Uniformly Hyperfinite Algebras (UHF-Algebras) and their classification

Consider the finite-dimensional matrix algebras

$$M_n(\mathbb{C}) := \{n \times n \text{ matrices over } \mathbb{C}\}$$

for $n \in \mathbb{N}$. If we have $n, d, m \in \mathbb{N}$ with $m = dn$, there exist unital $*$-embeddings

$$\varphi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C}) = M_{dn}(\mathbb{C})$$

$$a \mapsto \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}$$

i.e. $\varphi(a) = \underbrace{a \oplus \cdots \oplus a}_{d} = a \otimes I_d$

As a consequence, for any sequence of positive integers $k_1, k_2, \ldots$ satisfying $k_n | k_{n+1}$ $\forall n \in \mathbb{N}$, we have an associated sequence

$$M_{k_1}(\mathbb{C}) \hookrightarrow M_{k_2}(\mathbb{C}) \hookrightarrow M_{k_3}(\mathbb{C}) \hookrightarrow \cdots$$

of unital $*$-embeddings.

Throughout we assume $k_n < k_{n+1}$ $\forall n$ as well.

If we use these embeddings to realize each algebra as a unital $*$-subalgebra of the following algebras, we consider the C^*-algebra

$$A^{(k_n)} = A^{(k_n)_{n=1}^{\infty}} := \overline{\bigcup_{n=1}^{\infty} M_{k_n}(\mathbb{C})}$$

where the closure is with respect to the norm.

Definition: A unital C^*-algebra A is a UHF-algebra (uniformly hyperfinite) if there exists a (strictly) increasing sequence $(A_n)_{n=1}^{\infty}$ of simple finite-dimensional C^*-subalgebras such that

$$\bigcup_{n=1}^{\infty} A_n \text{ is norm-dense in } A$$

and $1_A \in A_n \ \forall n$

Since simple finite-dimensional C^*-algebras are always isomorphic to $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, UHF-algebras are always of the form

$$\bigcup_{n=1}^{\infty} M_{k_n}(\mathbb{C})$$

by considering the diagram

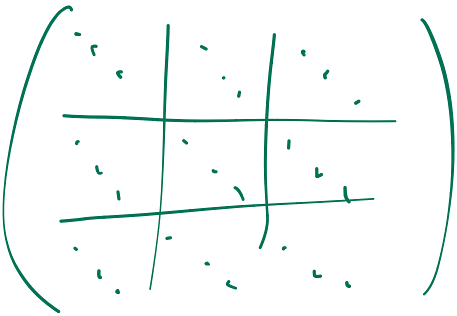
$$\begin{tikzcd}
A_1 \arrow[r, "\cong"] \arrow[d, hook] & M_{k_1}(\mathbb{C}) \arrow[d, "\varphi"] \\
A_2 \arrow[r, "\cong"] \arrow[d, hook] & M_{k_2}(\mathbb{C}) \arrow[d] \\
\vdots \arrow[r, "\cong"] & \vdots
\end{tikzcd}$$

Idea:

① Ring isomorphisms $A_2 \to M_{k_2}(\mathbb{C})$ are in bijective correspondence with the linear bases $\{W_{u,v}\}_{u,v=1}^{k_2}$ of $M_{k_2}(\mathbb{C})$ sat. $W_{1,1} + W_{2,2} + \dots + W_{k_2,k_2} = I_{k_2}$

② For every basis $\{V_{a,b}\}_{a,b=1}^{k_1}$ of $M_{k_1}(\mathbb{C})$ s.t. $V_{1,1}+\dots+V_{k_1,k_1}=I_{k_1}$ there exists a basis $\{W_{u,v}\}_{u,v=1}^{k_2}$ of $M_{k_2}\mathbb{C}$ with $W_{1,1}+\dots+W_{k_2,k_2}=I_{k_2}$ and

$$\varphi(V_{a,b}) = \sum_{j=1}^{k_2/k_1} W_{(a-1)\frac{k_2}{k_1}+j,\,(b-1)\frac{k_2}{k_1}+j}$$



$$\rightsquigarrow \quad \bigcup_{n=1}^{\infty} A_n \xrightarrow{\cong} \bigcup_{n=1}^{\infty} M_{k_n}(\mathbb{C})$$

By standard nonsense, this extends to a C^*-isomorphism

$$A = \overline{\bigcup_{n=1}^{\infty} A_n} \xrightarrow{\cong} \overline{\bigcup_{n=1}^{\infty} M_{k_n}(\mathbb{C})}$$

We wish to find a complete invariant of UHF-algebras.

For the sequence $k_1 | k_2 | k_3 | k_4 \cdots$

we set $\varepsilon_{(k_n)}(p) = \sup\{ n \in \mathbb{Z}_{\geq 0} : p^n | k_j$ for some $j \in \mathbb{N} \} \in \{0, 1, \dots, \infty\}$

for p prime.

" $\varepsilon_{(k_n)}(p)$ is the number of factors of p in the sequence $(k_n)_{n=1}^{\infty}$ "

Example: If $k_n = 2 \cdot 3^n$, then

$$\varepsilon_{(k_n)}(2) = 1$$

$$\varepsilon_{(k_n)}(3) = +\infty$$

$$\varepsilon_{(k_n)}(p) = 0 \quad \forall p \neq 2, 3$$

The big result is...

Theorem: If $k_1 | k_2 | \cdots$ and $l_1 | l_2 | \cdots$ are two sequences as described above, then

$$A^{(k_n)} \cong A^{(l_n)}$$

if and only if $\varepsilon_{k_n} \equiv \varepsilon_{l_n}$

That is, ε defines a complete invariant for UHF-algebras.

$$\begin{array}{cccc} \varepsilon_{(q_n)} & & A^{(q_n)} & \\ \rotatebox{90}{$\leq$} \quad \rotatebox{90}{$\geq$} & & \rotatebox{45}{$\cong$} \quad \rotatebox{90}{$\cong$} & \end{array}$$

To show the implication $\varepsilon_{(k_n)} \equiv \varepsilon_{(l_n)} \Rightarrow A^{(k_n)} \cong A^{(l_n)}$ we proceed by showing that UHF algebras can be put in a standard form.

Define the sequence of integers

$$q_n := \prod_{j=1}^{n} p_j^{\min(n, \varepsilon_{(k_n)}(p_j))} \qquad n = 1, 2, \ldots$$

where $p_1, p_2, \ldots$ is an enumeration of the primes.

$\Longrightarrow q_1 | q_2 | q_3 | \cdots$ and $\varepsilon_{(k_n)} \equiv \varepsilon_{(q_n)}$

Fix any $n = 1, 2, \dots$. If $k_n = p_1^{e_1} \cdots p_i^{e_i}$, $e_1, \dots, e_i \geq 0$, then $k_n \mid q_f$ whenever $f \geq i, e_1, \dots, e_i$.

Thus we can embed

$$\begin{tikzcd}
M_{k_1}(\mathbb{C}) \arrow[r, "\pi_1"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & M_{q_{f_1}}(\mathbb{C}) \arrow[d, hook] \\
M_{k_2}(\mathbb{C}) \arrow[r, "\pi_2"] \arrow[d, hook] & M_{q_{f_2}}(\mathbb{C}) \arrow[d, hook] \\
\vdots & \vdots
\end{tikzcd}$$

$(f_{n+1} > f_n)$

$[\pi_j = \varphi]$

This induces an embedding

$$A^{(k_n)} = \overline{\bigcup_{n=1}^{\infty} M_{k_n}(\mathbb{C})} \xrightarrow{\pi} \overline{\bigcup_{n=1}^{\infty} M_{q_{f_n}}(\mathbb{C})} = A^{(q_n)}$$

To show that this is surjective, note that, for each $n = 1, 2, \dots$,

$$q_n = \prod_{j=1}^{n} p_j^{\min(n, \varepsilon_{(k_n)}(p_j))}$$

which divides some k_r, i.e. $q_n \mid k_r$ for sufficiently large r.

<u>Reason</u>: for each $j=1,\dots,n$, we have 2 cases:

① $\min(n, \varepsilon_{(k_n)}(p_j)) = n$: then $\varepsilon_{(k_n)}(p_j) \geq n$

so there is some r for which $p_j^n \mid k_r$

② $\min(n, \varepsilon_{(k_n)}(p_j)) = \varepsilon_{(k_n)}(p_j)$:

then $\varepsilon_{(k_n)}(p_j) < \infty$, so for some r

we have $p_j^{\varepsilon_{(k_n)}(p_j)} \mid k_r$

Either way, taking r at least as large as described for each $j=1,\dots,n$ separately, we have $q_n \mid k_r$.

Then in our above diagram we have

$$\begin{tikzcd} & & M_{q_n}(\mathbb{C}) \arrow[dll, "\varphi"'] \arrow[d, hook, "\psi"] \\ M_{k_r}(\mathbb{C}) \arrow[rr, leftrightarrow, "\pi_r", "\psi"'] & G & M_{q_{f_r}}(\mathbb{C}) \end{tikzcd} \implies \begin{array}{c} M_{q_n}(\mathbb{C}) \\ \cap\mid \\ \pi_r(M_{k_r}(\mathbb{C})) \end{array}$$

$$\cap\!|$$

$$\mathrm{im}(\pi) = \mathrm{im}\left[A^{(k_n)} \xrightarrow{\pi} A^{(q_n)}\right)$$

Since this holds for each n,

$$\underbrace{\bigcup_{n=1}^{\infty} M_{q_n}(\mathbb{C})}_{\text{norm-dense}} \subseteq \mathrm{im}\Big(A^{(k_n)} \underbrace{\xrightarrow{\pi}}_{\text{isometric}} A^{(q_n)}\Big]$$

Hence $A^{(k_n)} \overset{\pi}{\cong} A^{(q_n)}$, as desired.

We will now show that

$$A^{(k_n)} \cong A^{(l_n)} \implies \mathcal{E}_{(k_n)} \equiv \mathcal{E}_{(l_n)}$$

To do this, we need a couple of lemmas:

<u>Lemma</u> 1: Let A be a C^*-algebra and $a \in A_{sa}$ such that

$$\|a - a^2\| < 1/4$$

Then $\exists$ projection $p \in A$ with

$\|a-p\|<1/2$

Proof: Since the C^*-algebra generated by a is abelian, it suffices to take A abelian, i.e.

$$A = C_0(\Omega), \quad \Omega \text{ locally compact, Hausdorff}$$

Since $\|a-a^2\|<1/4$,

$$1/2 \notin \mathrm{im}(|a|)$$

[if $|a|(x)=1/2$, then $|a(x)-a^2(x)|$
$$= \tfrac{1}{2}\cdot|1-a(x)|$$
$$\geq \tfrac{1}{2}\left|1-\tfrac{1}{2}\right| = \tfrac{1}{4},$$
contradicting $\|a-a^2\|<\frac{1}{4}$]

$$\Longrightarrow \quad |a|^{-1}(1/2,\infty) = |a|^{-1}[1/2,\infty) =: S$$

compact & open

$|a| \in C_0(\Omega)$,

so $|a|^{-1}[r,\infty)$ compact

$\forall\, r>0$

Thus $p := \chi_S$ is a projection in A satisfying

$\|a-p\|_\infty < \frac{1}{2}$ ✓

<u>Lemma ②</u>: Let p, q be projections in a unital C^*-algebra A such that

$$\|q-p\| < 1$$

Then there is a unitary $u \in A$ such that $q = upu^*$

and $\|1-u\| \leq \sqrt{2}\, \|q-p\|$

<u>Proof</u>: Set $v = 1 - p - q + 2qp$

$$\implies v^*v = 1-(q-p)^2$$

$$vv^* = 1-(p-q)^2 = v^*v$$

Now $\|q-p\|<1 \implies \|(q-p)^2\| = \|q-p\|^2 < 1$

Which implies v^*v invertible.

Hence there is some $w \in A$ with

$$1 = wv^*v = v^*vw = vv^*w$$

So v has a left & a right inverse, hence is invertible.

We set $u = v|v|^{-1}$ unitary

We check this works:

$$vp = qp = qv$$

$$\implies pv^* = v^*q$$

& $\underline{pv^*v} = v^*qv = \underline{v^*vp}$

So that p commutes with $|v|^2$,

hence with $|v|$

hence with $|v|^{-1}$

$$\Rightarrow \quad up = v|v|^{-1}p = vp|v|^{-1} = qv|v|^{-1} = qu$$

$$\Rightarrow \quad upu^* = q$$

Next, $\mathrm{Re}(v) = 1-(q-p)^2 = |v|^2$

So $\mathrm{Re}(u) = \mathrm{Re}(v)\cdot|v|^{-1} = |v|$

Hence $\|1-u\|^2 = \|(1-u)(1-u^*)\| = 2\|1-\mathrm{Re}(u)\| = 2\|1-|v|\|$

$1-t \le 1-t^2$
$\forall\ t \in [0,1]$

$[0 \le |v| \le 1]$

$$\le 2\,\|1-|v|^2\| = 2\,\|q-p\|^2$$

Which is the desired

$$\|1-u\| \le \sqrt{2}\,\|q-p\|$$

✓

We may now prove the remainder of the theorem.

It remains to show:

($\star$) Assuming $A^{(k_n)} \underset{\pi}{\cong} A^{(\ell_n)}$, we have $\mathcal{E}_{(k_n)} \le \mathcal{E}_{(\ell_n)}$

Equivalently, we wish to show that

$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}$ s.t. $k_n \mid \ell_m$

$$\tau(a^*a) = \tau(aa^*)$$

To this end, fix $n \in \mathbb{N}$. Since each $M_{k_n}(\mathbb{C})$ has a unique tracial state, and these states are preserved by our isometric embeddings φ, we see that $A^{(k_n)}$ has a unique tracial state τ.

Fix p a rank-one projection in $M_{k_n}(\mathbb{C})$.

Writing φ^n for the inclusion

$$M_{k_n}(\mathbb{C}) \xrightarrow{\varphi^n} A^{(k_n)}$$

we see that $\tau\varphi^n$ is a tracial state on $M_{k_n}(\mathbb{C})$, hence $\tau\varphi^n = \frac{tr}{k_n}$. In particular,

$$\tau(\varphi^n(p)) = \frac{1}{k_n}$$

Since $\pi\varphi^n p$ is a projection in $A^{(l_n)}$, we may find $m \in \mathbb{N}$ and $a \in M_{l_m}(\mathbb{C})_{sa}$ such that

$$\|\pi\varphi^n p - \psi^m a\| < 1/8$$

and

$$\|\pi\varphi^n p - \psi^m a^2\| < 1/8$$

(This follows from the density of

$$\bigcup_{r=1}^{\infty} M_{l_r}(\mathbb{C}) \overset{\text{dense}}{\subseteq} A^{(l_n)}$$

, and the fact that arbitrary elements of C^*-algebras can be written as the linear combination of two self-adjoint elements.)

Then

$$\|a-a^2\| = \|\psi^m(a) - \psi^m(a^2)\|$$
$$\leq \|\psi^m(a) - \pi\varphi^n(p)\| + \|\pi\varphi^n(p) - \psi^m(a^2)\|$$
$$< 1/4$$

By lemma 1, $\exists q \in M_{l_n}(\mathbb{C})$ projection s.t. $\|a-q\| < 1/2$.

Hence,

$$\|\pi\varphi^n(p) - \psi^m(q)\| \leq \|\pi\varphi^n(p) - \psi^m(a)\| + \|\psi^m(a) - \psi^m(q)\|$$
$$< 1/8 + 1/2 < 1$$

By lemma 2, $\exists u \in A^{(l_n)}$ unitary with $\psi^m q = u\pi\varphi^n p u^*$.

Letting τ' denote the unique tracial state on $A^{(l_n)}$,

we have

$$\tau'\psi^m q = \tau'(u\pi\varphi^n p u^*)$$

$$= \tau'(\pi\varphi^n p) \quad \text{(since } \tau' \text{ tracial)}$$

$$= \tau\varphi^n p \quad \text{(since } \tau = \tau'\pi\text{)}$$

$$= 1/k_n$$

But also $\tau'\psi^m = \frac{1}{l_m}\mathrm{tr}$, so

$$\tau'\psi^m q = \frac{d}{l_m} \quad \text{some } d = 0, 1, \ldots, l_m$$

and so

$$k_n d = l_m$$

as desired. ✓

Corollary: There exist an uncountable number of UHF algebras that are not $*$-isomorphic.

Pf: Indeed, any distinct functions

$$\varepsilon, \varepsilon' : \{\text{primes}\} \to \{0, 1, \ldots, \infty\}$$

(with $\sum_{p \text{ prime}} \varepsilon(p) = \sum_{p \text{ prime}} \varepsilon'(p) = \infty$)

define two UHF algebras that are not isomorphic.