Background: For any countably infinite amenable group G, Ornstein & Weiss '87 demonstrated that G-Bernoulli shifts (G, K^G, κ^G) are completely classified by the quantity $H(\kappa)$, defined by

$$H(\kappa) = -\sum_{x \in K'} \kappa(\{x\}) \log \kappa(\{x\})$$

if K has a full-measure countable set $K' \subseteq K$, $H(\kappa) = +\infty$ otherwise.

(Here "classified" is with regards to the relation of being measurably conjugate)

- One wishes to similarly classify other G-systems, for G a countable discrete group, by "entropy" type quantities. Of interest to us here is the class of G-Bernoulli shifts for G a $\boxed{\text{countable discrete}}$ sofic group

all groups for today will be this.

A crash course in soficity: Roughly speaking, sofic groups are groups G for which "most of G can mostly be embedded in finite symmetric groups".

Precisely: For $m \geq 1$, let $\mathrm{Sym}(m)$ be the full symmetric group on $\{1, \dots, m\}$.

Let $\sigma : G \to \mathrm{Sym}(m)$ be a function (not necessarily a homomorphism).

For $F \subseteq G$ finite subset, let $V(F) \subseteq \{1, \dots, m\}$ be the set of those v such that, for each $f_1, f_2 \in F$,

$$\sigma(f_1)\sigma(f_2)v = \sigma(f_1 f_2)v \quad \text{and} \quad \sigma(f_1)v \neq \sigma(f_2)v \text{ if } f_1 \neq f_2$$

If $|V(F)| \geq (1-\varepsilon)m$, then σ is an (F,ε)-approximation to G. "v thinks that $\sigma|_F$ is an injective homomorphism"

A sofic approximation to G is a sequence $\Sigma = (\sigma_j)_{j=1}^{\infty}$, where σ_j is an (F_i, ε_i)-approximation with $F_i \subseteq F_{i+1}$ $\forall i$, $\bigcup_{i=1}^{\infty} F_i = G$, and $\lim_{i\to\infty} \varepsilon_i = 0$.

Finally, G is <u>sofic</u> if it admits a sofic approximation.

<u>Remark</u>: G is sofic $\iff$ $G < \left(\prod_\alpha (\mathrm{Sym}(n_\alpha), d_{hamm})\right)_{\mathcal{U}}$

"G is isomorphic to a subgroup of a metric ultraproduct of $\mathrm{Sym}(n)$ groups with Hamming distance"

The main result $\checkmark$ of interest to us today of an investigation into entropy invariants for sofic Bernoulli systems are as follows:

<u>Thm</u>: Let G be sofic and (K_1, κ_1), (K_2, κ_2) be standard Borel prob. spaces s.t. $H(\kappa_1) + H(\kappa_2) < \infty$.
Then the following are sufficient conditions for $H(\kappa_1) = H(\kappa_2)$:

1. (G, K_1^G, κ_1^G) and (G, K_2^G, κ_2^G) are measurably conjugate] this is what we'll prove
2. $G = G_1 \times G_2$ has no nontrivial normal subgroups, G_1, G_2 infinite, G_1 non-amenable, and (G, K_1^G, κ_1^G) is OE to (G, K_2^G, κ_2^G)
3. G ICC property (T) and (G, K_1^G, κ_1^G) is uOE to (G, K_2^G, κ_2^G)

In the case that G is Ornstein, then (a) the finiteness assumptions may be removed,
(b) the converses hold. Thus:

Thm: Suppose G is countable, sofic, and Ornstein. Suppose (K_1, κ_1), (K_2, κ_2) are std Borel prob. spaces.

Then: (1) (G, K_1^G, κ_1^G) is msbly conjugate to $(G, K_2^G, \kappa_2^G) \iff H(\kappa_1) = H(\kappa_2)$

(2) If $G = G_1 \times G_2$ inf. countable, G_1 non-amen., and G has no nontrivial finite normal subgrps, then (G, K_1^G, κ_1^G) OE to $(G, K_2^G, \kappa_2^G) \iff H(\kappa_1) = H(\kappa_2)$

(3) If G ICC prop. (T), then (G, K_1^G, κ_1^G) vNE to $(G, K_2^G, \kappa_2^G) \iff H(\kappa_1) = H(\kappa_2)$

Rmk:
- Ornstein groups must be countably infinite
- All countably infinite groups are Ornstein (much later)

The structure of the argument: We will first construct a "mean sofic entropy" for (G, K^G, κ^G) with G sofic,

$$h(\Sigma, \alpha)$$

Sofic approximation

partition of K^G

generates the σ-alg. of K^G up to null, using action of G

It will be clear that any measure conjugacy $\Phi : (G, K_1^G, \kappa_1^G) \to (G, K_2^G, \kappa_2^G)$ preserves Σ and sends such a "generating partition" α of K_1^G to a generating partition β of K_2^G.

+ both generating

We will show that, if $H(\alpha) + H(\beta) < \infty$, then $h(\Sigma, \alpha) = h(\Sigma, \beta)$, and we conclude that the quantities $h(\Sigma, \alpha)$ are invariants of (K^G, κ^G).

Finally, if $H(\kappa) < \infty$, we will find a special partition α_κ giving $h(\Sigma, \alpha_\kappa) = H(\kappa)$.

Sofic entropy: We construct sofic entropy by counting microstates, which are "partitions of $\{1, \dots, m\}$ that almost match partitions on K^G: abbreviate $K^G = X$, $\kappa^G = \mu$ (G, X, μ)

- Fix $\sigma : G \to \mathrm{Sym}(m)$, ζ unif. prob. on $\{1, \dots, m\}$, $\begin{cases} \alpha = (A_1, A_2, \dots) \text{ pttn of } X \\ \beta = (B_1, B_2, \dots) \text{ pttn of } \{1, \dots, m\} \end{cases}$
- For each $F \subseteq G$ finite & $\phi : F \to \mathbb{N}$, denote

$$A_\phi = \bigcap_{f \in F} f A_{\phi(f)} \qquad B_\phi = \bigcap_{f \in F} \sigma(f) B_{\phi(f)}$$

we set $$d_F(\alpha, \beta) = \sum_{\phi : F \to \mathbb{N}} |\mu(A_\phi) - \zeta(B_\phi)|$$

We will be interested in $|AP(\sigma, \alpha : F, \varepsilon)|$, where

$$AP(\sigma, \alpha : F, \varepsilon) = \{\beta \text{ pttn of } \{1, \dots, m\},\ |\beta| = |\alpha|,\ d_F(\alpha, \beta) < \varepsilon\}$$

$F \subseteq G$ finite

$\alpha = (A_1, \_)$ pttn of X

$\beta = (B_1, \_)$ pttn of $\{1, \_, m\}$

$$A_\phi = \bigcap_{f \in F} f A_{\phi(f)}$$

$$B_\phi = \bigcap_{f \in F} \sigma(f) B_{\phi(f)}$$

$\sigma : F \to Sym(m)$ fctn

$\phi : F \to \mathbb{N}$ fctn

First, some examples:

- $F = \{e\}$, then

$$d_F(\alpha, \beta) = \sum_{j=1}^{\infty} |\mu(A_j) - \zeta(\sigma(e) B_j)|$$

$\Rightarrow$ elements of $AP(\sigma, \alpha : \{e\}, \varepsilon)$ are partitions of $\{1, \_, m\}$ whose atoms are almost equal in mass to atoms of α

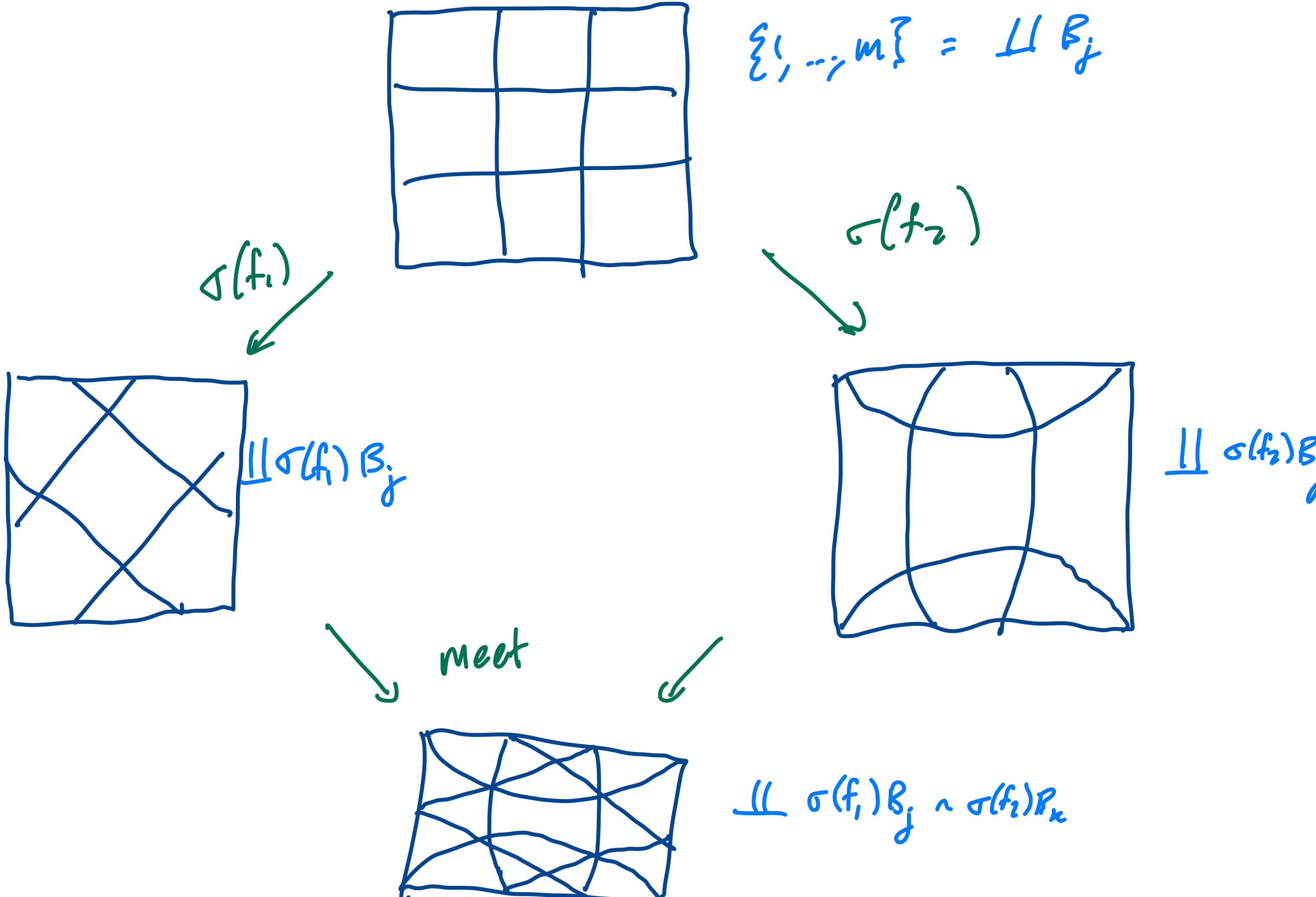
- $F = \{f_1, f_2\}$, then

$$d_F(\alpha, \beta) = \sum_{j,k=1}^{\infty} \left| \mu\big(f_1(A_j) \cap f_2(A_k)\big) - \zeta\Big(\sigma(f_1)(B_j) \cap \sigma(f_2)(B_k)\Big)\right|$$

$\Longrightarrow$ elements of $AP(\sigma, \alpha : \{f_1, f_2\}, \varepsilon)$ are partitions $\{1, \dots, m\} = \coprod_{j,k} C_{j,k}$ such that

- size $(f_1(A_j) \cap f_2(A_k)) \approx$ size$(C_{j,k})$
- $C_{j,k} = \sigma(f_1)(B_j) \cap \sigma(f_2)(B_k)$, i.e. $\mathcal{C} = \sigma(f_1)\beta \vee \sigma(f_2)\beta$

⋮


$\{1, \ldots, m\} = \coprod B_j$
$\sigma(f_1)$
$\sigma(f_2)$
$\coprod \sigma(f_1) B_j$
$\coprod \sigma(f_2) B_j$
meet
$\coprod \sigma(f_1) B_j \cap \sigma(f_2) B_k$

Sofic entropy, ct'd: Let $\Sigma = (\sigma_j)_{j=1}^{\infty}$ be a sofic approximation for G.

$\sigma_i : G \to \mathrm{Sym}(m_i)$

For α finite partition of X, $\varepsilon > 0$, and $F \subseteq G$ finite, set

- $H(\Sigma, \alpha : F, \varepsilon) = \limsup_{i \to \infty} \frac{1}{m_i} \log |AP(\sigma_i, \alpha : F, \varepsilon)|$

- $H(\Sigma, \alpha : F) = \lim_{\varepsilon \downarrow 0} H(\Sigma, \alpha : F, \varepsilon)$

- $h(\Sigma, \alpha) = \inf_{F \subseteq G} H(\Sigma, \alpha : F)$ mean Σ-entropy of α

(If α is infinite and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha$ is a sequence of finite ptns refining to α, set $H(\Sigma, \alpha : F) = \lim_{n \to \infty} H(\Sigma, \alpha_n : F)$ well-defined)

Partitions, their structure, and entropy functions thereof: The space $\mathcal{P}$ of msble partitions with $H(\alpha) < \infty$ of a prob. sp. (X, μ) (up to a.e. equivalence of atoms) admits the following structures:

- Conditional entropy fns $H(\alpha | \beta) := H(\alpha | \sigma(\beta))$, $H(\alpha) := H(\alpha | \{X, \emptyset\})$
- Order structure: $\beta \leq \alpha$ if $\forall A \in \alpha \ \exists B \in \beta \ \ \mu(A \setminus B) = 0$
- Joinings: $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$
- (Rohlin) metric: $d(\alpha, \beta) = H(\alpha|\beta) + H(\beta|\alpha)$
- Action by G: $g \cdot \alpha = \{gA : A \in \alpha\}$

Putting together the joinings / G-action: for $F \subseteq G$ finite & $\alpha \in \mathcal{P}$, set

$$\alpha^F = \bigvee_{f \in F} f\alpha$$

Some facts:

- $H(\alpha \vee \beta) = H(\alpha) + H(\beta|\alpha)$
- if $\beta \leq \gamma$, $H(\alpha|\gamma) \leq H(\alpha|\beta)$
- $\alpha \mapsto \alpha^F$ is cts on $\mathcal{P}$
- If $F_1 \subseteq F_2$ then $d_{F_1} \leq d_{F_2}$

$$I(\alpha | \mathcal{F})(x) = -\log(\mathbb{E}[A_x | \mathcal{F}](x))$$

$$H(\alpha | \mathcal{F}) = \int_X I(\alpha|\mathcal{F})(x) \, d\mu(x)$$

Further, we consider so-called "S-splittings", where $S \subseteq G : P \to \sigma$ is a simple S-splitting of α

if $\exists s \in S, \beta \leq \alpha$ s.t. $\sigma = \alpha \vee s\beta$. σ is an S-splitting of α if $\exists \alpha = \alpha_0, \dots, \alpha_m = \sigma$ with α_{i+1} a simple S-splitting of α_i, $\forall i \in \{0, \dots, m-1\}$.

If α, β have a common S-splitting, then they are said to be S-equivalent.

The utility of these notions is as follows:

Theorem: Let G be a (countable) group and $S \subseteq G$ a generating set. Let (G, X, μ) be a G-system. Let P be the space of finite-entropy partitions of X. Suppose $f : P \to \mathbb{R} \cup \{-\infty\}$ is upper semicts and that $f(\beta) = f(\alpha)$ whenever β is an S-splitting of α.

Then, if $\alpha, \beta \in P$ are generating (i.e. $\sigma_{G\text{-invt}}(\alpha) = \mathrm{Meas}(\mu) = \sigma_{G\text{-invt}}(\beta)$ up to nulls) we have $f(\alpha) = f(\beta)$

(Pf is straightforward)

So, in order to justify the equality $h(\Sigma,\alpha) = h(\Sigma,\beta)$ for α, β generating, it suffices to verify the following:

1. $h(\Sigma, \cdot) : \mathcal{P} \to \mathbb{R} \cup \{-\infty\}$ is upper semicts

2. $h(\Sigma,\beta) = h(\Sigma,\alpha)$ if β is a simple S-splitting of α.

We'll check each of these in turn.

① Upper semicontinuity of $h(\Sigma, \cdot)$: We first consider the case where all partitions are finite and have the same number of atoms.

$H(\Sigma, \alpha : F, \varepsilon) = \limsup_{i \to \infty} \frac{1}{n_i} \log |AP|$

$H(\Sigma, \alpha : F) = \lim_{\varepsilon \downarrow 0} H(\Sigma, \alpha : F, \varepsilon)$

$h(\Sigma, \alpha) = \inf_{F \subseteq G} H(\Sigma, \alpha : F)$

Suppose $\alpha \in P$ and $P \ni \beta^i \to \alpha$ in (P, d) as $i \to \infty$. By rearranging, this means (for $\alpha = (A_1, A_2, \ldots)$, $\beta^i = (B_1^i, B_2^i, \ldots)$),

$$\lim_{i \to \infty} \mu(A_j \triangle B_j^i) = 0 \quad \forall j$$

Then, defining $d_F(\alpha, \beta^i) = \sum_{\phi : F \to N} |\mu(A_\phi) - \mu(B_\phi^i)|$, we have $d_F(\alpha, \beta^i) \to 0$ by continuity under splittings. By the triangle inequality, for each σ, $\varepsilon > 0$, $i \geq 0$,

$$AP(\sigma, \alpha : F, \varepsilon + d_F(\alpha, \beta^i)) \supseteq AP(\sigma, \beta^i : F, \varepsilon)$$

Then, for each $c>1$, since $c d_F(\alpha, \beta^i) \geq \varepsilon + d_F(\alpha, \beta^i)$ for small $\varepsilon > 0$,

$$H(\Sigma, \alpha : F, c d_F(\alpha, \beta^i)) \geq H(\Sigma, \alpha : F, \varepsilon) \quad \text{for suff small } \varepsilon,$$

and hence $H(\Sigma, \alpha : F, c d_F(\alpha, \beta^i)) \geq H(\Sigma, \beta^i : F)$

for each $c>1$. Thus, taking $i \to \infty$,

$$H(\Sigma, \alpha : F) \geq \limsup_{i\to\infty} H(\Sigma, \beta^i : F)$$

as claimed. $\blacksquare$

2 Invariance of $h(\Sigma, \cdot)$ under splittings

This takes the form of two lemmas:

a: If $F \subseteq G$ finite and α is an F-splitting of β, then

$$H(\Sigma, \alpha : F) \leq H(\Sigma, \beta : F)$$

b: For each $\beta \in \mathcal{P}$, $t \in G$,

$$h(\Sigma, \beta \vee t\beta) \geq h(\Sigma, \beta)$$

Both a, b will follow from counting microstates.

(a): Here $F \subseteq G$ finite, α is F-splitting of β. Want: $H(\mathcal{E}, \alpha : F) \leq H(\mathcal{E}, \beta : F)$

We assume first α, β finite. It suffices to take $\alpha = \beta \vee f\xi$ where $f \in F$, $\xi \leq \beta$.

We want to show that, if α has lots of approximating partitions, then so does β.

The method: Show that

- each approximating partition of α determines an approximating partition of β
- Any $\bar{\alpha}$ determining a $\bar{\beta}$ must be "close to" a special $\bar{\alpha}_{\bar{\xi}}$ constructed using $\alpha = \beta \vee f\xi$; thus we can't have too many $\bar{\alpha}$ giving a single $\bar{\beta}$

(a): Here $F \subseteq G$ finite, α is F-splitting of β.

We assume first α, β finite. It suffices to take $\alpha = \beta \vee f\xi$ where $f \in F$, $\xi \leq \beta$.

Then we write

$$\beta = (B_1, \dots, B_v)$$
$$\xi = (X_1, \dots, X_w)$$
$$\alpha = (A_1, \dots, A_u)$$

$$B_j = \bigcup_{i : b(i) = j} A_i$$

and, for each $1 \leq i \leq u$, $A_i = B_{b(i)} \cap f X_{x(i)}$ for some $b(i) \in \{1, \dots, v\}$, $x(i) \in \{1, \dots, w\}$

As before, this defines a coarsening map

$$\Phi : AP(\sigma, \alpha : F, \varepsilon) \longrightarrow AP(\sigma, \beta : F, \varepsilon)$$
$$(\bar{A}_1, \dots, \bar{A}_u) = \bar{\alpha} \longmapsto \Phi(\bar{\alpha}) = \bar{\beta} = (\bar{B}_1, \dots, \bar{B}_v)$$
$$\bar{B}_j = \bigcup_{i : b(i) = j} \bar{A}_i$$

(as we checked prior, this is well-defined)

If each $|\Phi^{-1}(\bar{\beta})|$ is small, then we conclude

$$|AP(\sigma, \alpha : F, \varepsilon)| = \sum_{\bar{\beta} \in AP(\sigma, \beta : F, \varepsilon)} |\Phi^{-1}(\bar{\beta})| \leq (\text{small}) \times |AP(\sigma, \beta : F, \varepsilon)|$$

which is what we're looking for.

Since $\alpha = \beta \vee f\xi$, let's push ξ to $\{1, \ldots, m\}$ as well:

$\xi \leq \beta$, so $B_i \subseteq X_{t(i)}$ for a well-defined $t(i) \in \{1, \ldots, \omega\}$.

Set $\bar{\xi} = (\bar{X}_1, \ldots, \bar{X}_\omega)$ by $\bar{X}_j = \bigcup_{i : t(i) = j} \bar{B}_i$

and $\bar{\alpha} = (\bar{A}_1, \ldots, \bar{A}_n)$ by $\bar{A}_i = \bar{B}_{b(i)} \cap \sigma(f) \bar{X}_{x(i)}$

$\Longrightarrow \Phi(\bar{\alpha}) = \bar{\beta}$

Further, any other $\bar{\alpha}' \in \Phi^{-1}(\bar{\beta})$ satisfies

unif. $\mathbb{P}$ on $\{1,\dots,m\}$

$$\ell\Big(\bigcup_{i=1}^{u} \bar{A}_i \,\triangle\, \bar{A}_i' \Big) \le \varepsilon$$

and hence one gets $\bar{\alpha}'$ from $\bar{\alpha}$ by "moving" at most $\lceil \varepsilon m \rceil$ of the elements of $\{1,\dots,m\}$ from one atom to another. Thus

$$|\Phi^{-1}(\bar{\beta})| \le \binom{m}{\lceil m\varepsilon \rceil} u^{\lceil m\varepsilon \rceil} \qquad (\varepsilon < 1/4)$$

and, adding up + taking logs, + using Stirling's,

$$H(\Sigma, \alpha : F, \varepsilon) \le 2\varepsilon \log\left(\frac{1}{2\varepsilon}\right) + (1-2\varepsilon)\log\left(\frac{1}{1-2\varepsilon}\right) + 2\varepsilon \log u + H(\Sigma, \beta : F, \varepsilon)$$

and for $\varepsilon \downarrow 0$

$$H(\Sigma, \alpha : F) \le H(\Sigma, \beta : F)$$

For α, β not necessarily finite, the full inequality follows by

$$\alpha_n \xrightarrow{fin} \alpha \quad , \quad \beta_n \xrightarrow{fin} \beta \quad , \quad \xi_n \xrightarrow{fin} \xi$$

$$\alpha_n = \beta_n \vee f\xi_n$$

$\blacksquare$

(b): We wish to show

$$h(\Sigma, \beta \vee t\beta) \geq h(\Sigma, \beta) \qquad t \in G$$

As usual, take $\beta \in \mathcal{P}$ to be a finite partition.

Set
$$\alpha := \beta \vee t\beta = (A_1, \dots, A_u)$$
$$\beta = (B_1, \dots, B_v)$$

$$A_i = B_{x(i)} \wedge t B_{y(i)}$$

$x(i), y(i) \in \{1, \dots, v\}$

injective

assume $(x, y) : [u] \to [v]^2$

surjection:

Some A_i may be $\emptyset$

As before, we'll push these partition relations to the approximating partitions:

$$A_i = B_{x(i)} \cap t B_{y(i)}$$

→ Set $F \subseteq G$ finite with $e, t \in F$

→ Fix $\varepsilon, \delta > 0$ and $\sigma : G \to \mathrm{Sym}(m)$ a (F, δ)-approximation to G, i.e.

if $V(F) = \left\{ v \in \{1, \dots, m\} : \forall f_1, f_2 \in F,\ \sigma(f_1) v \neq \sigma(f_2) v \text{ and } \sigma(f_1)\sigma(f_2) v = \sigma(f_1 f_2) v \right\}$

then $|V(F)| \geq (1-\delta) m$.

→ Define $\Psi : AP(\sigma, \beta : F \cup Ft, \varepsilon) \longrightarrow AP(\sigma, \beta \vee \beta t : F, \varepsilon + 5|F|\delta)$

by $\Psi(\bar{\beta}) = \bar{\alpha} = (\bar{A}_1, \dots, \bar{A}_n)$, $\bar{A}_i = \bar{B}_{x(i)} \cap \sigma(t) \bar{B}_{y(i)}$

It is very important that ψ is well-defined, but we'll skip it. Instead,

note that ψ is injective: if $\bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_v) \neq \bar{\beta}' = (\bar{\beta}_1', \dots, \bar{\beta}_v')$

then some $\bar{\beta}_w \neq \bar{\beta}_w'$

$$\bigcup_{i:\, \alpha(i)=w} \bar{A}_i = \bigcup_{r \in \{1,\dots,v\}} \bar{\beta}_w \cap \sigma(t)\bar{\beta}_r \qquad \bigcup_{r\in\{1,\dots,v\}} \bar{\beta}_w' \cap \sigma(t)\bar{\beta}_r' = \bigcup_{i:\, \alpha(i)=w} \bar{A}_i'$$

so $\bar{A}_i \neq \bar{A}_i'$ some i, so $\psi(\bar{\beta}) \neq \psi(\bar{\beta}')$.

Thus

$$|AP(\sigma, \beta : F \cup Ft, \varepsilon)| \leq |AP(\sigma, \beta \cup t\beta : F, \varepsilon + 5|F|\delta)|$$

for each choice of $\varepsilon, \delta > 0$, $e, t \in {}^{\forall}F \overset{\text{fin}}{\subseteq} G$, $\forall \sigma : G \to \mathrm{Sym}(n)$ (F,δ)-approx.

Since G is sofic, there is some $\Sigma = \{\sigma_i\}_{i=1}^{\infty}$ sofic approximation with σ_i a (F, δ_i)-approx. for G with $0 \le \delta_i \to 0$.

If $c > 1$ arbitrary, then

$$\varepsilon + 5|F|\delta_i \le c\,\varepsilon \quad \text{for } i \text{ large}$$

and hence

$$H(\Sigma, \beta : F \cup Ft, \varepsilon) \le H(\Sigma, \beta \cup \beta t : F, c\,\varepsilon)$$

so

$$H(\Sigma, \beta : F \cup Ft) \le H(\Sigma, \beta \cup t\beta : F)$$

and, taking an infimum over $e, t \in F \overset{\text{fin}}{\subseteq} G$

gets $$h(\mathcal{E}, \beta) \leq h(\mathcal{E}, \beta \cup t\beta)$$

when β finite.

Considering chains $\beta_1 \subseteq \beta_2 \subseteq \cdots \subseteq \beta$ finite, converging to β,

we also get
$$h(\mathcal{E}, \beta) \leq h(\mathcal{E}, \beta \cup t\beta)$$

for all $\beta \in \mathcal{P}$. $\blacksquare$

Finally we conclude: let S generate G.

→ If α is a simple S-splitting of β, i.e. $\alpha = \beta \vee t\xi$ with $t \in S$, then for each $F \subseteq G$ finite with $t \in F$ we have

$$H(\Sigma, \alpha : F) \leq H(\Sigma, \beta : F)$$

and hence

$$h(\Sigma, \alpha) \leq h(\Sigma, \beta)$$

→ One may demonstrate that $\beta \vee t\beta$ is an S-splitting of α, so similarly

$$h(\Sigma, \beta \vee t\beta) \leq h(\Sigma, \alpha) \leq h(\Sigma, \beta)$$

→ On the other hand,

$$h(\Sigma, \beta \vee t\beta) \geq h(\Sigma, \beta)$$

and thus $h(\Sigma, \alpha) = h(\Sigma, \beta)$

→ Thus $h(\Sigma, \cdot)$ is invariant under simple S-splittings.

Inducting we get that

$h(\Sigma, \alpha) = h(\Sigma, \beta)$ whenever α is an S-splitting of β

Thus, $h(\Sigma, \cdot) : P \to \{-\infty\} \cup \mathbb{R}$ is

- upper semicontinuous
- Invariant under S-splittings

and hence $h(\Sigma, \alpha) = h(\Sigma, \beta)$ for all generating partitions α, β

with $H(\alpha) + H(\beta) < \infty$.

$\blacksquare$

Thus, if (G, K_1^G, κ_1^G) and (G, K_2^G, κ_2^G) are measurably conjugate

and $\left.\begin{matrix} \alpha \in P(K_1^G, \kappa_1^G) \\ \beta \in P(K_2^G, \kappa_2^G) \end{matrix}\right\}$ are finite-entropy partitions generating (with the action of G) the full σ-alg's up to null sets, we conclude $h(\Sigma, \alpha) = h(\Sigma, \beta)$.

Hypotheses:

- (K_j, κ_j) std Borel
- $H(\kappa_j) < \infty$

It remains to replace $h(\Sigma, \alpha)$ by $H(\kappa_1)$

$h(\Sigma, \beta)$ by $H(\kappa_2)$.

To do this, assuming $H(\kappa_1) < \infty$ (i.e. $\exists K' \subseteq K_1$ countable, full measure,

$$H(\kappa_1) = -\sum_{k \in K'} \kappa_1(\{k\}) \log \kappa_1\{k\} < \infty \Big)$$

we set α_{κ_1} to be the canonical partition of K_1^G:

$$\alpha_{K_1} = \{ \alpha_{K_1}^{(\ell)} \}_{\ell \in K} :$$

$$\alpha_{K_1}^{(\ell)} = \{ \underline{x} \in K_1^G : \underline{x}(e) = \ell \}$$

Then ① α_{K_1} is generating for $(G, K_1^G, \mu_{K_1}^G)$

② $h(\Sigma, \alpha_{K_1}) = H(K_1)$

and hence $H(K_1)$ is a measure-conjugacy invariant, as long as it is finite.

If G is Ornstein, "finiteness of $H(\kappa)$" can be removed; we conclude:

If G is countably infinite + sofic, and if $(K_1, \kappa_1), (K_2, \kappa_2)$ are std Borel, then

$$(K_1^G, \kappa_1^G) \simeq (K_2^G, \kappa_2^G) \iff H(\kappa_1) = H(\kappa_2)$$

$\blacksquare$