r > 0$. Write $f(y) = 1_{[x-r,x+r]}(y)\frac{1}{\omega(y)}$ and $\lambda = \frac{1}{8r}\int_{x-r}^{x+r} \frac{1}{\omega(y)}dy$, which we may assume is positive. Then

$$\omega(\{y \in \mathbb{R} : |(Mf)(y)| > \lambda\}) \leq \frac{32Cr}{\frac{1}{2r}\int_{x-r}^{x+r} \frac{1}{\omega(y)}dy}$$

If $|y-x| \le r$,

$$Mf(y) \ge \frac{1}{4r}\int_{x-r}^{x+r} \frac{1}{\omega(t)}dt = 2\lambda$$

so we reach the conclusion that $\{y \in \mathbb{R} : |(Mf)(y) > \lambda\}$ contains all of $[x-r, x+r]$. Thus

$$\omega(\{y \in \mathbb{R} : |(Mf)(y) > \lambda\}) \ge \int_{x-r}^{x+r} \omega(y)dy$$

and we conclude by rearranging the first inequality that

$$\left(\frac{1}{2r}\int_{x-r}^{x+r} \omega(y)dy\right)\left(\frac{1}{2r}\int_{x-r}^{x+r} \omega(y)\right) \le 16C$$

Since the bound holds for all choices of x, r, we are done. □

Fall 2023 Problem 6. Consider the following sequence of functions:

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \frac{\sin(n^4 x)}{n^3 x}$$

(a) Prove that f_n does *not* converge to zero in $L^4(\mathbb{R})$.

(b) Prove that f_n does converge to zero weakly in $L^4(\mathbb{R})$, that is, for any $\phi \in L^{4/3}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f_n(x)\phi(x)dx \to 0 \quad \text{as } n \to \infty.$$

Proof. (a): Note that, for some $\varepsilon > 0$,

$$\left|\frac{\sin x}{x}\right| \ge \varepsilon 1_{[-\varepsilon,\varepsilon]}(x) \quad \forall x \in \mathbb{R}$$

so that, changing variables,

$$\int \left|\frac{\sin(n^4 x)}{n^3 x}\right|^4 dx = \int \left|\frac{\sin u}{u}\right|^4 du \gtrsim \varepsilon^5 > 0$$

for all n; consequently, $\|f_n\|_4$ is uniformly bounded away from 0, showing the result.

(b): For each ϕ and n,

$$\begin{aligned}\int f_n(x)\phi(x) &= \int \frac{\sin(n^4 x)}{n^3 x}\phi(x)dx \\ &= n^{-4}\int \frac{\sin u}{u}\phi(n^{-4}u)du\end{aligned}$$

Consider the special case $\phi = 1_{[a,b]}$ for some $0 < a < b$. Then

$$\begin{aligned}\left|\int f_n(x)\phi(x)dx\right| &= n^{-4}\left|\int_{n^4 a}^{n^4 b} \frac{\sin u}{u}du\right| \\ &\lesssim n^{-4} + n^{-4}\int_{n^4 a}^{n^4 b}\left|\frac{\cos u}{u^2}\right| du \to 0\end{aligned}$$

as $n \to \infty$.

Next, if $\phi = 1_{[0,a]}$ for some $a > 0$,

$$\left| \int f_n(x)\phi(x)dx \right| = n^{-4} \left| \int_0^{n^4 a} \frac{\sin u}{u} du \right|$$
$$\lesssim n^{-4} + n^{-4} \int_0^{n^4 a} \left| \frac{\cos u}{u^2} \right| du \to 0$$

as $n \to \infty$. By symmetry and linearity, it follows that $\int f_n \phi \to 0$ whenever ϕ is the indicator of an interval. By linearity, the same holds for linear combinations of such.

Next, for any $\phi \in L^{4/3}(\mathbb{R})$ and any $\varepsilon > 0$, we may find ψ a finite linear combination of indicators of intervals for which $\|\phi - \psi\|_{4/3} < \varepsilon$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \left| \int f_n \phi \right| &\le \limsup_{n\to\infty} \left| \int f_n \psi \right| + \limsup_{n\to\infty} \int |f_n(\phi - \psi)| \\
&\le 0 + \limsup_n \|f_n\|_4 \|\phi - \psi\|_{4/3} \\
&\le \varepsilon \limsup_n \|f_n\|_4
\end{aligned}$$

using Hölder. By the calculation in (a), the f_n have constant L^4 norm in n, so in fact

$$\limsup_{n\to\infty} \left| \int f_n \phi \right| \lesssim \varepsilon$$

Since this holds for arbitrary $\varepsilon > 0$, the desired result follows. □

Fall 2023 Problem 7. Show that for every real number x that is not an integer, the series

$$\sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$$

is absolutely convergent, and sums to $\frac{\pi \cot(\pi x)}{2x} - \frac{1}{2x^2}$.

Proof. We begin by demonstrating that the series converges absolutely for each $x \in \mathbb{R} \setminus \mathbb{Z}$. For such an x and $n > 2|x|$,

$$\left| \frac{1}{x^2 - n^2} \right| \le n^{-2} \frac{1}{1 - (x/n)^2} \le 2n^{-2}$$

so that the series in question converges at least as fast as the sum over $n \mapsto n^{-2}$, which converges.

Next, we start verifying the identity by computing the residues of $\frac{\pi \cot(\pi z)}{2z}$ at $z \in \mathbb{Z}$. If $n \in \mathbb{Z}$ is nonzero, then we compute

$$\text{Res}\left[\frac{\pi \cot(\pi z)}{2z}, n \right] = \left. \frac{\pi \cos(\pi z)/z}{2\pi \cos(\pi z)} \right|_{z=n} = \frac{1}{2n}$$

We now evaluate the residue at zero. Taking Laurent expansions,

$$\frac{\pi \cot(\pi z)}{2z} = \frac{\pi}{2z} \frac{1 - \frac{\pi^2 z^2}{2} + O(z^4)}{\pi z + O(z^3)} = \frac{1}{2z^2} \left(1 - \frac{\pi^2 z^2}{2} + O(z^3) \right)$$

which has zero residue. It is clear that the function in question has no other singularities in $\mathbb{C}$.

We apply this to a Cauchy integral for the function under consideration. Let $N \in \mathbb{N}$ and define the paths as follows: $\gamma_N^{(1)}$ is the straight-line path from $-(N+\frac{1}{2}) - iN$ to $(N+\frac{1}{2}) - iN$; $\gamma_N^{(2)}$ is the straight-line path from $(N+\frac{1}{2}) + iN$; $\gamma_N^{(3)}$ is the straight-line path from $(N+\frac{1}{2}) + iN$ to $-(N+\frac{1}{2}) + iN$; $\gamma_N^{(4)}$ is the straight-line path from $-(N+\frac{1}{2}) - iN$. Lastly, use Γ_N to denote the concatenated path formed by $\gamma_N^{(1)} \to \gamma_N^{(2)} \to \gamma_N^{(3)} \to \gamma_N^{(4)}$.

If $z \in \mathbb{C} \setminus \mathbb{Z}$ is enclosed by Γ_N, then by the preceding residue calculation

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\pi \cot(\pi w)}{2w(w-z)} dw = \frac{\pi \cot(\pi z)}{2z} - \frac{1}{2z^2} + \sum_{0<|n|\leq N} \frac{1}{2n(n-z)}$$

Observe also that

$$\sum_{0<|n|\leq N} \frac{1}{2n(n-z)} = \sum_{n=1}^{N} \frac{1}{n^2 - z^2}$$

We claim that the contour integral has limit 0 as $N \to \infty$; we may assume that $|z| \leq \frac{1}{2}N$. We control the contributions along the $\gamma_N^{(i)}$. First we study $\gamma_N^{(1)}$. For any w along $\gamma_N^{(1)}$, we may write $w = x - iN$ with $-(N+\frac{1}{2}) \leq x \leq N + \frac{1}{2}$. Then

$$\frac{1}{\sin \pi w} = \frac{2i}{e^{\pi i(x-iN)} - e^{-\pi i(x-iN)}} = e^{-\pi N} \frac{2i}{e^{\pi i x} - e^{-\pi i x} e^{-2\pi N}}$$

and

$$\cos(\pi w) = \frac{e^{\pi N} e^{i\pi x} - e^{-\pi N} e^{-i\pi w}}{2} = e^{\pi N} \frac{e^{i\pi x} - e^{-2\pi N} e^{-i\pi w}}{2}$$

so that, for $z \in \mathbb{R} \setminus \mathbb{Z}$,

$$\left| \frac{\pi \cot(\pi w)}{2w(w-z)} \right| \leq \frac{\pi}{2N^2} \frac{2}{1 - e^{-2\pi N}} \leq \frac{\pi}{N^2}$$

by elementary estimates. Similarly, along $\gamma_N^{(3)}$ we have, for $z \in \mathbb{R} \setminus \mathbb{Z}$,

$$\left| \frac{\pi \cot(\pi w)}{2w(w-z)} \right| \leq \frac{\pi}{N^2}$$

so that

$$\int_{\gamma_N^{(1)} \cup \gamma_N^{(3)}} \left| \frac{\pi \cot(\pi w)}{2w(w-z)} \right| |dw| \leq 8\pi N^{-1}$$

Along $\gamma_N^{(2)}$, we have $w = (N + \frac{1}{2}) + iy$ with $-N \leq y \leq N$. Then

$$\frac{1}{\sin \pi w} = \frac{2i}{e^{\pi i((N+\frac{1}{2})+iy)} - e^{-\pi i((N+\frac{1}{2})+iy)}} = \frac{2(-1)^N}{e^{-\pi y} + e^{\pi y}}$$

and

$$\cos \pi w = \frac{e^{-\pi y} e^{i\pi(N+\frac{1}{2})} - e^{\pi y} e^{-i\pi(N+\frac{1}{2})}}{2} = i(-1)^N (e^{\pi y} + e^{-\pi y})$$

so that, by elementary estimates, assuming $|z| < N$,

$$\left| \frac{\pi \cot(\pi w)}{2w(w-z)} \right| \leq \frac{\pi}{N(N + \frac{1}{2} - |z|)}$$

and

$$\int_{\gamma_N^{(2)}} \left| \frac{\pi \cot(\pi w)}{2w(w-z)} \right| |dw| \leq \frac{2\pi}{N + \frac{1}{2} - |z|}$$

which limits to zero as $N \to \infty$. By the same token,

$$\lim_{N\to\infty} \int_{\gamma_N^{(4)}} \left| \frac{\pi \cot(\pi w)}{2w(w-z)} \right| |dw| = 0$$

In total, we conclude that

$$\lim_{N\to\infty} \int_{\Gamma_N} \frac{\pi \cot(\pi w)}{2w(w-z)} dw = 0$$

i.e.

$$\lim_{N\to\infty} \left(\frac{\pi \cot(\pi z)}{2z} - \frac{1}{2z^2} + \sum_{n=1}^{N} \frac{1}{n^2 - z^2} \right) = 0$$

for any $z \in \mathbb{R} \setminus \mathbb{Z}$. Rearranging and substituting real-variable notation, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} = \frac{\pi \cot(\pi x)}{2x} - \frac{1}{2x^2}$$

for each $x \in \mathbb{R} \setminus \mathbb{Z}$, as was to be shown.

□

Fall 2023 Problem 8. For z_1, z_2 in the unit disk $D(0,1) := \{z \in \mathbb{C} : |z| < 1\}$, define the quantity

$$\Delta(z_1, z_2) := \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|.$$

(i) Let $\alpha \in D(0,1)$, and let $g : D(0,1) \to D(0,1)$ denote the Möbius transformation

$$g(z) := \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

Show that $\Delta(g(z_1), g(z_2)) = \Delta(z_1, z_2)$ for all $z_1, z_2 \in D(0,1)$.

(ii) If $f : D(0,1) \to D(0,1)$ is holomorphic and z_1, z_2 are elements of $D(0,1)$, establish the inequality

$$\Delta(f(z_1), f(z_2)) \leq \Delta(z_1, z_2).$$

(iii) Determine when equality occurs in (ii).

Proof. (i): We first demonstrate the case $z_2 = 0$. Then

$$g(z_2) = -\alpha$$

and

$$\Delta(z_1, z_2) = |z_1|$$

On the other hand,

$$\begin{aligned}
\Delta(g(z_1), g(z_2)) &= \left|\frac{\frac{z_1-\alpha}{1-\bar{\alpha}z_1}+\alpha}{1+\frac{\bar{z_1}-\bar{\alpha}}{1-\alpha\bar{z_1}}\alpha}\right| \\
&= \left|\frac{\frac{z_1-\alpha}{1-\bar{\alpha}z_1}+\alpha}{1+\frac{z_1-\alpha}{1-\bar{\alpha}z_1}\bar{\alpha}}\right| \\
&= \left|\frac{z_1-\alpha+\alpha-|\alpha|^2 z_1}{1-\bar{\alpha}z_1+\bar{\alpha}z_1-|\alpha|^2}\right| \\
&= |z_1| = \Delta(z_1, z_2)
\end{aligned}$$

as was to be validated. Thus the invariance holds for arbitrary Möbius automorphism of the disk when z_2. For any other $z_2 \in D(0,1)$, let

$$h : D(0,1) \to D(0,1), \quad h(z) = \frac{z+z_2}{1+\bar{z}_2 z}$$

Then

$$\Delta(z_1, z_2) = \Delta(h(h^{-1}(z_1)), h(0)) = \Delta(h^{-1}(z_1), 0) = \Delta((g\circ h)(h^{-1}(z_1)), (g\circ h)(0)) = \Delta(g(z_1), g(z_2))$$

as was to be verified.

(ii): Write $g(z) = \frac{z-f(z_1)}{1-\overline{f(z_1)}z}$ and $h(z) = \frac{z+z_1}{1+\overline{z_1}z}$. Then

$$\Delta(f(z_1), f(z_2)) = \Delta(g(f(z_1), g(f(z_2))) = |g(f(z_2))|$$

Observe that $g \circ f \circ h$ is a holomorphic map $D(0,1) \to D(0,1)$ for which $0 \mapsto 0$. By the Schwarz lemma,

$$|g(f(h(z)))| \leq |z| \quad \forall z \in D(0,1)$$

from which we have

$$|g(f(z_2))| \leq |h^{-1}(z_2)| = \Delta(0, h^{-1}(z_2)) = \Delta(h(0), z_2)$$

i.e.

$$\Delta(f(z_1), f(z_2)) \leq \Delta(z_1, z_2)$$

(iii): Inspection of the proof in (ii) reveals that equality holds if and only if

$$|g(f(h(z)))| = |z| \quad \text{at } z = h^{-1}(z_2)$$

Since $z_1 \neq z_2$, $h^{-1}(z_2) \neq 0$. Thus equality holds if and only if $g \circ f \circ h$ is of the form $z \mapsto e^{i\theta}z$ for some $\theta \in \mathbb{R}$. Since this is a special case of a Möbius transformation, by pre- and post-composing the functional identity by h^{-1} and g^{-1}, respectively, we see that f must be a Möbius transformation fixing the boundary of the unit disk. By (i), this necessary condition is also sufficient. $\square$

Fall 2023 Problem 9. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and suppose that there exist two complex numbers ω_1, ω_2, linearly independent over the reals, as well as complex constants c_1, c_2, such that $f(z+\omega_1) = f(z) + c_1$ and $f(z+\omega_2) = f(z) + c_2$ for all complex numbers z. Show that f is a linear polynomial, that is to say there exist complex numbers a, b such that $f(z) = az + b$.

(ii) For any real number a, let R_a denote the closed rectangle

$$R_a := \{x + iy : 0 \le x \le a; 0 \le y \le 1\}$$

Suppose there is a homeomorphism $\phi : R_a \to R_b$ which is holomorphic on the interior of R_a, and maps each of the four sides of R_a to the corresponding side of R_b (for instance, ϕ maps the right side $\{a + iy : 0 \le y \le 1\}$ of R_a to the right side $\{b + iy : 0 \le y \le 1\}$ of R_b). Show that $a = b$. (*Hint:* is there somehow a way to enlarge the domain of ϕ? You may find part (i) to be useful.)

Proof. (i) Fix $z \in \mathbb{C}$. Let $C > 0$ be such that $|\omega_1|, |\omega_2| \in (C^{-1}, C)$. Let $R \gg_C 1$ be sufficiently large. Let $K > 0$ be such that $|f(w)| \le K$ on $|w - z| \le 10C$. Then, by Cauchy,

$$f''(z) = \frac{2}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(z-w)^3} dw$$

By repeatedly invoking the functional identities and the local upper bound, we conclude that

$$|f(w)| \lesssim K + CR \quad \forall |w - z| = R$$

and hence

$$|f''(z)| \lesssim R(K + CR)R^{-3}$$

which decays to 0 as $R \to \infty$. Thus $f''(z) = 0$ for all $z \in \mathbb{C}$, so certainly f is linear.

(ii): Sketch, because this argument is annoying to write precisely. We perform a Schwarz reflection on ϕ. Write $f_1(z) = 2a - \bar{z}$ and $f_2(z) = 2b - \bar{z}$. Then $f_2 \circ \phi \circ f_1$ is holomorphic on the interior of $f_1(R_a) = f_1^{-1}(R_a)$, and for any $0 \le y \le 1$ we have

$$(f_2 \circ \phi \circ f_1)(a + iy) = (f_2 \circ \phi)(a + iy) = f_2(b + iy') = b + iy'$$

where y' is such that $\phi(a + iy) = b + iy'$. Thus ϕ and $f_2 \circ \phi \circ f_1$ agree on the common boundary $R_a \cap f_1(R_a)$, so extend to a homeomorphism $R_{2a} \to R_{2b}$ which is analytic on the interior (and still has the boundary mapping properties assumed).

Repeating this extension, first to the right to infinity and then to the left, we obtain an extension of ϕ to the strip $\{x + iy : 0 \le y \le 1\}$ defining an auto-homeomorphism on that domain, holomorphic on the interior. Repeating the procedure again on the top/bottom edges, using reflection maps of the form $g_1 = 2i + \bar{z}$ and $g_2 = -2i + \bar{z}$, we conclude that ϕ extends to an entire function.

We next verify that ϕ satisfies the hypotheses of (i). Write again $f_1(z) = 2a - \bar{z}$ and $f_2(z) = 2b - \bar{z}$, and next $h_1(z) = 4a - \bar{z}$ and $h_2(z) = 4b - \bar{z}$. Then, in doing the Schwarz reflection, we have arranged for ϕ to satisfy the functional equation

$$\phi = h_2 \circ f_2 \circ \phi \circ f_1 \circ h_1$$

Note that

$$f_1(h_1(z)) = -2a + z, \quad h_2(f_2(z)) = 2b + z$$

so

$$\phi(z) = \phi(z - 2a) + 2b$$

Initially we may only have this relation on the interior of R_a, but by uniqueness we conclude that this holds on all of $\mathbb{C}$. Similarly,

$$\phi(z) = \phi(2i + z) - 2i$$

on all of $\mathbb{C}$. Thus the hypotheses of (i) are satisfied, so ϕ is linear. It is straightforward to verify that we must have $a = b$. □

Fall 2023 Problem 10. Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function with compact support, and let $F : \mathbb{C}\backslash\mathbb{R} \to \mathbb{R}$ be the Cauchy integral

$$F(z) := \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(y)dy}{y-z}.$$

(i) Explain why F is holomorphic on $\mathbb{C} \setminus \mathbb{R}$.

(ii) For any real number x, show that the principal value integral

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{f(y)dy}{y-x} := \lim_{\varepsilon\to 0^+}\int_{|y-x|\geq\varepsilon} \frac{f(y)dy}{y-x}$$

exists, and establish the *Sokhotski-Plemelj formulae*

$$\lim_{\varepsilon\to 0^+} F(x+i\varepsilon) = \frac{1}{2}f(x) + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(y)dy}{y-x}$$

and

$$\lim_{\varepsilon\to 0^+} F(x-i\varepsilon) = -\frac{1}{2}f(x) + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(y)dy}{y-x}$$

Proof. (i): Let T be any triangle in $\mathbb{C} \setminus \mathbb{R}$; it suffices to consider the case T is in the upper half-plane. Since T is compact, there is $\varepsilon > 0$ such that $\mathrm{Im}(z) > \varepsilon$ for all $z \in T$. Consequently, we have an upper bound

$$\left|\frac{f(y)}{y-z}\right| \leq \varepsilon^{-1}|f(y)| \quad \forall (y,z) \in \mathbb{R}\times T.$$

Thus

$$\int_T\int_{-\infty}^{\infty}\left|\frac{f(y)}{y-z}\right||dz|dy \leq \text{length}(T)\|f\|_1\varepsilon^{-1} < \infty$$

so, by Fubini-Tonelli, we may commute integrals:

$$\int_T F(z)dz = \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(y)\int_T \frac{1}{y-z}dzdy$$

By Cauchy, and the fact that T is contained in the upper half-plane, the inner integral is zero. Thus the integral of $F(z)$ over any triangle in $\mathbb{C} \setminus \mathbb{R}$ is zero, so F is holomorphic there.

(ii): We would like to show that the principal value integral exists. Translating, it suffices to take $x = 0$. Let $0 < \delta < \varepsilon < 1$. Since f is smooth, there is $\alpha > 0$ such that $|f(x) - f(0)| \leq \alpha|x|$ for all $|x| \leq 1$. Then

$$\begin{aligned}\left|\int_{\delta\leq|y|\leq\varepsilon}\frac{f(y)dy}{y}\right| &= \left|\int_{\delta\leq|y|\leq\varepsilon}\frac{f(y)-f(0)dy}{y}\right| \quad \left(\text{using } y\mapsto\frac{1}{y}\text{ odd}\right)\\ &\leq \alpha\int_{\delta\leq|y|\leq\varepsilon} dy = 2\alpha(\varepsilon-\delta)\end{aligned}$$

The left-most expression is just

$$\int_{|y|>\delta}\frac{f(y)dy}{y} - \int_{|y|>\varepsilon}\frac{f(y)dy}{y}$$

so that the family

$$r \mapsto \int_{|y|>r} \frac{f(y)dy}{y}$$

is Cauchy along any decreasing sequence $r_n \to 0$. It follows that the limit of the previous function of r exists as $r \to 0^+$, which is to say that the principal value integral exists.

We turn to the formulae. Note first the change-of-variable calculation

$$\begin{aligned} F(-z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(y)dy}{y+z} \\ &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Rf(-y)dy}{-y-z} \\ &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Rf(y)dy}{y-z} \end{aligned}$$

where we write R for the reflection transform: $Rf(y) = f(-y)$. Consequently, if we were to verify the first Sokhotski-Plemelj formula for all $f \in C_c^\infty(\mathbb{R})$, we could conclude that

$$\begin{aligned} \lim_{\varepsilon \to 0^+} F(x - i\varepsilon) &= -\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Rf(y)dy}{y - (-x + i\varepsilon)} \\ &= -\frac{1}{2} Rf(-x) - \frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{Rf(y)dy}{y+x} \\ &= -\frac{1}{2} f(x) + \frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(-y)dy}{-y-x} \\ &= -\frac{1}{2} f(x) + \frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)dy}{y-x} \end{aligned}$$

which is the second Sokhotski-Plemelj formula.

We now turn to the first. By translating f, we may assume $x = 0$. For $y \in \mathbb{R}$, we note that

$$\frac{1}{y - i\varepsilon} = \frac{y}{y^2 + \varepsilon^2} + i \frac{\varepsilon}{y^2 + \varepsilon^2}$$

and hence

$$F(i\varepsilon) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(y) \frac{y}{y^2 + \varepsilon^2} dy + \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \frac{\varepsilon}{y^2 + \varepsilon^2} dy$$

We will demonstrate that each summand has a limit as $\varepsilon \to 0^+$, corresponding to one of the two summands of the Sokhotski-Plemelj formula.

First, note that the family of functions

$$y \mapsto \frac{\varepsilon}{y^2 + \varepsilon^2}$$

form an approximate identity, up to a constant. Indeed,

$$\frac{\varepsilon}{y^2 + \varepsilon^2} = \frac{1}{\varepsilon} \frac{1}{1 + (y/\varepsilon)^2}$$

which is the standard L^1 rescaling of the model function $y \mapsto \frac{1}{1+y^2}$, which has L^1 mass π. It follows directly that

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\frac{\varepsilon}{y^2+\varepsilon^2}dy = \frac{1}{2}f(0)$$

We now handle the other term. Since $y \mapsto \frac{y}{y^2+\varepsilon^2}$ is odd, we may modify f by subtracting off an even bump function that takes value $f(0)$ on a neighborhood of 0 to reduce to the case that $f(0) = 0$. Since f is smooth, there is $\alpha > 0$ such that $|f(x)| \le \alpha|x|$ when $|x| \le 1$. Then we observe that

$$\begin{aligned}\left|\int_{|y|\le\varepsilon^{1/2}} f(y)\frac{y}{y^2+\varepsilon^2}dy\right| &\le \alpha\int_{|y|\le\varepsilon^{1/2}} \frac{y^2}{y^2+\varepsilon^2}dy \\ &\le 2\alpha\varepsilon^{1/2}\end{aligned}$$

so that

$$\lim_{\varepsilon\to 0^+}\int_{|y|\le\varepsilon^{1/2}} f(y)\frac{y}{y^2+\varepsilon^2}dy = 0$$

On the complementary regime $|y| > \varepsilon^{1/2}$, we note that

$$\left|\frac{1}{y} - \frac{y}{y^2+\varepsilon^2}\right| = \frac{\varepsilon^2}{|y|(y^2+\varepsilon^2)} \le \varepsilon^{1/2}$$

so that

$$\left|\int_{|y|>\varepsilon^{1/2}} \frac{f(y)}{y}dy - \int_{|y|>\varepsilon^{1/2}} f(y)\frac{y}{y^2+\varepsilon^2}dy\right| \le \int_{\mathbb{R}} |f(y)|\varepsilon^{1/2}dy \to 0$$

as $\varepsilon \to 0^+$, recalling that f has finite L^1 mass. Thus we have

$$\begin{aligned}\frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(y)dy}{y} &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}\int_{|y|>\varepsilon}\frac{f(y)dy}{y} \\ &= \frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}\left[\int_{|y|>\varepsilon} f(y)\frac{\varepsilon^2}{y(y^2+\varepsilon^2)}dy + \int_{|y|>\varepsilon} f(y)\frac{y}{y^2+\varepsilon^2}dy\right] \\ &= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{|y|>\varepsilon} f(y)\mathrm{Re}\left(\frac{1}{y-i\varepsilon}\right)dy\end{aligned}$$

in particular validating that last limit exists. Since we have also found that

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(y)i\mathrm{Im}\left(\frac{1}{y-i\varepsilon}\right)dy = \frac{1}{2}f(0)$$

we may combine the previous two displays to obtain

$$\lim_{\varepsilon\to 0^+} F(i\varepsilon) = \frac{1}{2}f(0) + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(y)dy}{y-x}$$

as was to be shown. By the remarks and reductions earlier in this solution, we are done. □

Fall 2023 Problem 11. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function which is not a polynomial. Show that the expression

$$\frac{1}{\log R}\int_0^{2\pi} \max(\log|f(Re^{i\theta})|, 0)d\theta$$

diverges to infinity as $R \to +\infty$.

Proof. Under the assumption, we may find $|\alpha| \le 1$ such that $f_\alpha := f - \alpha$ has infinitely many zeroes, and 0 is not one of them. For $R > 1$ such that f_α is nonvanishing on $|z| = R$, let $B_R(z)$ be the (rescaled) Blaschke factor

$$\prod_{j=1}^{n} \frac{(z/R) - (z_j/R)}{1 - \overline{(z_j/R)}(z/R)}$$

where $z_1, \ldots, z_n$ are the zeroes of f_α on $\{|z| < R\}$. Then there exists a zero-free holomorphic function g_R defined on a neighborhood of $\{|z| \le R\}$ for which

$$f_\alpha(z) = g_R(z) B_R(z)$$

Observe that $|B_R(Re^{i\theta})| = 1$ for all θ. Consequently,

$$\log |f_\alpha(Re^{i\theta})| = \log |g_R(Re^{i\theta})|$$

Notice that $z \mapsto \log |g_R(z)|$ is harmonic. Thus

$$\int_0^{2\pi} \log |g_R(Re^{i\theta})| d\theta = 2\pi \log |g_R(0)| = 2\pi \log |f_\alpha(0)| - 2\pi \sum_{j=1}^{n} \log \frac{|z_j|}{R}$$

Notice that each $|z_j| \le R$, so after incorporating the minus sign, each of the last summands is nonnegative. Removing some of the zeroes, we have a lower bound

$$\int_0^{2\pi} \log |g_R(Re^{i\theta})| d\theta \ge 2\pi \log |f_\alpha(0)| + 2\pi \sum_{z_j : |z_j| \le R^{1/2}} \log \frac{R}{|z_j|}$$

Each of the last summands is at least $\frac{1}{2} \log R$, so we obtain the lower bound

$$\int_0^{2\pi} \log |g_R(Re^{i\theta})| d\theta \ge 2\pi \log |f_\alpha(0)| + \pi \log R \cdot \#\{z \in \mathbb{C} : f_\alpha(z) = 0, |z| \le R^{1/2}\}$$

Dividing out by $\log R$,

$$\frac{1}{\log R} \int_0^{2\pi} \log |g_R(Re^{i\theta})| d\theta = \frac{2\pi}{\log R} \log |f_\alpha(0)| + \pi \cdot \#\{z \in \mathbb{C} : f_\alpha(z) = 0, |z| \le R^{1/2}\}$$

The quantity being enumerated is unbounded as $R \to +\infty$. Consequently,

$$\lim_{R \to +\infty} \int_0^{2\pi} \log |f_\alpha(Re^{i\theta})| d\theta = +\infty$$

Finally, observe that, for each z,

$$\max(\log |f(z) - \alpha|, 0) \le \max(\log |f(z)|, 0) + \max(\log |\alpha|, 0) + \log 2 = \max(\log |f(z)|, 0) + \log 2$$

so that

$$\int_0^{2\pi} \max(\log |f(Re^{i\theta})|, 0) \ge -2\pi \log 2 + \int_0^{2\pi} \log |f_\alpha(Re^{i\theta})| d\theta$$

from which the desired conclusion follows. $\square$

Fall 2023 Problem 12. Show that the improper Fourier integral

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{i\xi x} dx = \lim_{\varepsilon \to 0^+, R \to \infty} \int_{\varepsilon \leq |x| \leq R} \frac{\sin(x)}{x} e^{i\xi x} dx$$

is equal to π when ξ is a real number with $|\xi| < 1$, and vanishes when ξ is a real number with $|\xi| > 1$.

Proof. Suppose first that $|\xi| < 1$. For $0 < \varepsilon < 1 < R < \infty$, write $\Gamma_{\varepsilon,R,+}$ to be the contour which makes a clockwise half-circle of radius ε from $-\varepsilon$ to ε along $\gamma^{(1)}_{\varepsilon,R,+}$, makes a line segment from ε to R along $\gamma^{(2)}_{\varepsilon,R,+}$, makes a counterclockwise half-circle of radius R from R to $-R$ along $\gamma^{(3)}_{\varepsilon,R,+}$, and makes a line segment from $-R$ to $-\varepsilon$ along $\gamma^{(4)}_{\varepsilon,R,+}$. We take $\gamma^{(2)}_{\varepsilon,R,-} = \gamma^{(2)}_{\varepsilon,R,+}$ and $\gamma^{(4)}_{\varepsilon,R,-} = \gamma^{(4)}_{\varepsilon,R,+}$. Set $\gamma^{(1)}_{\varepsilon,R,-}$ and $\gamma^{(3)}_{\varepsilon,R,-}$ to be the reflections of $\gamma^{(1)}_{\varepsilon,R,+}$ and $\gamma^{(3)}_{\varepsilon,R,+}$ under $z \mapsto \bar{z}$, respectively. Set $\Gamma_{\varepsilon,R,-}$ to be the corresponding concatenated contour.

Note that

$$\int_{\Gamma_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz = 0 = \int_{\Gamma_{\varepsilon,R,-}} \frac{e^{-i(\xi+1)z}}{z} dz$$

We now evaluate each of the components. By the fractional residue theorem,

$$\lim_{\varepsilon \to 0^+} \int_{\gamma^{(1)}_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz = -\pi i$$

and

$$\lim_{\varepsilon \to 0^+} \int_{\gamma^{(1)}_{\varepsilon,R,-}} \frac{e^{i(\xi-1)z}}{z} dz = \pi i$$

On the upper component,

$$\left| \frac{e^{i(\xi+1)Re^{i\theta}}}{Re^{i\theta}} \right| = R^{-1} e^{-R(\xi+1)\sin\theta}$$

Since $|\xi| < 1$, there is some $\delta > 0$ such that $\xi + 1 > \delta$. Consequently,

$$\int_{\gamma_{\varepsilon,R,+}} \left| \frac{e^{i(\xi+1)z}}{z} \right| dz \leq \int_0^{\pi} e^{-R\delta \sin\theta} d\theta$$

By the dominated convergence theorem,

$$\lim_{R \to +\infty} \int_0^{\pi} e^{-R\delta \sin\theta} d\theta = 0$$

so that

$$\lim_{R \to +\infty} \int_{\gamma_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz = 0$$

By the same argument,

$$\lim_{R \to +\infty} \int_{\gamma_{\varepsilon,R,-}} \frac{e^{i(\xi-1)z}}{z} dz = 0$$

Putting it all together,

$$\begin{aligned}
0 &= \int_{\Gamma_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz + \int_{\Gamma_{\varepsilon,R,-}} \frac{e^{i(\xi-1)z}}{z} dz \\
&= 2i \int_{\varepsilon \leq |x| \leq R} \frac{\sin(x)}{x} e^{i\xi x} dx \\
&+ \int_{\gamma^{(1)}_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz + \int_{\gamma^{(1)}_{\varepsilon,R,-}} \frac{e^{i(\xi-1)z}}{z} dz \\
&+ \int_{\gamma^{(3)}_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz + \int_{\gamma^{(3)}_{\varepsilon,R,-}} \frac{e^{i(\xi-1)z}}{z} dz
\end{aligned}$$

from which we extract the limit

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{i\xi x} dx = \pi$$

We now consider the setting $|\xi| > 1$. Since the kernel $x \mapsto \frac{\sin(x)}{x}$ is even, we may assume $\xi > 1$. Then as before

$$\lim_{\varepsilon \to 0^+} \int_{\gamma^{(1)}_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz = -\pi i$$

and

$$\lim_{R \to +\infty} \int_{\gamma^{(3)}_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz = 0$$

We substitute a different set of estimates, identical to those above up to rewriting:

$$\lim_{\varepsilon \to 0^+} \int_{\gamma^{(1)}_{\varepsilon,R,+}} \frac{e^{i(\xi-1)z}}{z} dz = -\pi i$$

$$\lim_{R \to +\infty} \int_{\gamma^{(3)}_{\varepsilon,R,+}} \frac{e^{i(\xi-1)z}}{z} dz = 0$$

Putting the above together,

$$\begin{aligned}
0 &= \int_{\Gamma_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz + \int_{\Gamma_{\varepsilon,R,+}} \frac{e^{i(\xi-1)z}}{z} dz \\
&= 2i \int_{\varepsilon \leq |x| \leq R} \frac{\sin(x)}{x} e^{i\xi x} dx \\
&+ \int_{\gamma^{(1)}_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz + \int_{\gamma^{(1)}_{\varepsilon,R,+}} \frac{e^{i(\xi-1)z}}{z} dz \\
&+ \int_{\gamma^{(3)}_{\varepsilon,R,+}} \frac{e^{i(\xi+1)z}}{z} dz + \int_{\gamma^{(3)}_{\varepsilon,R,+}} \frac{e^{i(\xi-1)z}}{z} dz
\end{aligned}$$

which now supplies the alternate principal value

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} e^{i\xi x} dx = 0$$

as was to be established.

□

11 Spring 2024

Spring 2024 Problem 1. For each $n \in \mathbb{N}$, let $a_n : \mathbb{R} \to [0, \infty)$ be a Borel-measurable function. Show that

$$\left\{(x, y) \in \mathbb{R} \times [0, \infty) : \sum_{n=1}^{\infty} a_n(x)y^n < \infty\right\}$$

is Borel-measurable.

Proof. Let A be the set in question. Then we may write

$$A = \bigcup_{k=1}^{\infty} A_k, \quad A_k = \left\{(x, y) \in \mathbb{R} \times [0, \infty) : \sum_{n=1}^{\infty} a_n(x)y^n \leq k\right\},$$

and

$$A_k = \bigcap_{j=1}^{\infty} A_{k,j}, \quad A_{k,j} = \left\{(x, y) \in \mathbb{R} \times [0, \infty) : \sum_{n=1}^{j} a_n(x)y^n \leq k\right\}.$$

For each j, the function

$$f_j(x, y) = \sum_{n=1}^{j} a_n(x)y^n$$

is Borel-measurable. Thus,

$$f_j^{-1}([0, k]) = A_{k,j} \in \mathcal{B},$$

so A is Borel, as was to be shown. $\square$

Spring 2024 Problem 2. Let $K \subseteq \mathbb{R}$ be a compact set of positive Lebesgue measure, that is, $|K| > 0$. For each $n \in \mathbb{N}$, we define sets K_n and Borel measures μ_n as follows:

$$K_n := \{x \in \mathbb{R} : \operatorname{dist}(x, K) \leq \frac{1}{n}\} \quad \text{and} \quad \mu_n(A) = \frac{|A \cap K_n|}{|K_n|}.$$

Suppose $\mu_n \to \mu$ in the weak-$*$ topology. Show that $|\operatorname{supp}(\mu)| = |K|$.

Proof. We first demonstrate that $\operatorname{supp}(\mu) \subseteq K$. If $x \notin K$, fix $N \in \mathbb{N}$ such that $\operatorname{dist}(x, K) > \frac{2}{N}$. Let f_n be a sequence of functions satisfying

$$f_n(\mathbb{R}) \subseteq [0, 1], \quad f \equiv 1 \text{ on } B(x, N^{-1}), \quad \operatorname{supp}(f_n) \subseteq B(x, \frac{3}{2N}).$$

Then $f_n \to 1_{\overline{B(x,N^{-1})}}$ as $n \to \infty$. By the dominated convergence theorem,

$$\langle f_n, \mu\rangle \to \mu(\overline{B(x, N^{-1})}).$$

For $k > N$,

$$0 \leq \langle f_n, \mu_k\rangle \leq \mu_k(B(x, 2/N)) = 0.$$

It follows that

$$0 = \lim_{k\to\infty} \langle f_n, \mu_k\rangle = \langle f_n, \mu\rangle,$$

so $\mu(B(x, N^{-1})) = 0$. Thus $x \notin \operatorname{supp}(\mu)$, so $\operatorname{supp}(\mu) \subseteq K$.

Next, we show that $\operatorname{supp}(\mu)$ contains every Lebesgue point of K. It follows $|\operatorname{supp}(\mu)| = |K|$. To this end, let $x \in K$ be Lebesgue for K. Let $k \in \mathbb{N}$; we wish to show that $\mu(B(x, 1/k)) > 0$. Let f be nonnegative with support in $B(x, 1/k)$ such that $f \equiv 1$ on $B(x, 1/2k)$; it suffices to show $\langle \mu, f \rangle > 0$.

Since x is Lebesgue, we may find $\varepsilon > 0$ such that $|K \cap B(x, \varepsilon)| \geq 0.99|B(x, \varepsilon)|$ and $\varepsilon < \frac{1}{2k}$. In particular, for each n, we have

$$\frac{|K_n \cap B(x, \varepsilon)|}{|K_n|} \geq \frac{0.99|B(x, \varepsilon)|}{|K_n|}$$

Since $f \geq 1_{B(x,\varepsilon)}$, we have

$$\langle \mu_n, f \rangle \geq \frac{0.99|B(x, \varepsilon)|}{|K_n|}$$

By continuity from above of Lebesgue measure and the assumption of weak-$*$ convergence,

$$\langle \mu, f \rangle \geq \frac{0.99|B(x, \varepsilon)|}{|K|} > 0$$

and we are done.

$\square$

Spring 2024 Problem 3. Fix $f \in L^3(\mathbb{R}^2)$ with respect to Lebesgue measure. Show that

$$f_n(x, y) := \int_0^1 \int_0^{2\pi} f\left(x + \frac{r\cos(\theta)}{n}, y + \frac{r\sin(\theta)}{n}\right) [1 - 2r] d\theta dr$$

converges to zero as $n \to \infty$ in the following two ways:

(a) almost everywhere.

(b) in $L^3(\mathbb{R}^2)$.

Proof. Let T_n be the nth integral operator specified above for $f \in L^3(\mathbb{R}^2)$. Then $T_n = A_n - B_n$, where

$$A_n f(x, y) = \int_0^1 \int_0^{2\pi} f\left(x + \frac{r\cos(\theta)}{n}, y + \frac{r\sin(\theta)}{n}\right) d\theta dr$$

$$B_n f(x, y) = \int_0^1 \int_0^{2\pi} f\left(x + \frac{r\cos(\theta)}{n}, y + \frac{r\sin(\theta)}{n}\right) 2r d\theta dr$$

Let (x, y) be any Lebesgue point for f. Then

$$B_n f(x, y) \to 2\pi f(x, y)$$

as $n \to \infty$, since B_n is just the usual average up to a constant factor.

To study A_n, we note that for each nonnegative $g \in L^3(\mathbb{R}^2)$

$$\begin{aligned}
A_n g(x,y) &= \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} r^{-1} \int_0^{2\pi} g\left(x + \frac{r\cos(\theta)}{n}, y + \frac{r\sin(\theta)}{n}\right) r d\theta dr \\
&\leq 2 \sum_{k=0}^{\infty} n 2^k \int_{2^{-k-1}}^{2^{-k}} \int_0^{2\pi} g\left(x + \frac{r\cos(\theta)}{n}, y + \frac{r\sin(\theta)}{n}\right) (r/n) d\theta dr \\
&\leq 2 \sum_{k=0}^{\infty} n 2^k \int_{\|z-(x,y)\| \leq \frac{1}{n}2^{-k}} g(z) dz \\
&\leq \frac{10}{n} \sum_{k=0}^{\infty} 2^{-k} \mathcal{A}_{\frac{1}{n}2^{-k}} g(x,y)
\end{aligned}$$

where

$$\mathcal{A}_r g(x,y) = \frac{1}{|B((x,y),r)|} \int_{B((x,y),r)} g(z) dz$$

Thus, if (x,y) is a Lebesgue point for f, and for arbitrary $\varepsilon > 0$ we choose $\delta > 0$ such that

$$\frac{1}{|B((x,y),r)|} \int_{\|z-(x,y)\| \leq r} |f(z) - f(x,y)| dz < \varepsilon \quad \forall r < \delta$$

then for each n such that $\frac{1}{n} < \delta$ we conclude

$$A_n[|g - g(x,y)|](z) \leq \frac{10}{n} \sum_{k=0}^{\infty} 2^{-k} \varepsilon \leq \frac{10}{n} \varepsilon.$$

Thus, if (x,y) is a Lebesgue point for f with $f(x,y) = 0$, we have $A_n f(x,y) \to 0$. On the other hand,

$$A_n[1](x) = \int_0^1 \int_0^{2\pi} d\theta dr = 2\pi,$$

so that for general Lebesgue point for f

$$A_n f(x,y) \to 2\pi f(x,y).$$

Thus $T_n f(x,y) = A_n f(x,y) - B_n f(x,y) \to 0$ for each Lebesgue point. Since almost every point is Lebesgue, we conclude that the convergence happens almost everywhere.

It remains to consider norm convergence. It is easy to see that $T_n g \to 0$ uniformly, for each $g \in C_c^\infty(\mathbb{R}^2)$. Thus, if $\varepsilon > 0$ is arbitrary and $g \in C_c^\infty(\mathbb{R}^2)$ is such that $\|f - g\|_{L^3(\mathbb{R}^2)} < \varepsilon$,

$$\begin{aligned}
\|\limsup_n |T_n f|\|_{L^3(\mathbb{R}^2)} &\leq \|\limsup_n |T_n g|\|_{L^3(\mathbb{R}^2)} + \|\limsup_n |T_n[f-g]|\|_{L^3(\mathbb{R}^2)} \\
&= \|\limsup_n |T_n[f-g]|\|_{L^3(\mathbb{R}^2)}.
\end{aligned}$$

We bound:

$$|T_n[f-g]| \leq A_n[|f-g|] + B_n[|f-g|]$$

by the triangle inequality. Note that $B_n[|f-g|] \leq M[|f-g|]$, where M is the Hardy-Littlewood maximal function. Repeating the analysis from part (a),

$$A_n[|f-g|] \leq \frac{10}{n} \sum_{k=0}^{\infty} 2^{-k} \mathcal{A}_{\frac{1}{n}2^{-k}}[|f-g|]$$

so that $A_n[|f-g|] \leq \frac{20}{n} M[|f-g|]$. Since M is $L^3 \to L^3$ bounded,

$$\| \limsup_n |T_n[f-g]| \|_{L^3(\mathbb{R}^2)} \leq 30 \|M\|_{L^3 \to L^3} \|f-g\| = O(\varepsilon)$$

It follows that

$$\| \limsup_n |T_n f| \|_{L^3(\mathbb{R}^2)} = O(\varepsilon)$$

for all $\varepsilon > 0$, so

$$\| \limsup_n |T_n f| \|_{L^3(\mathbb{R}^2)} = 0.$$

In particular, $T_n f \to 0$ in $L^3(\mathbb{R}^2)$. □

Spring 2024 Problem 4. Fix $f \in L^1(\mathbb{R})$ that is non-negative and satisfies $\int f(x)dx = 1$. We then define the n-fold convolution

$$f_n(x) := \int \cdots \int f(x - y_1 - y_2 - \cdots - y_n) f(y_1) \cdots f(y_n) dy_1 dy_2 \cdots dy_n$$

of f with itself. Show that the sequence $f_n(x)$ does *not* converge in $L^1(\mathbb{R})$.

Proof. Suppose for the sake of contradiction that $f_n \to g$ in $L^1(\mathbb{R})$. Note that each $f_n \geq 0$ and $\int f_n = 1$, so the same holds for g. Then we have

$$0 = g - g \leftarrow f_n * f_n - f_n \to g * g - g,$$

by continuity of $*$ on $L^1(\mathbb{R})$. Thus, g is a nontrivial nonnegative L^1 function with the property that $g * g = g$.

On the other hand, the Fourier transform $\hat{g}$ of g satisfies

$$\hat{g}^2 = \hat{g}, \quad \hat{g}(0) = 1, \quad \hat{g} \in C_0(\mathbb{R}),$$

where the last set is the family of continuous functions that limit to zero at ∞. The first equality is an equality of L^∞ functions, but since $\hat{g}$ is continuous we conclude that the equality holds pointwise. Finally, this equality implies that $\hat{g}$ only takes the values 0 and 1. But this contradicts the statements $\hat{g}(0) = 1, \hat{g}(\xi) \to 0$, and we are done. □

Spring 2024 Problem 5. Fix $1 \leq p < q < \infty$. Throughout this problem, $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ are defined using Lebesgue measure and $|A|$ denotes the Lebesgue measure of the set A.

(a) Suppose $f \in L^p(\mathbb{R})$ satisfies $\int_A |f(x)|^q dx < \infty$ for every Borel subset A with $|A| = 1$. Show that $f \in L^q(\mathbb{R})$.

(b) Show that there exists $f \in L^p(\mathbb{R})$ so that $\int_a^{a+1} |f(x)|^q dx < \infty$ for every $a \in \mathbb{R}$ but $f \notin L^q(\mathbb{R})$.

Proof. (a): Let $U = \{x \in \mathbb{R} : |f(x)| \geq 1\}$ and $L = \{x \in \mathbb{R} : |f(x)| < 1\}$. Since $f \in L^p(\mathbb{R})$, we have $|U| < \infty$, so we may find measurable sets $U_1, \ldots, U_n$ with $U = \bigcup_{k=1}^n U_k$ and $|U_k| \leq 1$. Then we have

$$\int_U |f(x)|^q dx \leq \sum_{k=1}^n \int_{U_k} |f(x)|^q dx < \infty$$

and

$$\int_L |f(x)|^q dx \leq \int_L |f(x)|^p dx < \infty$$

so $f \in L^q(\mathbb{R})$.

(b): Ugly example, sketch. Let $\beta = \frac{2q}{p}$, and let f be the function

$$f(x) = \sum_{n=1}^\infty a_n 1_{(2n,2n+1)}(x) \cdot |x - 2n|^{-\frac{n^\beta - 1}{n^\beta q} + \frac{1}{n^q}}.$$

Then

$$\int |f(x)|^p dx = \sum_{n=1}^\infty |a_n|^p \int_0^1 x^{-\frac{n^\beta-1}{n^\beta}\frac{p}{q} + \frac{p}{n^\beta}} dx = \sum_{n=1}^\infty |a_n|^p \frac{n^\beta q}{n^\beta(q-p) + p + pq}$$

and

$$\int_{2n}^{2n+1} |f(x)|^q dx = |a_n|^q \int_0^1 x^{-\frac{n^\beta - 1}{n^\beta} + \frac{q}{n^\beta}} = |a_n|^q \frac{n^\beta}{1+q}$$

so that

$$\int |f(x)|^q dx = \frac{1}{1+q} \sum_{n=1}^\infty n^\beta |a_n|^q.$$

Choosing $a_n = n^{-2/p}$ will do.

□

Spring 2024 Problem 6. Let $\mathcal{H}$ be a Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ a unitary operator, that is, U is bounded, linear, and invertible with inverse equal to its adjoint U^*.

(a) Prove that $\ker(U - I)^\perp = \overline{\operatorname{ran}(U - I)}$, where $\operatorname{ran}(U - I)$ denotes the range of $U - I$ and I is the identity operator on $\mathcal{H}$.

(b) Let P denote the (orthogonal) projection of $\mathcal{H}$ onto $\ker(U - I)$. Prove that, for any vector $v \in \mathcal{H}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k v \to Pv \quad \text{in the } \mathcal{H}\text{-norm, as } n \to \infty.$$

Proof. (a): Observe that, for $A \in B(\mathcal{H})$,

$$\begin{aligned} x \in \ker(A) \quad &\iff \quad \langle Ax, y\rangle = 0\, \forall y \\ &\iff \quad \langle x, A * y\rangle = 0\, \forall y \\ &\iff x \in \operatorname{ran}(A^*)^\perp \end{aligned}$$

Since $(L^\perp)^\perp = \overline{L}$, we conclude that

$$\ker(A)^\perp = \overline{\operatorname{ran}(A^*)}.$$

We specialize to $A = U - I$; we have $A^* = U^* - I = U^{-1} - I = -U^{-1}(U - I)$. Since $-U^{-1}$ is an isomorphism, $\operatorname{ran}(-U^{-1}(U-I)) = \operatorname{ran}(U - I)$, and we are done.

(b): By (a), we may write $v = v_1 + v_2$ with $v_1 \in \ker(U - I)$ and $v_2 \in \overline{\operatorname{ran}(U - I)}$ and $\langle v_1, v_2 \rangle = 0$. Then, for each n,

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k v_1 = \frac{1}{n}\sum_{k=0}^{n-1} v_1 = v_1 = Pv_1.$$

Let $\varepsilon > 0$ be arbitrary and fix $w = (U - I)x$ such that $\|v_2 - w\| < \varepsilon$. Then we have

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k w = \frac{1}{n}\sum_{k=0}^{n-1} U^k (U - I)x = \frac{1}{n}\sum_{k=1}^{n} U^k x - \frac{1}{n}\sum_{k=0}^{n-1} U^k x = \frac{U^n x - x}{n}.$$

Since U is unitary, $\|U^n x\| = \|x\|$, so

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} U^k w \right\| \leq (2/n)\|x\|.$$

Finally, wince $\|v_2 - w\| < \varepsilon$, we have

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} U^k (v_2 - w) \right\| \leq \frac{1}{n}\sum_{k=0}^{n-1} \|v_2 - w\| < \varepsilon,$$

and we conclude that

$$\limsup_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} U^k v_2 \right\| \leq \varepsilon.$$

Since ε was arbitrary, we conclude that

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k v_2 \to 0 = Pv_2,$$

where the convergence is in norm, and the final equality uses (a). Summing, we conclude that

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k v = \frac{1}{n}\sum_{k=0}^{n-1} U^k v_1 + \frac{1}{n}\sum_{k=0}^{n-1} U^k v_2 = Pv_1 + Pv_2 = Pv,$$

as was to be shown.

□

Spring 2024 Problem 7. Rigorously evaluate

$$\int_{-\infty}^{\infty} \frac{\log|x + i|}{x^2 + 4} dx.$$

Proof. Let $\log$ be the branch of the logarithm defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ satisfying $\operatorname{Im}(\log z) \in (-\pi, \pi)$ for all z in its domain. Let $R > 2$ be arbitrary, and let Γ_R be the contour composed of the following pieces:

$$\Gamma_R^1(t) = t \quad (-R \leq t \leq R)$$

$$\Gamma_R^2(t) = Re^{it} \quad (0 \leq t \leq \pi)$$

Then we may integrate $\frac{\log(z+i)}{z^2+4}$ on Γ, to obtain by the residue theorem

$$\int_\Gamma \frac{\log(z+i)}{z^2+4} dz = 2\pi i \frac{\log(3i)}{4i} = \frac{\pi}{2}(\log(3) + i\theta),$$

where $\theta \in (-\pi, \pi)$ is such that $\tan\theta = 3$. Note as well that

$$\left| \int_{\Gamma_2} \frac{\log(z+i)}{z^2+4} dz \right| \leq \frac{\pi R(\log(R^2+1) + \pi)}{R^2 - 4} \lesssim \frac{\log R}{R} \to 0$$

as $R \to \infty$, so that

$$\lim_{R\to\infty} \int_\Gamma \frac{\log(z+i)}{z^2+4} dz = \int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+4} dx.$$

Note that the latter integrand is absolutely integrable over $\mathbb{R}$, so the limit is indeed valid. Taking real parts, we conclude

$$\int_{-\infty}^{\infty} \frac{\log|x+i|}{x^2+4} dx = \frac{\pi \log 3}{2}.$$

□

Spring 2024 Problem 8. For each $n \in \mathbb{N}$, suppose $f_n : \mathbb{D} \to (-1, 1)$ is harmonic. (Here $\mathbb{D}$ denotes the unit disk in the complex plane.)

(a) Show that there is a subsequence of the functions f_n that converges uniformly on compact subsets of $\mathbb{D}$.

(b) Suppose $f(z)$ is such a subsequential limit and that $f(0) = 1$. Show that $f(z) = 1$ for all $z \in \mathbb{D}$.

Proof. (a): For each n, choose $h_n : \mathbb{D} \to \mathbb{C}$ analytic with Re $(h_n) = f_n$ and $h_n(0) \in \mathbb{R}$. Then each h_n takes values in the strip $\{z : \text{Re}\,(z) \in (-1, 1)\}$, so by Montel's theorem $\{h_n\}_n$ is normal. Thus we may find $h : \mathbb{D} \to \mathbb{C}$ holomorphic and a subsequence $k \mapsto n_k$ such that $h_{n_k} \to h$ locally uniformly. Since

$$|h(z) - h_{n_k}(z)| \geq |\text{Re}\,(h(z)) - \text{Re}\,(h_{n_k}(z))|,$$

we conclude that $f_{n_k} \to \text{Re}\,(h)$ locally uniformly, as was to be shown.

(b): By the work in (a), we see that f is harmonic. Since each f_n takes values in $(-1, 1)$, f takes values in $[-1, 1]$. By the maximal principle, f is constant, i.e. $f(z) = 1$ for all $z \in \mathbb{D}$. □

Spring 2024 Problem 9. Find all entire functions f with the property that if one writes $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and u, v are the real and imaginary parts of f, then for all $x, y \in \mathbb{R}$ we have

$$u(x, y) + v(x, y) \leq x + y.$$

Proof. Consider one such f. Let $g(z) = (1 - i)(f(z) - z)$. Then we have

$$\text{Re}\,(g(x + iy)) = u(x, y) + v(x, y) - x - y \leq 0$$

for all $x, y \in \mathbb{R}$. Thus the entire function g omits the right half-plane $\{z : \text{Re}\,(z) > 0\}$ as values, hence is constant by the little Picard theorem. It follows that f must be of the form $f(z) = cz$ for some constant c. Writing $c = a + ib$,

$$u(x, y) = ax - by, \quad v(x, y) = ay + bx$$

Testing the assumed inequality along $y = 0$, we must have $a + b = 1$; along $x = 0$, $a - b = 1$. Thus we must have $c = 1$, i.e. $f(z) = z$. Since this clearly satisfies the inequality, we are done. □

Spring 2024 Problem 10. Let $h : (-\infty, 0] \to \mathbb{R}$ be continuous and define

$$\Gamma = \{x + ih(x) : -\infty < x \leq 0\}.$$

Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous and that f is holomorphic in $\mathbb{C} \setminus \Gamma$. Show that if $f(z) = 0$ for all $z \in \Gamma$, then $f(z) = 0$ for all $z \in \mathbb{C}$.

Proof. We argue via Morera's theorem. It suffices to consider axis-parallel rectangles. It is easy to see that we may assume that the rectangle is contained in the half-plane $\{x + iy : x \leq 0\}$. Let $\{c_j = a_j + ib_j : 1 \leq j \leq 4\}$ be the four vertices, listed in CCW order starting in the top left corner (thus: $a_1 = a_2, b_2 = b_3, a_3 = a_4, b_4 = b_1$). We assume for simplicity that $b_2 < f(a_1) < b_1$ and $b_3 < f(a_3) < b_4$; the other cases may be handled similarly. Choose $\varepsilon > 0$ arbitrary. Let $\delta > 0$ be such that the set

$$N = \{x + iy : a_1 \leq x \leq a_3, |y - h(x)| \leq \delta\}$$

satisfies the estimate

$$\sup\{|f(z) - f(w)| : z, w \in N, \mathrm{Re}\,(z) = \mathrm{Re}\,(w)\} < \varepsilon;$$

this is possible by compactness. Then, if R denotes the (boundary) rectangle, we have

$$\int_R f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \int_{\Gamma_3} f(z)dz,$$

where:

- Γ_1 is the contour which traverses $a_1 + ib_1$ to $a_1 + ih(a_1) + i\delta$, then follows $t + ih(t) + i\delta$ over $a_1 \leq t \leq a_3$, then goes from $a_3 + ih(a_3) + i\delta$ to $a_4 + ib_4$, then $a_4 + ib_4$ to $a_1 + ib_1$;
- Γ_2 is the contour which travels from $a_1 + ih(a_1) + i\delta$ to $a_1 + ih(a_1) - i\delta$, then follows $t + ih(t) - i\delta$ from $t = a_1$ to $t = a_3$, then goes from $a_3 + ih(a_3) - i\delta$ to $a_3 + ih(a_3) + i\delta$, then follows $t + ih(t) + i\delta$ from $t = a_3$ to $t = a_1$;
- Γ_3 is the contour which traverses $a_1 + ib_1 - i\delta$ to $a_2 + ib_2$, then to $a_3 + ib_3$, then to $a_3 + ih(a_3) - i\delta$, then follows $t + ih(t) - i\delta$ from $t = a_3$ to $t = a_1$.

Both Γ_1, Γ_3 lie in $\mathbb{C} \setminus \Gamma$ which is simply-connected, so the integral contributions vanish. We expand the remaining contribution:

$$\begin{aligned}
\int_{\Gamma_2} f(z)dz &= \int_{-\delta}^{\delta} f(a_1 + ih(a_1) - it)dt \\
&\quad + \int_{-\delta}^{\delta} f(a_3 + ih(a_3) + it)dt \\
&\quad + \int_{a_1}^{a_3} f(t + ih(t) - i\delta)dt \\
&\quad + \int_{a_3}^{a_1} f(t + ih(t) + i\delta)dt \\
&= I + II + III + IV
\end{aligned}$$

Rearranging,

$$III + IV = \int_{a_1}^{a_3} f(t + ih(t) - i\delta) - f(t + ih(t) + i\delta) dt$$

and by the definition of δ,

$$|III + IV| \le |a_3 - a_1|\varepsilon.$$

For the other two, note that

$$|I|, |II| \le 2\delta\varepsilon,$$

by again the definition of δ. Thus

$$\left| \int_{\Gamma_2} f(z) dz \right| \le \varepsilon \left(|a_3 - a_1| + 2\delta \right).$$

Thus we have

$$\left| \int_{R} f(z) dz \right| \le \varepsilon \left(|a_3 - a_1| + 2\delta \right)$$

for arbitrary choice of ε. It follows that $\int_R f(z)dz = 0$. Since R was arbitrary, we conclude that f is entire. Since f has a more-than-discrete set of zeroes, it follows that $f \equiv 0$, as was to be shown. □

Spring 2024 Problem 11. Let $f : \mathbb{C} \to \mathbb{C}$ be the unique holomorphic function with

$$f(0) = 0, \quad f'(z) = e^{z^2}.$$

Show that this function admits a convergent expansion

$$f(z) = z \exp(cz^2) \prod_n \left[\left(1 - \frac{z^2}{z_n^2} \right) \exp\left(\frac{z^2}{z_n^2} \right) \right]$$

Proof. By the Weierstrass factorization theorem, there is a convergent expansion

$$f(z) = z^m e^{g(z)} \prod_n E_{p_n}\left(\frac{z}{a_n}\right)$$

where $m \ge 0$ is an integer, $g(z)$ is entire, $\{a_n\}_n$ are the zeroes other than 0 listed with multiplicity, $\{p_n\}_n$ is a sequence of nonnegative integers, and E_p is the elementary factor

$$E_0(z) = (1 - z), \quad E_p(z) = (1 - z) \exp\left(\frac{z}{1} + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right) \quad (p \ne 0).$$

Since $f(0) = 0$ and $f'(0) = 1 \ne 0$, we have $m = 1$. Since f' is even, f is odd (easy to see by considering the convergent power series); thus the zeroes $\{a_n\}_n$ come in $\pm$ pairs.

We consider the order of f. For a given r and $|z| \le r$,

$$|f(z)| \le \int_0^r e^{t^2} dt \le re^{r^2}$$

so that

$$\log \|f\|_{L^\infty(B(0,r))} \le \log r + r^2$$

and

$$\frac{\log\log\|f\|_{L^\infty(B(0,r))}}{\log r} \leq \frac{\log 2}{\log r} + 2,$$

i.e. f has order at most 2. Thus the genus of f is at most 2, and we may take $p_n = 2$ for every n (Hadamard). Thus we may write

$$f(z) = ze^{g(z)} \prod_n (1 - \frac{z^2}{z_n^2}) \exp\left(\frac{z^2}{z_n^2}\right)$$

(where the opposite-sign a_n are bundled into the z_n, which cancel in the linear term in the final exponential). It remains to consider g. Since f has order 2, g is a polynomial of degree ≤ 2. Since f is odd, g has zero linear term. Since $f'(0) = 1$, and

$$ze^{g(z)} \prod_n (1 - \frac{z^2}{z_n^2}) \exp\left(\frac{z^2}{z_n^2}\right)\Bigg|_{z=0} = e^{g(0)},$$

we conclude that $g(0) \in 2\pi i\mathbb{Z}$. Since shifting by such elements does not affect the exponential terms, we may assume $g(0) = 0$, so $g(z) = cz^2$ for a suitable constant c.

□

Spring 2024 Problem 12. Consider the following polynomial of $z, w \in \mathbb{C}$:

$$P(w, z) := w^3(w-2)^3 + z.$$

(a) Find an explicit $\delta > 0$ so that $w \mapsto P(w, z)$ has precisely three zeroes (counted by multiplicity) in the unit disk whenever $|z| < \delta$.

(b) Let us write $w_1(z), w_2(z), w_3(z)$ for these three zeroes. Show that

$$z \mapsto w_1(z) + w_2(z) + w_3(z)$$

defines a holomorphic function on $|z| < \delta$.

Warning: each individual $w_i(z)$ will *not* be holomorphic!

Proof. (a): We choose $\delta = 1$. Fix any z with $|z| < \delta$. Then we have, for each $|w| = 1$,

$$|w^3(w-2)^3| \geq 1 = \delta > |z|,$$

so by Rouché's theorem, $w \mapsto P(w, z)$ and $w \mapsto w^3(w-2)^3$ have the same number of zeroes in $|w| < 1$. The result follows.

(b): Observe that

$$\frac{1}{2\pi i}\int_{|w|=1} w\frac{\partial_w P(w,z)}{P(w,z)}dw = w_1(z) + w_2(z) + w_3(z)$$

by the argument principle. The left-hand side is holomorphic in $|z| < 1$ by an application of Morera and Fubini-Tonelli, and we are done. □