These are solutions to old analysis qualifying exams for UCLA, accessible on the math UCLA website. Varying levels of detail are presented. Please send any corrections to johnsrude (at) math.ucla.edu; if you did not obtain this document directly from [the source](#), please first check that I have not already made that correction.

Note that [Adam Lott](#) has compiled solutions back to 2009, accessible [here](#). The solutions presented here are my own; the solutions presented there are the compiled work of many individuals and should probably be treated as rather more reliable.

### 1 Spring 2020

**Problem 1.** Assume $f \in C^\infty_c(\mathbb{R})$ satisfies

$$
\int_\mathbb{R} e^{-tx^2} f(x) dx = 0 \quad \text{for any} \quad t \geq 0
$$

Show that $f(x) = -f(-x)$ for any $x \in \mathbb{R}$.

**Proof.** Taking the even part of $f$, that is, $\frac{f(x) + f(-x)}{2}$, we see that the claim holds if and only if

$$
f \text{ even}, \quad C^\infty_c(\mathbb{R}) \text{ orthogonal to centered Gaussians} \implies f = 0
$$

Suppose $f$ satisfies the left-hand side of the above implication. Note that

$$
0 = \int_\mathbb{R} e^{-tx^2} f(x) dx = \sqrt{\frac{\pi}{t}} \int_\mathbb{R} e^{-\pi^2 \xi^2 / t} \hat{f}(\xi) d\xi
$$

for all $t > 0$, which implies that $\hat{f}$ is also orthogonal to centered Gaussians. Since $f$ is even and real-valued, $\hat{f}$ is also even and real-valued. Since $f$ is compactly supported, the integral

$$
\hat{f}(z) = \int_\mathbb{R} e^{-2\pi i z x} f(x) dx
$$

is well-defined and continuous for $z \in \mathbb{C}$. If $\Delta$ is any triangle in $\mathbb{C}$, then Fubini provides

$$
\int_\Delta \hat{f}(z) dz = \int_\Delta \int_\mathbb{R} e^{-2\pi i z x} f(x) dx dz = \int_\mathbb{R} \int_\Delta e^{-2\pi i z x} f(x) dz dx \quad \text{since the integrand is continuous and compactly-supported}
$$

$$
= 0 \quad \text{since } e^{-2\pi i z x} \text{ is analytic in } z \text{ for each } x \in \mathbb{R}
$$
so by Morera we have that \( \hat{f} \) extends to an entire function. Thus \( \hat{f} \) on \( \mathbb{R} \) is given by a convergent real power series

\[
\hat{f}(\xi) = \sum_{n=0}^{\infty} a_n \xi^n
\]

and, since \( \hat{f} \) is even, \( a_n = 0 \) for all odd \( n \). But then, for any \( t > 0 \),

\[
0 = \int_\mathbb{R} e^{-t\xi^2} \hat{f}(\xi) d\xi = \sum_{n=0}^{\infty} a_n \int_\mathbb{R} e^{-t\xi^2} \xi^n d\xi = \sqrt{\frac{\pi}{t}} \sum_{k=0}^{\infty} a_{2k} \frac{(2k - 1)!!}{(2t)^k}
\]

Thus the real power series

\[
\sum_{k=0}^{\infty} a_{2k} \frac{(2k - 1)!!}{(2t)^k}
\]

which converges uniformly in a neighborhood of \( \infty \), and hence defines an analytic function there, is identically 0 on a non-discrete set, and hence has zero coefficients. Thus each \( a_n \) is equal to 0, which implies that \( f \) was zero from the start.

\[\Box\]

**Problem 2.** Assume \( f_n : \mathbb{R} \to \mathbb{R} \) is a sequence of differentiable functions satisfying

\[
\int_\mathbb{R} |f_n(x)| dx \leq 1 \quad \text{and} \quad \int_\mathbb{R} |f_n'(x)| dx \leq 1.
\]

Assume also that for any \( \epsilon > 0 \) there exists \( R(\epsilon) > 0 \) such that

\[
\sup_n \int_{|x| \geq R(\epsilon)} |f_n(x)| dx < \epsilon
\]

Show that there exists a subsequence of \( \{f_n\} \) that converges in \( L^1(\mathbb{R}) \).

**Proof.** Note that the second condition implies that the \( \{f_n\} \) have total variation bounded by 1. Since each \( f_n \) is absolutely integrable, \( |f_n| \leq 1 \) everywhere since otherwise the total variation condition would imply that \( |f_n| > \epsilon > 0 \) everywhere, contradicting integrability. Thus the \( \{f_n\} \) are uniformly bounded.

Now let \( \{\phi_\epsilon\}_{\epsilon} \) be approximations to the identity. Then, for \( I \subseteq \mathbb{R} \) compact,

\[
\|f_n - f_n \ast \phi_\epsilon\|_{L^1(I)} = \int_I \left| \int_\mathbb{R} (f_n(x) - f_n(x - y))\phi_\epsilon(y) dy \right| dx
\]

\[
= \int_I \left| \int_\mathbb{R} (f_n(x) - f_n(x - \epsilon y))\phi(y) dy \right| dx
\]

\[
\leq \int_\mathbb{R} \int_I |f_n(x) - f_n(x - \epsilon y)| dx |\phi(y) dy|
\]

\[
\leq \epsilon |I|
\]

Thus, for each \( \epsilon > 0 \), we choose \( I \subseteq \mathbb{R} \) compact so that all \( f_n \) satisfy

\[
\sup_n \int_{\mathbb{R} \setminus I} |f_n| < \epsilon
\]

\[\text{keyword: Helly's selection theorem}\]
and for this choice of $I$, choose $\delta_0 > 0$ such that
\[
\|f_n - f_n * \phi_\delta\|_{L^1(I)} < \varepsilon \quad \forall \delta_0 \geq \delta > 0
\]
For each such $\delta > 0$, the sequence $f_n * \phi_\delta$ is uniformly bounded and equicontinuous: first, by Young,
\[
\|f_n * \phi_\delta\|_{L^1(I)} \leq \|f_n\|_{L^\infty(\mathbb{R})} \|\phi_\delta\|_{L^1(\mathbb{R})} \leq 1
\]
Secondly,
\[
\|(f_n * \phi_\delta)'\|_{L^\infty(I)} = \|f_n * \phi_\delta\|_{L^\infty(I)} \leq \|f_n\|_{L^\infty(\mathbb{R})} \|\phi_\delta\|_{L^1(\mathbb{R})} \leq \delta^{-2} \|\phi'\|_{L^1(\mathbb{R})} < \infty
\]
with the latter expression independent of $n$. Thus the family $\{f_n * \phi_\delta\}_n$ is uniformly bounded and equicontinuous for each $\delta > 0$, and hence by Arzelà-Ascoli there is a continuous function $f_\delta$ on $I$ and a subsequence $n_k$ such that $f_{n_k} * \phi_\delta \to f_\delta$ uniformly on $I$. As such, there is a $K \in \mathbb{N}$ such that, for all $k > K$,
\[
\|f_{n_k} * \phi_\delta - f_\delta\|_{L^1(I)} < \varepsilon
\]
All together, we see that, for $j, k > K$,
\[
\|f_{n_j} - f_{n_k}\|_{L^1(I)} \leq \|f_{n_j} - f_{n_j} * \phi_\delta\|_{L^1(I)} + \|f_{n_j} * \phi_\delta - f_\delta\|_{L^1(I)} + \|f_\delta - f_{n_k} * \phi_\delta\|_{L^1(I)} + \|f_{n_k} * \phi_\delta - f_{n_k}\|_{L^1(I)} < 4\varepsilon
\]
and hence
\[
\|f_{n_j} - f_{n_k}\|_{L^1(I)} < 5\varepsilon
\]
for sufficiently large $j, k$. Thus we may construct the convergent subsequence as desired. \hfill \Box

**Problem 3.** Prove that $L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ is a Borel subset of $L^3(\mathbb{R}^n)$.

**Proof.** Note that, for $f \in L^3(\mathbb{R}^n)$,
\[
f \in L^\infty(\mathbb{R}^n) \iff \exists K \in \mathbb{N} \text{ such that } m(\{|f| > K\}) = 0
\]
We claim that
\[
m(\{|f| > K\}) = 0 \iff \int_q^r |f| \leq K(r - q) \forall q < r \in \mathbb{Q}
\]
(1)
The forward implication is clear. For the reverse implication, suppose that there is some bounded set $S \subseteq \mathbb{R}$ with $m(S) > 0$ such that $|f| > K + \varepsilon$ on $S$, where $\varepsilon > 0$. Then for every $\delta > 0$, the definition of Lebesgue measure (with some minor tweaks) supplies a finite disjoint union of open intervals with rational endpoints
\[
U = (q_1, r_1) \cup \cdots \cup (q_n, r_n)
\]
such that
\[
S \subseteq U, m(U) < m(S) + \delta
\]
Then we have
\[
\int_U |f| = \int_S |f| + \int_{U \setminus S} |f| > (K + \varepsilon)m(S) = (K + \varepsilon)\sum_{j=1}^n (r_j - q_j)
\]
If the right-hand side of (1) still holds, the above provides

\[ K \sum_{j=1}^{n} (r_j - q_j) + \int_{U \setminus S} |f| > (K + \varepsilon) \sum_{j=1}^{n} (r_j - q_j) \]

or

\[ \int_{U \setminus S} |f| > \varepsilon \sum_{j=1}^{n} (r_j - q_j) = \varepsilon m(S) > 0 \]

However, the left-hand side of the above can be expanded via Hölder as

\[ \int_{U \setminus S} |f| = \| f \chi_{U \setminus S} \|_{L^1} \leq \| f \|_{L^3} \| \chi_{U \setminus S} \|_{L^{3/2}} = \| f \|_{L^3} m(U \setminus S)^{2/3} < \| f \|_{L^3} \delta^{2/3} \]

so we conclude that, for our given \( f \in L^3(\mathbb{R}^n) \), there is some \( m(S) > 0 \) and \( \varepsilon > 0 \) such that for every \( \delta > 0 \) we have

\[ \| f \|_{L^3(\mathbb{R}^n)} \delta^{2/3} > \varepsilon m(S) > 0 \]

Since \( \| f \|_{L^3(\mathbb{R}^n)} < +\infty \), we may send \( \delta \to 0 \) to obtain a contradiction.

Thus we have shown (1). Since each \( \chi_{(q,r)} \) belongs to \( L^{3/2}(\mathbb{R}^n) = (L^3(\mathbb{R}^n))^* \), we see that integrating \( f \) against \( \chi_{(q,r)} \) is a continuous functional on \( L^3(\mathbb{R}^n) \). Thus the collection of \( f \) satisfying the RHS of (1) is Borel; taking a countable union over \( K \in \mathbb{N} \) provides that \( L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n) \) is a Borel subset of \( L^3(\mathbb{R}^n) \), as desired.

\[ \square \]

**Problem 4.** Fix \( f \in L^1(\mathbb{R}) \). Show that

\[ \lim_{n \to \infty} \int_{0}^{2} f(x) \sin(x^n)dx = 0 \]

**Proof.** For each \( n \),

\[ \int_{0}^{2} f(x) \sin(x^n)dx = \int_{0}^{1} f(x) \sin(x^n)dx + \int_{1}^{2} f(x) \sin(x^n)dx \]

We analyze each term separately. Note that

\[ \sin(x^n) \xrightarrow{n} 0 \text{ pointwise for } x \in (0, 1) \]

and so

\[ f(x) \sin(x^n) \xrightarrow{n} 0 \text{ pointwise for } x \in (0, 1) \]

Thus, by DCT, since \( f(x) \sin(x^n) \leq |f(x)| \in L^1(\mathbb{R}) \) for each \( n \),

\[ \lim_{n \to \infty} \int_{0}^{1} f(x) \sin(x^n)dx = \int_{0}^{1} \lim_{n \to \infty} f(x) \sin(x^n)dx = 0 \]

which is the desired result for the first term. Note that

\[ \frac{1}{i n x^{n-1}} \frac{d}{dx} e^{i x^n} = e^{i x^n} \]

2 keyword: oscillatory integral
For the second term, assuming first that $f \in C_c^\infty((1, 2))$,

\[
\int_1^2 f(x) \sin(x^n) dx = \text{Im} \left[ \int_1^2 f(x)e^{inx} dx \right]
\]

\[
= \text{Im} \left[ \int_1^2 f(x) \frac{d}{dx}e^{inx} dx \right]
\]

\[
= -\text{Im} \left[ \int_1^2 e^{inx} \frac{d}{dx} \left( \frac{f(x)}{inx^n-1} \right) dx \right]
\]

\[
= -\text{Im} \left[ \int_1^2 e^{inx} f'(x) \frac{1}{inx^n-1} dx + \frac{(n-1)}{in} \int_1^2 e^{inx} \frac{f(x)}{x^n} \right] dx
\]

\[
= \int_1^2 \cos(x^n) f'(x) \frac{1}{nx^n-1} dx + \frac{(n-1)}{n} \int_1^2 \cos(x^n) \frac{f(x)}{x^n} dx
\]

and hence, by the triangle inequality and DCT,

\[
\left| \int_1^2 f(x) \sin(x^n) dx \right| \leq \left| \int_1^2 \cos(x^n) f'(x) \frac{1}{nx^n-1} dx \right| + \frac{(n-1)}{n} \int_1^2 \cos(x^n) \frac{f(x)}{x^n} dx
\]

and as $n \to 0$.

Thus we have the desired limit in the $f^2$ term for all $f \in C_c^\infty(1, 2)$. By a standard fact, for general $f \in L^1((1, 2))$ we may find a sequence $\{f_n\}_{n=1}^\infty$ in $C_c^\infty((1, 2))$ such that $f_n \to f$ in $L^1$. For each $\varepsilon > 0$, let $j \in \mathbb{N}$ be such that $\|f_j - f\|_{L^1} < \varepsilon/2$. By the above, there is some $n \in \mathbb{N}$ such that, for all $k \geq n$,

\[
\left| \int_1^2 f_j(x) \sin(x^n) dx \right| < \varepsilon/2
\]

All together we have

\[
\left| \int_1^2 f(x) \sin(x^n) dx \right| \leq \left| \int_1^2 f_j(x) \sin(x^n) dx \right| + \left| \int_1^2 [f_j(x) - f(x)] \sin(x^n) dx \right|
\]

\[
< \varepsilon/2 + \int_1^2 |f_j(x) - f(x)| dx \quad \text{since } |\sin(t)| \leq 1 \text{ everywhere}
\]

\[
< \varepsilon
\]

for all $k \geq n$; letting $\varepsilon \to 0$ we obtain the desired

\[
\int_1^2 f(x) \sin(x^n) dx \n \to 0
\]

which together with the limit on for $\int_0^1$ provides the desired result.

\[\square\]

**Problem 5.** Rigorously determine the infimum of

\[
\int_{-1}^1 |P(x) - |x||^2 dx
\]

over all choices of polynomials $P \in \mathbb{R}[x]$ of degree not exceeding three.
Proof. Some details omitted. Write a general degree $\leq 3$ polynomial as $P_{abcd}(x) = ax^3 + bx^2 + cx + d$. We claim first that, for any particular choice of $(a, b, c, d)$,

$$E(a, b, c, d) = \int_{-1}^{1} |P_{abcd}(x) - |x||^2 \geq \int_{-1}^{1} |P_{000d}(x) - |x||^2 = E(0, b, 0, d)$$

(2)

Differentiating $E$ by $a, c$ we see that the function

$$(a, c) \mapsto E(a, b, c, d)$$

has a unique critical point at $a = c = 0$. At this point, the Hessian of $E$ in $a, c$ is

$$H_{a,c}E = \begin{bmatrix} 4/7 & 4/5 \\ 4/5 & 4/3 \end{bmatrix}$$

which has positive determinant and trace, hence is positive definite. Thus (2) holds and we may restrict attention to even polynomials.

Now, differentiating $E$ against $b, d$,

$$\partial_b E = 4b/5 + 4d/3 - 1, \quad \partial_d E = 4b/3 + 4d - 2$$

which defines a vector field in the $b, d$ plane. The inner-product of this vector field with an outer-pointing vector field is given by

$$D_{b,d}E_{(b,d)} \cdot \frac{1}{\sqrt{2}}(b, d) = \frac{1}{\sqrt{2}} \left( \frac{4b^2}{5} + \frac{8bd}{3} + 4d^2 - b - 2d \right)$$

which is positive for sufficiently large $\| (b, d) \|$, which implies that $E$ achieves a global minimum somewhere. This happens when $D_{b,d}E_{(b,d)} = 0$, or when

$$b = \frac{15}{16}, \quad d = \frac{3}{16}$$

and here we achieve

$$\inf E = E(0, \frac{15}{16}, 0, \frac{3}{16}) = \frac{1}{96}$$

as the infimum value. \qed

Problem 6. Let us define a sequence of linear functionals on $L^\infty(\mathbb{R})$ as follows:

$$L_n(f) = \frac{1}{n!} \int_0^{\infty} x^n e^{-x} f(x) dx.$$

(a) Prove that no subsequence of this sequence converges weak-∗.
(b) Explain why this does not contradict the Banach-Alaoglu Theorem.

Proof. (a): Suppose $\{L_{n_k}\}_k$ is a subsequence; we show that this sequence does not converge weak-∗ly. Since each integrand $\frac{1}{n!} x^n e^{-x}$ converges locally uniformly to 0, we may choose a sequence of compact intervals $I_k \subseteq \mathbb{R}$ satisfying

$$\frac{1}{n!} \int_{\mathbb{R} \setminus I_k} x^n e^{-x} dx < \frac{1}{10}$$
and
\[ \inf I_k \xrightarrow{k \to \infty} \infty \]
Choose a subsequence \( \{I_{k_j}\}_j \) whose intervals are pairwise disjoint. Then
\[ f = \sum_{j=1}^{\infty} (-1)^j \chi_{I_{k_j}} \]
is in \( L^\infty(\mathbb{R}) \). Then, for each \( j \),
\[ L_{n_{k_j}}(f) = L_{n_{k_j}}((-1)^j \chi_{I_{k_j}}) + L_{n_{k_j}}\left(\sum_{\ell \neq j} (-1)^\ell \chi_{I_{k_\ell}}\right) = (-1)^j + O\left(\frac{1}{10}\right) \]
where the implicit constant is at most 2. Thus the sequence
\[ \{L_{n_{k_j}}(f)\}_k \]
is not Cauchy, and so does not converge. Thus \( \{L_{n_{k_j}}\}_k \) does not converge weak-$\star$ly, as desired.

(b): Since \( L^\infty(\mathbb{R}) \) is non-separable, Banach-Alaoglu only shows that \( (L^\infty(\mathbb{R}))^\star \) is compact, not sequentially compact. \( \square \)

**Problem 7.** Let \( \mathcal{F}_M \) be the set of functions holomorphic on \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \) and continuous on \( \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\} \) that satisfy
\[ \int_0^{2\pi} |f(e^{it})| dt \leq M < \infty. \]
Show that every sequence \( \{f_n\} \) in \( \mathcal{F}_M \) contains a subsequence that converges uniformly on compact subsets of \( \mathbb{D} \).

**Proof.** Note first that \( \mathcal{F}_M \) is locally uniformly bounded and locally equicontinuous: for \( K \subseteq \mathbb{D} \) a compact ball, set \( r = \text{dist}(K, \mathbb{D}) > 0 \). Then, for any \( f \in \mathcal{F}_M \), and any \( z \in K \),
\[
|f(z)| = \left| \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(w)}{w-z} \, dw \right| \\
\leq \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{|f(w)|}{|w-z|} \, ds(w) \\
\leq \frac{1}{2\pi r} \int_{\partial \mathbb{D}} |f(w)| ds(w) \leq \frac{M}{2\pi r}
\]
and
\[
|f'(z)| = \left| \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(w)}{(w-z)^2} \, dw \right| \\
\leq \frac{M}{2\pi r^2}
\]
\[ \text{keyword: normal family} \]
which implies, for any \( w \in K \),

\[
|f(z) - f(w)| \leq |z - w||f'(t)| \leq \frac{M}{2\pi r^2}|z - w|
\]

where \( t \) is some point in \( K \) on the segment connecting \( z \) and \( w \).

Thus, for each \( k \geq 2 \) natural, the sequence \( \{f_n\} \) is uniformly bounded and equicontinuous on \( B(0, 1 - \frac{1}{k}) \). By Arzelà-Ascoli, we may iteratively refine \( \{f_n\} \) (keeping the first \( k \) terms on each step) to be uniformly convergent on each of these balls; this provides the locally uniformly convergent subsequence as desired. \( \square \)

**Problem 8.** For each \( z \in \mathbb{C} \), let

\[
F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}.
\]

(a) Show that \( F \) is an entire function and satisfies \(|F(z)| \leq e^{|z|}\).

(b) Show that there is an infinite collection of numbers \( a_n \in \mathbb{C} \), so that

\[
F(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{a_n^2} \right)
\]

and the product converges uniformly on compact subsets of \( \mathbb{C} \).

**Proof.** (a): Note that

\[
|F(z)| \leq \sum_{n=0}^{\infty} \frac{|z|^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!} \frac{(2n)!}{(n!)^2} 2^{-2n} \leq \sum_{n=0}^{\infty} \frac{|z|^{2n}}{2^n (2n)!} \leq e^{|z|}
\]

using the fact that

\[
\sum_{j=0}^{2n} \binom{2n}{j} = 2^{2n}
\]

which implies that \( F \) converges absolutely on \( \mathbb{C} \), hence is an entire function satisfying the desired estimate.

(b): Define the function

\[
G(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^{2n}(n!)^2}
\]

Thus \( F(z) = G(z^2) \). If \( \{a_n\} \) are the zeros of \( F \) (\( n \) ranging over \( \mathbb{Z} \) \( \setminus \{0\} \) and \( a_{-n} = -a_n \)), then \( a_n^2 = b_n \) are the zeros of \( G \) (here \( n \) ranges over \( \mathbb{N} \)). From \(|F(z)| \leq e^{|z|}\) we see that \( F \) has order \( \leq 1 \), and thus

\[
\sum_{n \neq 0} \frac{1}{|a_n|^2} < \infty
\]

which implies

\[
\sum_{n=1}^{\infty} \frac{1}{|b_n|^2} < \infty
\]
so that $G$ has order $< 1$. Hadamard’s canonical representation then provides

$$G(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{b_n}\right)$$

and hence

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

converging locally uniformly in $\mathbb{C}$.

**Problem 9.** Let $f \in L^1(\mathbb{C}) \cap C^1(\mathbb{C})$. Show that the integral

$$u(z) = -\frac{1}{2\pi} \int \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\lambda(\zeta)$$

defines a $C^1$ function on the whole complex plane that satisfies

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u(x + iy) = f(x + iy)$$

In this problem, $d\lambda(\zeta)$ denotes (planar) Lebesgue measure on $\mathbb{C}$ and $C^1$ is meant in the real-variables sense.

**Proof.** Since $f$ is $L^1$ and $\frac{1}{\zeta - z}$ is bounded near $\infty$, the integrand is integrable near $\infty$. Since $f$ is $C^1$ and $\frac{1}{\zeta - z}$ is locally integrable, the integrand is locally integrable. Thus the integrand is globally integrable and so $u$ is well-defined for every $z \in \mathbb{C}$.

Note that, by Cauchy-Pompeiu,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi} \int_{D} (\partial_x + i \partial_y) f \frac{1}{\zeta - z} d\lambda(\zeta)$$

for any domain $D$ containing $z$. Since $f$ is $L^1$, there is a sequence of radii $R_j$ such that

$$\int_{|z-\zeta|=R_j} |f| \frac{ds}{2\pi R_j} = \text{avg}_{|z-\zeta|=R_j} |f| \to 0$$

If this weren’t true, then the above integral would be above $\varepsilon > 0$ for all sufficiently large $R > 0$; however, we would then have

$$\infty > \int_{|\zeta-z|\geq 1} \left| \frac{f(\zeta)}{\zeta - z} \right| d\lambda(\zeta) = \int_{0}^{2\pi} \int_{1}^{\infty} \frac{|f(z + Re^{i\theta})|}{R} RdRd\theta = \int_{1}^{\infty} 2\pi \text{avg}_{|z-\zeta|=R} |f| dR \geq 2\pi \varepsilon \int_{R_0}^{\infty} dR = \infty$$
a contradiction. Thus we have the desired sequence \( R_j \), and so
\[
\left| \frac{1}{2\pi i} \int_{|z-\zeta|=R_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right| \leq \frac{1}{2\pi R_j} \left| 2\pi R_j \text{avg}_{|z-\zeta|=R_j} |f| \to 0 \right|
\]
Setting \( D = D_j = \{ z : |z - \zeta| \leq R_j \} \) and taking a limit, we find
\[
f(z) = \lim_{j \to \infty} -\frac{1}{2\pi} \int_{D_j} (\partial_x + i\partial_y) f \frac{1}{\zeta - z} \, d\lambda(\zeta) = -\frac{1}{2\pi} \int_C (\partial_x + i\partial_y) f \frac{1}{\zeta - z} \, d\lambda(\zeta)
\]
where the latter integral is in the sense of principal value.
Assuming first that \( f \) is compactly supported,
\[
\frac{\partial}{\partial x} u(x + iy) = \lim_{\Re h \to 0} -\frac{1}{2\pi h} \int_C \frac{f(\zeta)}{\zeta - (z + h)} - \frac{f(\zeta)}{\zeta - z} \, d\lambda(\zeta)
\]
\[
= \lim_{\Re h \to 0} -\frac{1}{2\pi h} \int_C \frac{f(\zeta + h)}{\zeta - (z + h) - z} - \frac{f(\zeta)}{\zeta - z} \, d\lambda(\zeta)
\]
\[
= \lim_{\Re h \to 0} -\frac{1}{2\pi} \int_C \frac{f(\zeta + h) - f(\zeta)}{h} \frac{1}{\zeta - z} \, d\lambda(\zeta) \quad \text{by a change of variables}
\]
\[
= -\frac{1}{2\pi} \int_C \frac{\partial_x f(\zeta)}{\zeta - z} \, d\lambda(\zeta)
\]
where we justify the last exchange of integral and limit by DCT, since
\[
\left| \frac{f(\zeta + h) - f(\zeta)}{h} \frac{1}{\zeta - z} \right| \leq \sup_{w \in \text{supp}(f)} |f'(w)| \frac{1}{|\zeta - z|} \chi_{B_R(0)}(\zeta)
\]
for sufficiently large \( R \) and small \( h \). Similarly,
\[
\frac{\partial}{\partial y} u(x + iy) = -\frac{1}{2\pi} \int_C \frac{\partial_y f(\zeta)}{\zeta - z} \, d\lambda(\zeta)
\]
so that
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u(x + iy) = -\frac{1}{2\pi} \int_C \frac{(\partial_x + i\partial_y) f(\zeta)}{\zeta - z} \, d\lambda(\zeta) = f(x + iy)
\]
by the Cauchy-Pompeiu calculation above. If \( \chi_j \) is a sequence of smooth cutoff functions satisfying
\[
0 \leq \chi_j \leq \chi_{j+1}, \chi_j \downarrow 1, \chi_j \equiv 1 \quad \text{on } B_j(z), \text{supp}(\chi_j) \subseteq B_{j+1}(z)
\]
then, setting \( u_j \) to be the function constructed in the problem using the function \( f \chi_j \),
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u_j(x + iy) = -\frac{1}{2\pi} \int_C \frac{(\partial_x + i\partial_y) f(\zeta) \chi_j(\zeta)}{\zeta - z} \, d\lambda(\zeta)
\]
\[
= -\frac{1}{2\pi} \int_C \frac{f(\zeta)(\partial_x + i\partial_y) \chi_j(\zeta)}{\zeta - z} \, d\lambda(\zeta)
\]
\[
= -\frac{1}{2\pi} \int_C \frac{\chi_j(\zeta)(\partial_x + i\partial_y) f(\zeta)}{\zeta - z} \, d\lambda(\zeta)
\]
\[
= \chi_j(x + iy) f(x + iy)
\]
As \( j \to \infty \), the right-hand side of the above has local uniform limit \( f(x + iy) \). Thus

\[
\lim_{j \to \infty} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u_j(x + iy) = f(x + iy)
\]

whereas

\[
u_j \to u \quad \text{pointwise}
\]

Since \( \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u_j \) converges locally uniformly, it follows that

\[
f(x + iy) = \lim_{j \to \infty} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u_j(x + iy) = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u(x + iy)
\]
as desired.

\[\square\]

**Problem 10.** Evaluate the improper Riemann integral

\[
\int_{0}^{\infty} \frac{x^2 - 1 \sin x}{x^2 + 1} \frac{dx}{x}.
\]

Justify all manipulations.

**Proof.** Since the integrand is even,

\[
\int_{0}^{\infty} \frac{x^2 - 1 \sin x}{x^2 + 1} \frac{dx}{x} = \frac{1}{2} \int_{\mathbb{R}} \frac{x^2 - 1 \sin x}{x^2 + 1} \frac{dx}{x}
\]

Let \( C_R \) denote the upper half circle with radius \( R \), that is, the circular arc in the upper half plane connecting \( R \) to \( -R \). Let \( D_R \) denote the curve formed by a straight line from \( -R \) to \( -\frac{1}{R} \), travels a half-circle in the upper half plane to \( \frac{1}{R} \), and then travels by a straight line segment to \( R \). Let \( \Gamma_R \) denote the curve \( C_R \cup D_R \). Then for \( R \geq 2 \) the residue theorem provides

\[
\int_{\Gamma_R} \frac{z^2 - 1 e^{iz}}{z^2 + 1} \frac{dz}{z} = 2\pi i \text{Res} \left[ \frac{z^2 - 1 e^{iz}}{z^2 + 1}, i \right] = 2\pi i \frac{-2 e^{-1}}{i} = 2\pi e^{-1}
\]

By Jordan’s lemma,

\[
\left| \int_{C_R} \frac{z^2 - 1 e^{iz}}{z^2 + 1} \frac{dz}{z} \right| \leq \pi \max_{z \in C_R} \left| \frac{z^2 - 1}{z(z^2 + 1)} \right| \leq \frac{\pi}{R} \cdot 100
\]

so that

\[
\lim_{R \to \infty} \int_{C_R} \frac{z^2 - 1 e^{iz}}{z^2 + 1} \frac{dz}{z} = 0
\]
By the fractional residue theorem,

\[
\lim_{R \to \infty} \int_{D_R} \frac{z^2 - 1}{z^2 + 1} e^{iz} \frac{dz}{z} = -\lim_{R \to \infty} \int_{|z| = \frac{1}{R}, \text{Im}(z) > 0} \frac{z^2 - 1}{z^2 + 1} e^{iz} \frac{dz}{z} \\
+ \frac{1}{2} \pi i \text{Res} \left[ \frac{z^2 - 1}{z^2 + 1} e^{iz} \frac{dz}{z}, 0 \right] \\
+ \lim_{R \to \infty} \int_{(-R, -\frac{1}{R}) \cup (\frac{1}{R}, R)} \frac{z^2 - 1}{z^2 + 1} e^{iz} \frac{dz}{z} \\
+ i \lim_{R \to \infty} \int_{(-R, -\frac{1}{R}) \cup (\frac{1}{R}, R)} \frac{1}{z^2 + 1} \sin x \frac{dx}{x} \\
= \pi i + 0 \quad \text{(since the integrand is odd)} \\
+ i \lim_{R \to \infty} \int_{(-R, -\frac{1}{R}) \cup (\frac{1}{R}, R)} \frac{1}{z^2 + 1} \sin x \frac{dx}{x} \\
= \pi i + i \int_{\mathbb{R}} \frac{1}{x^2 + 1} \sin x \frac{dx}{x}
\]

Together we have

\[
\int_0^\infty \frac{x^2 - 1}{x^2 + 1} \frac{dx}{x} = \frac{1}{2} \int_{\mathbb{R}} \frac{x^2 - 1}{x^2 + 1} \frac{dx}{x} \\
= \frac{1}{2i} \left[ 2\pi e^{-i} - \pi i \right] \\
= \pi e^{-i} - \frac{\pi}{2}
\]

\[\blacklozenge\]

**Problem 11.** Let \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \) and let \( K \subseteq \mathbb{T} \) be a compact proper subset. 
(a) Show that there is a sequence of polynomials \( P_n(z) \) so that \( P_n(z) \to \bar{z} \) uniformly on \( K \).
(b) Show that there is no sequence of polynomials \( P_n(z) \) for which \( P_n(z) \to \bar{z} \) uniformly on \( \mathbb{T} \).

**Proof.** (a): Since \( \mathbb{C} \setminus K \) contains \( \mathbb{D}, \mathbb{C} \setminus \overline{\mathbb{D}} \), and some point of \( \partial \mathbb{D} \), we see that \( \mathbb{C} \setminus K \) is path connected and hence connected. By Runge’s theorem, we may find a sequence of polynomials \( \{ P_n \} \) such that \( P_n(z) \to \frac{1}{z} \) uniformly on \( K \). Since \( \bar{z} = \frac{1}{z} \) on \( K \subseteq \partial \mathbb{D} \), this is the desired result.

(b): Suppose for the sake of contradiction that \( P_n \) is a sequence of polynomials converging uniformly on \( \mathbb{T} \) to \( \bar{z} \). Then, for each \( z \in \mathbb{D} \),

\[
P_n(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{P_n(w)}{z-w} dw
\]

Thus, for each \( z \in \mathbb{D} \), uniform convergence of the \( P_n \) provides

\[
\lim_{n \to \infty} P_n(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{\bar{w}}{z-w} dw = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{1}{z-w} dw
\]

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locally uniformly. Since the locally uniform limit of holomorphic functions is holomorphic, the limit function

\[ P(z) = \lim_{n \to \infty} P_n(z) = \begin{cases} \frac{2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases} \]

is holomorphic in \( \mathbb{D} \). But note that \( P \) here is not even continuous, a contradiction. Thus no sequence of polynomials converges uniformly on \( \mathbb{T} \) to \( \mathbb{Z} \).

**Problem 12.** Let \( u \) be a continuous subharmonic function on \( \mathbb{C} \) that satisfies

\[ \limsup_{|z| \to \infty} \frac{u(z)}{\log |z|} \leq 0 \]

Show that \( u \) is constant on \( \mathbb{C} \).

**Proof.** Since subharmonic functions are preserved by conformal changes-of-coordinate, the function

\[ v(z) := u \left( \frac{1}{z} \right) \]

is subharmonic on \( \mathbb{C} \setminus \{0\} \) and satisfies

\[ v(z) = o(\log |z|) \quad \text{as } z \to 0 \]

For each \( \varepsilon > 0 \), the function \( v(z) - \varepsilon \log |z| \) satisfies the maximum principle on \( \mathbb{D} \setminus \{0\} \). By the decay estimate on \( v \) at 0,

\[ v(z) - \varepsilon \log |z| \to -\infty \quad \text{as } z \to 0 \]

and hence, for any \( z \in \mathbb{D} \setminus \{0\} \),

\[ v(z) - \varepsilon \log |z| \leq \max_{z \in \partial \mathbb{D}} v(z) \]

Since \( \varepsilon > 0 \) was arbitrary, we conclude

\[ v(z) \leq \max_{z \in \partial \mathbb{D}} v(z) \]

for all \( z \in \mathbb{D} \setminus \{0\} \). Thus \( v \) is bounded above near 0, so \( u \) is bounded above near \( \infty \). Since \( u \) was assumed to be continuous, it is locally bounded, and hence is globally bounded above. By a standard fact, globally bounded above subharmonic functions on \( \mathbb{C} \) are constant (by e.g. the super-averaging principle) and hence \( u \) is constant.

\[ \square \]

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\( ^{4} \)keyword: Phragmén-Lindelöf
2 Fall 2020

Problem 1. (a) Suppose \( f : [0, 1] \times [0, \infty) \rightarrow [0, 1] \) is continuous. Prove that \( F : [0, 1] \rightarrow [0, 1] \) defined by

\[
F(x) = \limsup_{y \to \infty} f(x, y)
\]

is Borel measurable.

(b) Show that for any Borel set \( E \subseteq [0, 1] \) there is a choice of continuous function \( f : [0, 1] \times \mathbb{R} \rightarrow [0, 1] \) so that \( F \) agrees almost everywhere with the indicator function of \( E \).

Proof. (a): Since half-open intervals of the form \((a, \infty)\) with \( a \in \mathbb{R} \) generate as a \( \sigma \)-algebra the full Borel \( \sigma \)-algebra \( \mathcal{B} \), it suffices to show that

\[
F^{-1}((a, \infty)) \in \mathcal{B}
\]

for arbitrary \( a \in \mathbb{R} \). The left-hand side may be written as

\[
F^{-1}((a, \infty)) = \{x \in [0, 1] : \exists n \in \mathbb{N} \forall q \in \mathbb{Q} \cap [0, \infty) \exists p \in \mathbb{Q} \cap (q, \infty) \text{ s.t. } f(x, p) \in (a + \frac{1}{n}, \infty)\}
\]

\[
= \bigcup_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q} \cap [0, \infty)} \bigcup_{p \in \mathbb{Q} \cap (q, \infty)} \{x \in [0, 1] : f(x, p) \in (a + \frac{1}{n}, \infty)\}
\]

\[
= \bigcup_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q} \cap [0, \infty)} \bigcup_{p \in \mathbb{Q} \cap (q, \infty)} f(\cdot, p)^{-1}(a + \frac{1}{n}, \infty)
\]

Since \( f \) is continuous, the latter set is Borel; hence \( F \) is Borel, as desired.

(b): We show that the set of all such \( E \) contains the set of closed intervals with nonempty interior in \([0, 1]\) and is a \( \sigma \)-algebra; the result follows. Suppose \( a < b \in [0, 1] \). Then the function

\[
f(x, y) = \begin{cases} 
0 & x \not\in (a - \frac{1}{1+y}, b + \frac{1}{1+y}) \\
(1+y)(x - (a - \frac{1}{1+y})) & x \in [a - \frac{1}{1+y}, a] \\
1 & x \in (a, b) \\
-(1+y)(x - b) & x \in [b, b + \frac{1}{1+y}]
\end{cases}
\]

is clearly continuous on \([0, 1] \times [0, \infty)\) and satisfies

\[
\lim_{y \to \infty} f(x, y) = \chi_{[a,b]}
\]

as claimed.

Now suppose that \( g \) is continuous on \([0, 1] \times [0, \infty)\) and induces \( G = \chi_B \) for some subset \( B \subseteq [0, 1] \) as in the problem statement. Then \( 1 - g(x, y) \) induces \( G' = \chi_{[0,1] \setminus B} \), so the family of sets achievable this way is closed under complements.

Now suppose \( f_1, f_2, \ldots \) are continuous functions on \([0, 1] \times [0, \infty)\) inducing \( F_k = \chi_{B_k} \) for subsets \( B_1, B_2, \ldots \subseteq [0, 1] \). Let \((a_n, b_n)\) be an enumeration of \( \mathbb{N} \times \mathbb{N} \), and define

\[
g(x, y) = \begin{cases} 
f_k(x, b_n + y - [y]) & [y] = 2n + 1, a_n = k \\
-(y - [y]) + 1/2 f_k(x, b_n + 1) & [y] = 2n, a_{n-1} = k, |y - [y]| \leq 1/2 \\
(y - [y] - 1/2) f_i(x, b_{n+1}) & [y] = 2n, a_n = l, |y - [y]| > 1/2
\end{cases}
\]
The upshot of this construction is that, for each fixed \( x \), \( g(x, y) \) realizes every subsequential limit in the \( y \)'s coming from the \( f_k \), while still being continuous. In particular we see that the limit function

\[
G(x) = \lim_{y \to \infty} g(x, y) = \chi_{\bigcup B_k}
\]

and we conclude that our family of sets realizable in this way is closed under countable union. Together with the presence of closed intervals and complements, we see that the sets realized through this procedure is all of \( B \). 

\[\square\]

**Problem 2.** Show that there is a constant \( c \in \mathbb{R} \) so that

\[
\lim_{n \to \infty} \int_0^1 f(x) \cos(\sin(n\pi x)) \, dx = c \int_0^1 f(x) \, dx
\]

for every \( f \in L^1([0, 1]) \). The limit is taken over those \( n \in \mathbb{N} \).

**Proof.** We first show that

\[
\int_0^1 \cos(\sin(r\pi x)) \, dx \xrightarrow{r \to \infty} c
\]

for some constant \( c \in \mathbb{R} \). To do this, define the functions

\[
G(r) := \int_0^1 \cos(\pi r x) \, dx = \frac{1}{r} \int_0^r \cos(\pi x) \, dx
\]

and

\[
F(r) = \int_0^r \cos(\pi x) \, dx = rG(r)
\]

for \( 0 < r \in \mathbb{R} \). Note that

\[
F(r + 1) = F(r) + \int_r^{r+1} \cos(\pi x) \, dx
\]

\[
= F(r) + \int_0^1 \cos(\pi (x - \lfloor r \rfloor)) \, dx
\]

\[
= F(r) + \int_0^1 \cos(\pm \sin(\pi x)) \, dx
\]

\[
= F(r) + F(1)
\]

from which we conclude

\[
F(r) = \lfloor r \rfloor F(1) + F(r - \lfloor r \rfloor)
\]

whenever \( r \not\in \mathbb{N} \), and

\[
F(n) = nF(1)
\]

for \( n \in \mathbb{N} \). Thus

\[
G(n) = \frac{F(n)}{n} = F(1)
\]

and, for \( r \not\in \mathbb{N} \),

\[
G(r) = \frac{F(r)}{r} = F(1) - \frac{\lfloor r \rfloor - r}{r} F(1) + \frac{F(r - \lfloor r \rfloor)}{r}
\]
from which we easily see that 
\[ G(r) \xrightarrow{r \to \infty} F(1) \in \mathbb{R} \]
which is our \( c \). Thus, for any interval \([a, b] \subseteq [0, 1]\) of positive length,
\[
\int_0^1 \chi_{[a,b]}(x) \cos(n \pi x) \, dx = \int_0^1 (\chi_{[a,b]} - \chi_{[a,a]}) \cos(n \pi x) \, dx
\]
\[
= bG(bn) - aG(an)
\]
\[
\xrightarrow{n \to \infty} (b-a)c = c \int_a^b \chi_{[a,b]}
\]
if \( a \neq 0 \), and
\[
\int_0^1 \chi_{[a,b]}(x) \cos(n \pi x) \, dx = bG(bn) \xrightarrow{n \to \infty} bc = c \int_{[0,b]}
\]
Summing, we see that for any simple function \( f \),
\[
\lim_{n \to \infty} \int_0^1 f(x) \cos(n \pi x) \, dx = c \int_0^1 f(x) \, dx
\]
Now suppose \( f \in L^1([0, 1]) \) and \( f_1, f_2, \ldots \to f \) are simple functions. Let \( \varepsilon > 0 \), and fix \( k \in \mathbb{N} \) such that \( \|f_k - f\|_{L^1} < \varepsilon/3 \) and \( \|f_k - f\|_{L^1} < \varepsilon/(3c) \). Lastly, pick \( N \in \mathbb{N} \) such that, for all \( n > N \),
\[
\left| \int_0^1 f_k(x) \cos(n \pi x) \, dx - c \int_0^1 f_k(x) \, dx \right| < \varepsilon/3 \tag{3}
\]
Then we have, for such \( n \),
\[
\int_0^1 f(x) \cos(n \pi x) \, dx = \int_0^1 f_k(x) \cos(n \pi x) \, dx + \int_0^1 [f - f_k](x) \cos(n \pi x) \, dx
\]
\[
= c \int_0^1 f_k(x) \, dx + \left( \int_0^1 f_k(x) \cos(n \pi x) \, dx - \int_0^1 f_k(x) \, dx \right)
\]
\[
+ \int_0^1 f_k(x) \cos(n \pi x) \, dx
\]
\[
= c \int_0^1 f(x) \, dx + c \int_0^1 [f - f_k](x) \, dx
\]
\[
+ \left( \int_0^1 f_k(x) \cos(n \pi x) \, dx - \int_0^1 f_k(x) \, dx \right)
\]
\[
+ \int_0^1 [f - f_k](x) \cos(n \pi x) \, dx
\]
\[
= c \int_0^1 f(x) \, dx + I + II + III
\]
Since \( \|f_k - f\|_{L^1} < \varepsilon/(3c) \), we see that \( |I| < \varepsilon/3 \). Since \( \|f_k - f\|_{L^1} < \varepsilon/3 \) and \( |\cos(n \pi x)| \leq 1 \) everywhere, we see that \( |III| < \varepsilon/3 \). Lastly, by \( |II| < \varepsilon/3 \). Thus, for all \( n > N \),
\[
\left| \int_0^1 f(x) \cos(n \pi x) \, dx - c \int_0^1 f(x) \, dx \right| < \varepsilon
\]
and so \( \int_0^1 f(x) \cos(n \pi x) \, dx \xrightarrow{n \to \infty} c \int_0^1 f(x) \, dx \) for arbitrary \( f \in L^1([0, 1]) \), as desired. \( \square \)
Problem 3. Let \( d\mu_n \) be a sequence of probability measures on \([0, 1]\) so that
\[
\int f(x) \, d\mu_n(x)
\]
converges for every continuous function \( f : [0, 1] \to \mathbb{R} \).

(a) Show that
\[
\iint g(x, y) \, d\mu_n(x) \, d\mu_n(y)
\]
converges for every continuous function \( g : [0, 1]^2 \to \mathbb{R} \).

(b) Show by example that under the above hypotheses, it is possible that
\[
\int_{0 \leq x \leq y \leq 1} d\mu_n(x) \, d\mu_n(y)
\]
does not converge.

Proof. (a): We first show this in the case \( g(x, y) = f(x)f'(y) \) for continuous functions \( f, f' \) on \([0, 1]\). In this case,
\[
\iint g(x, y) \, d\mu_n(x) \, d\mu_n(y) = \iint f(x)f'(y) \, d\mu_n(x) \, d\mu_n(y) = \left( \int_0^1 f(x) \, d\mu_n(x) \right) \left( \int_0^1 f'(y) \, d\mu_n(y) \right)
\]
is a product of convergent sequences, hence converges.

We claim that the algebra \( \mathcal{A} \) generated by such products are dense in \( C([0, 1]^2) \) with respect to the \( L^\infty \) norm. To show this we use Stone-Weierstrass: since \([0, 1]^2\) is a compact metric space, it suffices to show that \( \mathcal{A} \) contains the constants and separates points. Clearly we have all constants. Suppose \((x, y), (x', y') \in [0, 1]^2\) are distinct, say \( x \neq x' \). Then let \( f \) be continuous on \([0, 1]\) such that \( f(x) = 1, f(x') = 0 \). Then \( g(x, y) = f(x) \) is a product of two continuous functions on \([0, 1]\) which separates \((x, y)\) from \((x', y')\); thus \( \mathcal{A} \) is dense in \( C([0, 1]^2) \).

Since \( \mathcal{A} \) is in fact linearly generated by such tensor product pairs, we see from the computation
\[
\iint a g(x, y) + b g'(x, y) \, d\mu_n(x) \, d\mu_n(y) = a \iint g(x, y) \, d\mu_n(x) \, d\mu_n(y) + b \iint g'(x, y) \, d\mu_n(x) \, d\mu_n(y)
\]
that the desired convergence holds for all elements of \( \mathcal{A} \).

We conclude by approximation: suppose \( g \in C([0, 1]^2) \) and let \( g_1, g_2, \ldots \in \mathcal{A} \) such that \( g_j \mathop{\to}_{L^\infty} g \). If \( \varepsilon > 0 \), fix \( j \in \mathbb{N} \) such that \( \|g_j - g\|_{L^\infty} < \varepsilon/3 \). Then let \( N \in \mathbb{N} \) such that
\[
\left| \int g_j(x, y) \, d\mu_n(x) \, d\mu_n(y) - \int g_j(x, y) \, d\mu_m(x) \, d\mu_m(y) \right| < \varepsilon/3
\]

\(^5\text{keyword: Stone-Weierstrass}\)
for all \( n, m > N \). Then, for such \( n, m \),
\[
\left| \iint g(x, y)d\mu_n(x)d\mu_n(y) - \iint g(x, y)d\mu_m(x)d\mu_m(y) \right| \\
\leq \left| \iint g(x, y)d\mu_n(x)d\mu_n(y) - \iint g_j(x, y)d\mu_n(x)d\mu_n(y) \right| \\
+ \left| \iint g_j(x, y)d\mu_n(x)d\mu_n(y) - \iint g_j(x, y)d\mu_m(x)d\mu_m(y) \right| \\
+ \left| \iint g_j(x, y)d\mu_m(x)d\mu_m(y) - \iint g(x, y)d\mu_m(x)d\mu_m(y) \right| \\
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\]
from which we conclude that
\[
n \mapsto \iint g(x, y)d\mu_n(x)d\mu_n(y)
\]
is Cauchy, hence convergent. Thus the result holds for every continuous real-valued \( g \), as desired.

(b): Define \( \mu_n \) as
\[
\mu_n = \begin{cases} 
\delta_1 & n \text{ odd} \\
\sum_{k=1}^{n} \frac{1}{n} \delta_{1-k/n^2} & n \text{ even}
\end{cases}
\]
Then clearly every continuous function \([0, 1] \rightarrow \mathbb{R}\) satisfies
\[
\int f(x)d\mu_n(x) \rightarrow f(1)
\]
as \( n \rightarrow \infty \); however,
\[
\iint_{0 \leq x \leq y \leq 1} d\mu_n(x)d\mu_n(y) = 1
\]
for \( n \) odd, whereas for \( n \) even
\[
\iint_{0 \leq x \leq y \leq 1} d\mu_n(x)d\mu_n(y) = 1/2 + \sum_{k=1}^{n} \frac{1}{n^2} \\
= 1/2 + 1/n
\]
which has limit \( 1/2 \). Thus the above sequence fails to converge over \( n \in \mathbb{N} \).

**Problem 4.** Let \( X \) be a separable Banach space over \( \mathbb{R} \) and let \( F : X \rightarrow \mathbb{R} \) be norm-continuous and convex. Suppose now that a sequence \( x_n \) in \( X \) converges weakly to \( x \in X \). Show that
\[
F(x) \leq \sup_n F(x_n)
\]

**Proof.** By a standard fact of functional analysis, there are convex linear combinations
\[
y_n = t_{x_1}^n x_1 + \ldots + t_{x_n}^n x_n
\]
such that \( y_n \rightarrow x \) strongly. Since \( F \) is (strongly) continuous, \( F(y_n) \rightarrow F(x) \). By convexity,
\[
F(y_n) \leq t_{x_1}^n F(x_1) + \ldots + t_{x_n}^n F(x_n) \leq \sup_{1 \leq j \leq n} F(x_n)
\]
For every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that, for each $n > N$,

$$F(x) \leq F(y_n) - \varepsilon \leq \sup_{1 \leq j \leq n} F(x_n) - \varepsilon \leq \sup_n F(x_n) - \varepsilon$$

Sending $\varepsilon$ to 0, we conclude that

$$F(x) \leq \sup_n F(x_n)$$

as desired.

**Problem 5.**

Suppose $f \in L^1([0, 1])$ has the property that

$$\int_E |f(x)|\,dx \leq \sqrt{|E|}$$

for every Borel $E \subseteq [0, 1]$. Here $|E|$ denotes the Lebesgue measure of $E$.

(a) Show that $f \in L^p([0, 1])$ for all $p > 2$.

(b) Give an example of an $f$ satisfying $(\ast)$ that is not in $L^2([0, 1])$.

**Proof.** (a): Fix $p < 2$. Let $k \in \mathbb{Z}$ be such that $j - 1 > j(p - 1)$ for all $j \geq k$. We note the following dyadic decomposition of $f$:

$$\sum_{j \in \mathbb{Z}} 2^{j-1} \chi_{2^{j-1} \leq |f| < 2^j} \leq |f| \leq \sum_{j \in \mathbb{Z}} 2^j \chi_{2^{j-1} \leq |f| < 2^j}$$

Since $f \in L^1([0, 1])$, the provided inequality provides

$$\sum_{j \in \mathbb{Z}} \sqrt{\{2^{j-1} \leq |f| < 2^j\}} 2^{j-1} \leq \int |f| < \infty$$

Thus

$$\sum_{j \geq k} \sqrt{\{2^{j-1} \leq |f| < 2^j\}} 2^{j(p-1)} \leq \sum_{j \geq k} \sqrt{\{2^{j-1} \leq |f| < 2^j\}} 2^{j-1} < \infty \quad (4)$$

Now we may estimate

$$\int |f|^p = \int |f||f|^{p-1} \leq \sum_{j \in \mathbb{Z}} \int |f| 2^{j(p-1)} \chi_{2^{j-1} \leq |f| < 2^j} \leq \sum_{j \in \mathbb{Z}} \sqrt{\{2^{j-1} \leq |f| < 2^j\}} 2^{j(p-1)} = \sum_{j < k} \sqrt{\{2^{j-1} \leq |f| < 2^j\}} 2^{j(p-1)} + \sum_{j \geq k} \sqrt{\{2^{j-1} \leq |f| < 2^j\}} 2^{j(p-1)}$$

The first summand is the inner product of an $\ell^\infty$ vector and an $\ell^1$ vector, hence is finite. As we saw in 4, the second summand is finite. Thus $f \in L^p([0, 1])$. Since $p < 2$ was arbitrary, we have the desired result.

(b): 

\footnote{keyword: dyadic decomposition; Lorentz spaces}
Problem 6. Prove that the following inequality is valid for all odd $C^1$ functions $f : [-1, 1] \to \mathbb{R}$:

$$\int_{-1}^{1} |f(x)|^2 dx \leq \int_{-1}^{1} |f'(x)|^2 dx$$

By odd, we mean that $f(-x) = -f(x)$.

Proof. We compute

$$\int_{-1}^{1} f(x)^2 dx = \int_{-1}^{1} \left( \int_{0}^{x} f'(t) dt \right)^2 dx$$

$$\leq \left[ \int_{-1}^{1} |f'(t)||1 - t|^{1/2} dt \right]^2 \quad \text{by Minkowski}$$

$$\leq \left[ \int_{-1}^{1} f'(t)^2 dt \right] \left[ \int_{-1}^{1} |1 - t| dt \right] \quad \text{by Hölder}$$

$$= \int_{-1}^{1} f'(t)^2 dt$$

as desired. $\square$

Problem 7. Let $\Delta_j = \{ z : |z - a_j| \leq r_j \}, 1 \leq j \leq n$ be a collection of disjoint closed disks, with radii $r_j \geq 0$, all contained in the open unit disk $\mathbb{D}$ of the complex plane. Let $\Omega = \mathbb{D} \setminus (\bigcup_j \Delta_j)$, and let $u : \Omega \to \mathbb{R}$ be harmonic. Prove that there exist real numbers $c_1, \ldots, c_n$ such that

$$u(z) - \sum_{j=1}^{n} c_j \log |z - a_j|$$

is the real part of a (single valued) analytic function on $\Omega$. Show also that the choice of $c_1, \ldots, c_n$ is unique.

Proof. Consider the one-form

$$v(z) = \frac{\partial u}{\partial z} dz$$

on $\Omega$, where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$ is a Wirtinger derivative. Since $u$ is harmonic, $v$ is holomorphic (in the sense that the coefficient function is holomorphic). For each $1 \leq j \leq n$, set $C_j$ to be a counterclockwise loop in $\Omega$ around $\Delta_j$ that doesn’t enclose any other $\Delta_k$; by standard algebraic topology, $\{C_j\}_{1 \leq j \leq n}$ determine a basis for $H_1(\Omega; \mathbb{R})$.

Now, for each $j$, set

$$c_j = \frac{1}{\pi i} \int_{C_j} v(z) dz$$

A standard calculation shows

$$\int_{C_j} c_j \frac{\partial}{\partial z} \log |z - a_j| dz = \pi ic_j = \int_{C_j} v(z) dz$$

so that

$$w(z) := u(z) - \sum_{j=1}^{n} c_j \log |z - a_j|$$
is a harmonic function on $\Omega$ satisfying
\[ \int_{C_j} \frac{\partial}{\partial z} w(z) dz = 0 \]
for all $j$. Since $\{C_j\}$ generate all of $H_1(\Omega; \mathbb{R})$, we see that, for $h(z) = \frac{\partial}{\partial z} w(z)$, $h$ holomorphic on $\Omega$ and $\int_\gamma h(z) dz = 0$ for all piecewise smooth loops $\gamma$ in $\Omega$. Thus, for fixed $z_0 \in \Omega$, the path integral
\[ g(z) = w(z_0) + \int_{z_0}^z h(t) dt \]
is well-defined (that is, independent of path chosen) and is an analytic antiderivative for $h$ on $\Omega$. Thus we find $\frac{\partial}{\partial z} g(z) = h(z) = \frac{\partial}{\partial z} w(z)$, $g(z_0) = w(z_0)$ and so $w$ is equal to $g + a$ where $a$ is a conjugate-analytic function on $\Omega$ vanishing at $z_0$; since $w$ is purely real-valued, $\text{Im}(a) = -\text{Im}(g)$ and so $\bar{a}$ is an analytic function on $\Omega$ whose imaginary part agrees with that of $g$ everywhere on $\Omega$. Thus $a$ and $g$ agree up to a real additive constant; since $a(z_0) = 0$ we see that $w = g + \bar{g} - g(z_0)$. Thus $w = 2\text{Re}(g) - g(z_0)$, so $w$ is indeed the real part of an analytic function on $\Omega$.

Finally, if
\[ u(z) - \sum_{j=1}^n d_j \log |z - a_j| = \text{Re}(q) \]
for constants $d_1, \ldots, d_n$ and analytic function $q$ on $\Omega$, then
\[ \int_{C_j} \frac{\partial q}{\partial z} dz = 0 \]
for each $z$, by e.g. examining the Laurent series about each $a_j$. Taking real parts,
\[ \int_{C_j} \frac{\partial u}{\partial z} dz = d_j \pi i \]
which implies that $d_j = c_j$ from before, so these constants are unique.

Problem 8.\[\square\] Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and satisfy $f(\frac{1}{2}) = f(-\frac{1}{2}) = 0$. Show that $|f(0)| \leq \frac{1}{4}$.

Proof. Note that $\tilde{f}$, defined by
\[ \tilde{f}(z) = \frac{1 - z/2 + z/2}{z - 1/2} f(z) \]
is analytic in $\mathbb{D}$ and takes values in $\overline{\mathbb{D}}$, by standard Blaschke factor theory. Thus
\[ |f(0)| = \left| \left| \frac{1/2}{1} \right| \left| \frac{1/2}{1} \right| |\tilde{f}(0)| \right| \leq \frac{1}{4} \]
as desired.\[\square\]

\[7\text{keyword: Blaschke factors}\]
Problem 9. Consider the following region in the complex plane:

$$\Omega = \{x + iy : 0 < x < \infty \text{ and } 0 < y < \frac{1}{x}\}.$$ 

Exhibit an explicit conformal mapping $f$ of $\Omega$ onto $D = \{z \in \mathbb{C} : |z| < 1\}$.

Proof. We first claim that, for $f_1(z) = z^2$,

$$f_1(\Omega) = \{z \in \mathbb{C} : \text{Im}(z) \in (0, 2)\}$$

and that $f_1$ is a conformal map between these two domains. To show this, note first that $\Omega$ doesn’t contain any pairs $z_1, z_2$ with $z_2 = -z_1$; since $f_1$ is clearly analytic, $f_1$ is a conformal map $\Omega \to f_1(\Omega)$.

We show that its image is as claimed. First note that, for any $x + iy$ with $y < 1/x$,

$$f_1(x + iy) = x^2 - y^2 + i2xy$$

so $f_1(\Omega) \subseteq \{z : \text{Im}(z) < 2\}$. Similarly, since $x > 0, y > 0$ for all $x + iy \in \Omega$, $f_1(\Omega) \subseteq \{z : \text{Im}(z) > 0\}$, so we conclude that

$$f_1(\Omega) \subseteq \{z \in \mathbb{C} : \text{Im}(z) \in (0, 2)\}$$

We claim that every point of this domain lies in the image of $f_1$. If $\sqrt{i}$ denotes the branch of the inverse of $z \mapsto z^2$ for which $\sqrt{i} \in \Omega$, we see that

$$w = \sqrt{r}e^{i\theta} = \sqrt{r}(\cos \theta + i \sin \theta) = \sqrt{r}(\cos(\theta/2) + i \sin(\theta/2))$$

satisfies

$$\text{Re}(w), \text{Im}(w) > 0 \quad \text{and} \quad \text{Im}(w)\text{Re}(w) = r \cos(\theta/2) \sin(\theta/2) = \frac{r}{2} \sin(\theta) < 1$$

whenever $r \sin(\theta) = \text{Im}(w) < 2$, i.e. when $w$ is in the putative image of $f_1$. Thus $f_1(\Omega) = \{z \in \mathbb{C} : \text{Im}(z) \in (0, 2)\}$, as claimed.

The remainder of the problem is routine: $\Omega_1 := \frac{\pi}{2}f_1(\Omega)$ satisfies

$$z \mapsto \exp(z) : \Omega_1 \to \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

conformally, and finally the Möbius transformation

$$z \mapsto \frac{z - i}{z + i}$$

carries $\mathbb{H}$ onto $\mathbb{D}$ conformally. Thus the composition

$$z \mapsto \frac{e^{\pi/2}z^2 - i}{e^{\pi/2}z^2 + i}$$

maps $\Omega$ conformally onto $\mathbb{D}$, as required.
Problem 10. Let $K \subseteq \mathbb{C}$ be a compact set of positive area but empty interior and define a function $F : \mathbb{C} \to \mathbb{C}$ via

$$F(z) = \iint_K \frac{1}{w-z}d\mu(w),$$

where $d\mu$ denotes (planar) Lebesgue measure on $\mathbb{C}$.

(a) Prove that $F(z)$ is bounded and continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus K$.

(b) Prove that $\{F(z) : z \in \mathbb{C}\} = \{F(z) : z \in K\}$.

Hint: If $a \in F(\mathbb{C}) \setminus F(K)$ and $F^{-1}(a) = \{z_1, \ldots, z_n\} \subseteq \mathbb{C} \setminus K$, then the argument principle can be applied to $G(z) = \prod_{j=1}^{n}(z-z_j)$ to get a contradiction.

Proof. (a): We use the fact that translation is continuous in $L^p(\mathbb{C})$, for every $1 \leq p < \infty$. If $\tau_\varepsilon f(\cdot) = f(\cdot - \varepsilon)$, we note that

$$|(\tau_\varepsilon F - F)(z)| \leq \|\tau_\varepsilon \cdot - z - \cdot z\|_{L^1(\mathbb{C})}$$

$$= \|\cdot z - \cdot z(\tau_\varepsilon \cdot z - \varepsilon \cdot z)\|_{L^1(\mathbb{C})}$$

$$\leq \|\cdot z - \varepsilon \cdot z\|_{L^1(\mathbb{C})}$$

where $R > 0$ is sufficiently large so that $K \subseteq B(0, R - 1)$, and where the limit holds because $w \mapsto \frac{1}{w-z}$ is locally $L^{3/2}$ and translation is continuous in $L^3(\mathbb{C})$. Thus

$$F(z - \varepsilon) \to F(z)$$

as $\mathbb{C} \ni \varepsilon \to 0$, for arbitrary $z \in \mathbb{C}$; that is, $F$ is continuous on all of $\mathbb{C}$.

Next, we argue that $F$ is bounded on $\mathbb{C}$: if $R > 0$ is sufficiently large so that $K \subseteq B(0, R/2)$, then for every $z \notin B(0, R)$

$$|F(z)| \leq \frac{2}{R}\mu(K)$$

which implies that $F(z) \to 0$ as $z \to \infty$; thus $F$ extends to a continuous function on the compact space $\mathbb{C} \cup \{\infty\}$, so $F$ is bounded.

If $\Delta$ is a triangle in $\mathbb{C} \setminus K$ which does not bound any part of $K$, we compute

$$\int_\Delta F(z)dz = \int_\Delta \iint_K \frac{1}{w-z}d\mu(w)dz$$

$$= \iint_K \int_\Delta \frac{1}{w-z}dzd\mu(w)$$

by Fubini, since $\Delta$ and $K$ are compact

$$= 0$$

since $w \in K$ and $\Delta$ doesn't enclose any of $K$

and so by Morera’s theorem we conclude that $F$ is analytic on $\mathbb{C} \setminus K$.

(b): Note that, if $a \in F(\mathbb{C}) \setminus F(K)$, then $F^{-1}(a)$ is finite (since otherwise there would be an accumulation point inside $\mathbb{C} \setminus K$, which would imply that $F$ is constant, which contradicts $a \in F(\mathbb{C}) \setminus F(K)$). Thus we may assume $F^{-1}(a) = \{z_1, \ldots, z_n\}$ for some $n > 0$, listed with multiplicity; we will reach a
contradiction from this. We assume for simplicity that $0 \not\in K$; by translating, the general case will follow. Since $a \not\in F(K)$, and $F(K)$ is compact, we see that

$$G(z) = \frac{F(z) - a}{\prod(z - z_j)}$$

is continuous on all of $\mathbb{C} \cup \{\infty\}$ with $G(\infty) = 0$, hence bounded. Furthermore, since the $z_j$ were listed with multiplicity, $\infty$ is the unique zero of $G$. Thus, setting

$$H(z) = G(1/z)$$

the argument principle provides

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{H'(z)}{H(z)} dz = n$$

for sufficiently small $\varepsilon > 0$, and, changing variables,

$$\frac{1}{2\pi i} \int_{|z|=1/\varepsilon} \frac{G'(z)}{G(z)} dz = -n$$

Thus $G(\{\{|z| = \frac{1}{\varepsilon}\})$ is a (rectifiable) curve that has winding number $-n \neq 0$ with respect to 0. For each $0 < r \leq \frac{1}{\varepsilon}$, let $\gamma_r$ denote the curve $\{|z| = r\}$, traced counterclockwise. Since $G$ doesn’t vanish anywhere in $\mathbb{C}$, the winding number of $G \circ \gamma_r$ with respect to 0 is well-defined for all $r$, though we may need to understand “winding number” as coming from the identification $\pi_1 \mathbb{C} \setminus \{0\} \cong \mathbb{Z}$ due to lack of regularity. It is also continuous and integer-valued, so in particular is equal to $-n$ for all $r$. Since $0 \not\in K$, we have that $B(0, r) \subseteq K^c$ for sufficiently small $r > 0$. Since $G$ is analytic on $K^c$, we reach a contradiction from the conclusion that $G \circ \gamma_r$ has winding number $-n < 0$.

\[\square\]

**Problem 11.** Let $\{f_n\}$ be a sequence of analytic functions on a (connected) domain $\Omega$ such that $|f_n(z)| \leq 1$ for all $n$ and all $z \in \Omega$. Suppose the sequence $\{f_n(z)\}$ converges for infinitely many $z$ in a compact subset $K$ of $\Omega$. Prove that $\{f_n(z)\}$ converges for all $z \in \Omega$.

**Proof.** We claim that there is some analytic function $f$ on $\Omega$ such that every subsequence of $\{f_n\}$ has a further subsequence which converges locally uniformly to $f$; the result follows. Since $|f_n(z)| \leq 1$ uniformly, $\{f_n\}$ is a normal family, so every subsequence has a locally uniformly convergent further subsequence. Suppose $f^1, f^2$ are two such limit functions; they are clearly analytic. Since the collection of points $z \in K$ such that $f_n(z)$ converges is infinite, and that limit value must equal $f^1(z) = f^2(z)$, we see that $f^1(z) = f^2(z)$ for infinitely many points of $K$. Since $K$ is compact, $\{z : f^1(z) = f^2(z)\}$ has an accumulation point inside of $\Omega$. Thus by the uniqueness principle $f^1 = f^2$ on $\Omega$, and so any subsequence of the $\{f_n\}$ refines to a further subsequence which converges to the same limit function $f$.

\[\square\]

**Problem 12.** Let $\Omega = \{z \in \mathbb{C} : -2 < \text{Im} \ z < 2\}$. Show that there is a finite constant $C$ so that

$$|f(0)|^2 \leq C \int_{-\infty}^{\infty} [|f(x+i)|^2 + |f(x-i)|^2] dx$$

for every holomorphic $f : \Omega \to \mathbb{D}$ for which the right-hand side is finite.
Proof. By Cauchy’s integral theorem, for each $R > 0$,

$$f(0) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(z)}{z} \, dz$$

where $\Gamma_R$ is the counterclockwise-oriented rectangle in $\Omega$ with top and bottom along the lines $\{\text{Im}(z) = \pm 1\}$ and left/right edges along the lines $\{\text{Re}(z) = \pm R\}$. Let $L_R$ and $R_R$ denote the left/right hand sides of this rectangle; then

$$\left| \int_{L_R} \frac{f(z)}{z} \, dz \right| \leq \frac{2}{R} \to 0$$

and similarly for $R_R$. Thus

$$f(0) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{f(t + i)}{t + i} + \frac{f(-t - i)}{-t - i} \, dt$$

By Hölder’s inequality,

$$|f(0)| \leq \frac{1}{2\pi} \lim_{R \to \infty} \left( \int_{-R}^{R} |f(t + i)|^2 + |f(-t - i)|^2 \, dt \right)^{1/2} \left( \int_{-R}^{R} \frac{2}{|t + i|^2} \, dt \right)^{1/2}$$

$$= C \left( \int_{-R}^{R} |f(t + i)|^2 + |f(-t - i)|^2 \, dt \right)^{1/2}$$

where

$$C = \frac{1}{\sqrt{2\pi}} \left( \int_{R}^{1 + x^2} \, dx \right)^{1/2} = \frac{1}{\sqrt{2\pi}}$$

Thus

$$|f(0)|^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} [||f(x + i)|^2 + |f(x - i)|^2] \, dx$$

as desired. \qed