Setting:

- Recall that a II_1-factor M with trace τ is said to have property Gamma if $\exists (u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(M)$ with $\tau(u_n)=0$ and $\|x_n y - y x_n\|_2 \longrightarrow 0 \ \forall y \in M$

$$\left(\|z\|_2 := \tau(z z^*)^{1/2} \right)$$

- A countable group G is said to be inner amenable if G admits an inner-invariant mean, i.e. a $\varphi \in S(\ell^\infty(G))$ such that $\varphi \circ \mathrm{Ad}_g = \varphi$ for all $g \in G$.

- Thm: [Effros 1975] If LG is II_1 with Gamma, then G inner amenable.

Does the converse hold?

[Vaes 2009]: No! We construct the counterexample by forming G ICC, inner amenable, such that LG does not have Gamma.

Fix $p_0, p_1, p_2, \dots$ sequence of distinct primes.

Set $$H_n := \left(\frac{\mathbb{Z}}{p_n\mathbb{Z}}\right)^3, \quad K := \bigoplus_{n=0}^{\infty} H_n$$

Note that $\Lambda := SL(3,\mathbb{Z}) \curvearrowright H_n$

$\rightsquigarrow \Lambda \curvearrowright K$ diagonally

We denote by $K_N = \bigoplus_{n=N}^{\infty} H_n$ for each $N \geq 0$ the finite co-dimensional pieces of K.

Lastly, we consider the inductive construction

$$G_0 := K \rtimes \Lambda$$

$$G_N \hookrightarrow G_{N+1} := G_N *_{K_N} (K_N \times \mathbb{Z})$$

where $K_N < K < K \rtimes \Lambda = G_0 < G_N$

and we set $G = \varinjlim G_N$.

Comments:

$$\cdots < K_2 < K_1 < K < K \rtimes \Lambda = G_0 < G_1 < G_2 < \cdots$$

ICC

with • $K_N = K_{N+1} \oplus \left(\frac{\mathbb{Z}}{p_N \mathbb{Z}}\right)^?$

• Consider a reduced word

$$\cdots * g * (k,n) * g' * \cdots \in G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z})$$

$$\|$$

$$\cdots * g * [(k,0)\cdot(0,n)] * g' * \cdots \qquad \left\{ \begin{array}{l} g, g' \in G_N \\ k \in K_N \\ n \in \mathbb{Z} \end{array} \right\}$$

$$\|$$

$$\cdots * gk * (0,n) * g' * \cdots$$

If $n = 0$, this reduces further. So we can write words in G_{N+1} as alternating sequences of nontrivial elements of $G_N/\mathbb{Z}$.

We also have the relation

$$\cdots * gk * (0,n) * g' * \cdots$$
$$\shortparallel$$
$$\cdots * g * (k,n) * g' * \cdots$$
$$\shortparallel$$
$$\cdots * g * (0,n) * kg' * \cdots$$

But that's "it".

- Note also $K_n = K_{n+1} \oplus H_n$, so (regarding all sets as being subsets of G) $H_n < K_n$ and $H_n \cap K_{n+1} = \{e\}$

moreover, $H_n \lhd G_N$ for all $N < n$:

First, for $N=0$, $G_0 = K \rtimes \Lambda$ clearly contains H_n as a normal subgroup

$$H_n < \overset{\text{abelian}}{K} \lhd K \rtimes \Lambda$$

Inductively, if $H_n \lhd G_N$ and $N+1 < n$,

then from $G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z})$

we notice that elements of $H_n < K_N$ "commute across $*_{K_N}$", and that $K_N \times \mathbb{Z}$ commutes with H_n, and that $g H_n g^{-1} = H_n \quad \forall g \in G_N$.

Hence

$$(\cdots * g * (0,n) * g' * \cdots) H_n (\cdots (g')^{-1} * (0,-n) * g^{-1} * \cdots) = H_n$$

so $H_n \trianglelefteq G_{N,H}$ as well.

We'll show 3 facts:

1. G has ICC
2. G is inner amenable
3. LG does not have Gamma.

We show 1 first:

<u>Lemma 1</u>: For every $g \in G \setminus K$, the set $\{hgh^{-1} : h \in \Lambda\}$ is infinite.

We divide into cases depending on where g appears in the tower:

<u>Case A</u>: $g \in G_{N+1} \setminus G_N$ for some N.

Since $G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z})$,

the elements hgh^{-1} are distinct, for distinct $h \in \Lambda$: $G_N \not\ni g$ admits a reduced word representation

$$g = \cdots * g' * (0,n) * g'' * \cdots \quad (\text{perhaps } g' = g'' = e)$$

for some $n \in \mathbb{Z} \setminus \{0\}$; since, for $h \in \Lambda \setminus \{e\}$,

we have $h \notin K \supseteq K_N$, so

$$hg = h(\cdots * g' * (0,n) * g'' * \cdots)$$
$$\neq (\cdots * g' * (0,n) * g'' * \cdots) h$$

so $hgh^{-1} \neq g$ for all $h \in \Lambda \setminus \{e\}$.

It follows that $hgh^{-1} \neq \bar{h} g \bar{h}^{-1}$, $\forall h \neq \bar{h} \in \Lambda$.

<u>Case B</u>: If $g \in G_0 \setminus K$, we write

$$g = (k, h_0) \in K \rtimes \Lambda, \quad h_0 \neq e$$

Then $h g h^{-1} = (-, h h_0 h^{-1})$, $h \in \Lambda$

are all distinct, since Λ has ICC.

We now show ①, i.e. that G has ICC.

Suppose $g \neq e$ has finite conjugate class.

By the above, $g \in K$.

So for some N, $g \in K \setminus K_N \subseteq G_N \setminus K_N$.

But this in particular means that

$$g \in G_N *_{K_N} (K_N \times \mathbb{Z}) \setminus K_N$$

has finite conjugacy class, whereas

$(0,n) * g * (0,-n)$ are all distinct. $(n \in \mathbb{Z})$

Thus all $g \neq e$ have ICC.

We now show ②, that G is inner amenable.

Embed $LG \hookrightarrow \ell^2(G)$ by $x \mapsto x\delta_e$.

Set $\xi_n = p_n^{-3/2} \sum_{h \in H_n} \delta_h \in \ell^2(G)$

[where $H_n \leq K_n < G$]

Then $\langle \xi_n, \xi_n \rangle = p_n^{-3} \sum_{h,k \in H_n} \langle \delta_h, \delta_k \rangle = p_n^{-3} \sum_{h \in H_n} 1$

$= 1$

and

$\langle \delta_e, \xi_n \rangle = p_n^{-3/2} \longrightarrow 0$.

Take $n \geq N$ and $g \in G_N$. Then

$$u_g \xi_n u_g^* = p_n^{-3/2} \sum_{h \in H_n} u_g \delta_h u_g^*$$

$$= p_n^{-3/2} \sum_{h \in H_n} \delta_{ghg^{-1}}$$

$$= \xi_n$$

where we use $H_n \trianglelefteq G_N$.

Thus, for arbitrary $g \in G$, the sequence

$\| u_g \xi_n u_g^* - \xi_n \|_2$ is eventually 0.

Thus, for any $F \subseteq G$ finite, there is $\xi_n \in \ell^2 G$ with

$$\max_{g \in F} \| u_g \xi_n u_g^* - \xi_n \|_2 = 0 \quad \text{and}$$

$$\| \xi_n - \tau(\xi_n) \cdot 1 \|_2 \geq 1 - p_n^{-3/2}$$

so there is no $c > 0$ for which

$$\max_{g \in F} \| u_g \xi_n u_g^* - \xi_n \|_2 \geq c \cdot \| \xi_n - \tau(\xi_n) \|_2$$

and so $G \overset{Ad}{\curvearrowright} \ell^2(G)$ does not have spectral gap. Thus G is inner amenable.

It remains to show 3, that LG does not have Gamma.

By a slight rearrangement, it suffices to show:

Given $(x_n)_n \subseteq \mathcal{U}(LG)$ such that

$$\|x_n y - y x_n\|_2 \to 0 \quad \forall y \in LG$$

we must have

$$\|x_n - \tau(x_n)\cdot 1\|_2 \to 0$$

To prove this, we break it up as

$$\begin{aligned}\|x_n - \tau(x_n)\cdot 1\|_2 \le\ & \|x_n - y_n\|_2 \quad \mathrm{I} \\ & + \|y_n - E_{LK_N}(y_n)\|_2 \quad \mathrm{II} \\ & + \|E_{LK_N}(y_n) - \tau(y_n)\cdot 1\|_2 \quad \mathrm{III} \\ & + \|\tau(y_n)\cdot 1 - \tau(x_n)\cdot 1\|_2 \quad \mathrm{IV}\end{aligned}$$

for $y_n := E_{LG \cap (LA)'}(x_n)$.

We analyze each of I – IV separately.

Note that $\tau \circ E_B = \tau$, so in particular

IV $= 0$ always.

(I): Let $\pi : G \to \mathcal{U}(\ell^2 G)$ denote the adjoint rep $\pi_g \xi = u_g \xi u_g^*$.

Then $\xi_n := x_n \delta_e$ is π-almost invariant:

$$\|\pi_g \xi_n - \xi_n\|_2 = \| u_g x_n \delta_e u_g^* - x_n \delta_e\|_2$$
$$= \| \{u_g x_n u_g^* - x_n\} \delta_e \|_2$$
$$= \| u_g x_n u_g^* - x_n \|_2$$
$$= \| u_g x_n - x_n u_g \|_2 \longrightarrow 0$$

In particular, ξ_n is $\pi(\Lambda)$-almost invariant.
Since $\Lambda = SL(3,\mathbb{Z})$ has prop. (T), if P denotes the orthogonal projection in $\ell^2 G$ onto the space of $\pi(\Lambda)$-invariant vectors,

$$\| \xi_n - P(\xi_n) \| \longrightarrow 0$$

Thus, for $y_n = E_{LG \cap (L\Lambda)'}(x_n)$,

$$\|x_n - y_n\|_2 \longrightarrow 0 \quad \text{as} \quad n \to \infty .$$

I

$$\left[\begin{array}{l} y \in LG \cap (L\Lambda)' \iff u_g y u_g^* = y \\ \qquad\qquad\qquad\quad \iff \pi_h y = y \quad \forall h \in \Lambda \end{array}\right]$$

II: We first show that

$$L G \cap (L\Lambda)' = LK \cap (L\Lambda)'$$

For $x \in LG \cap (L\Lambda)'$, we may write

$$x = \sum_{g \in G} c_g u_g \qquad \text{(wo-limit)}$$

Then, for each $h \in \Lambda$, $u_h x = x u_h$

$$\sum_{g\in G} c_g u_{hg} = u_h x = x u_h = \sum_{g \in G} c_g u_{gh} = \sum_{g\in G} c_{hgh^{-1}} u_{hg}$$

[mult. is separately wo-cts]

Thus $c_g = c_{hgh^{-1}}$ for all $g \in G, h \in \Lambda$.

Since G is discrete and, for $g \in G \setminus K$,

$\{hgh^{-1} : h \in \Lambda\}$ is infinite,

we must have $c_g = 0 \quad \forall g \in G \setminus K$.

So $x = \sum_{g \in K} c_g u_g \in LK \cap (L\Lambda)'$, and hence

$$LG \cap (L\Lambda)' \subseteq LK \cap (L\Lambda)' \subseteq LG \cap (L\Lambda)'$$

as desired.

Thus $y_n \in LG \cap (L\Lambda)' = LK \cap (L\Lambda)'$

so (in particular) $y_n \in (LK)_1$.

To show $\mathrm{II} = \|y_n - E_{LK_N}(y_n)\|_2 \to 0$,

consider $g_{N+1} \in G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z})$

the canonical generator of the $\mathbb{Z}$.

Since g_{N+1} commutes with K_N,

$u_{g_{N+1}}$ commutes with LK_N, so

$$u_{g_{N+1}} y_n u_{g_{N+1}}^* - y_n = u_{g_{N+1}} (y_n - E_{LK_N}(y_n)) u_{g_{N+1}}^*$$

$$+ u_{g_{N+1}} E_{LK_N}(y_n) u_{g_{N+1}}^* - y_n$$

$$= \underbrace{u_{g_{N+1}} (y_n - E_{LK_N}(y_n)) u_{g_{N+1}}^*}_{(*)} + \underbrace{E_{LK_N}(y_n) - y_n}_{(**)}$$

We claim $(*), (**)$ orthogonal:

$g_{N+1}(K \setminus K_N) g_{N+1}^{-1}$ and K are disjoint

since $G_N *_{K_N} (K_N \times \mathbb{Z}) = G_{N+1} \hookrightarrow G$

Since $(*) \in \left[u_{g_{N+1}} \{ u_k : k \in K \setminus K_N \} u_{g_{N+1}^{-1}} \right]''$

and $(**) \in LK_N$

we get $(*), (**)$ orthogonal.

Thus

$$\| u_{g_{N+1}} y_n u_{g_{N+1}}^* - y_n \|_2 \geq \| \overset{(*)}{u_{g_{N+1}} (y_n - E_{LK_N}(y_n)) u_{g_{N+1}}^*} \|_2$$

$$\geq \| y_n - E_{LK_N}(y_n) \|_2$$

So it remains to control

$$\|u_{g_{N+1}} y_n u_{g_{N+1}}^* - y_n\|_2 = \| u_{g_{N+1}} x_n u_{g_{N+1}}^* - x_n$$

$$+ u_{g_{N+1}}(y_n - x_n) u_{g_{N+1}}^* - (y_n - x_n)\|_2$$

$$\leq \underbrace{\|u_{g_{N+1}} x_n u_{g_{N+1}}^* - x_n\|_2}_{\to 0 \text{ by def. of } x_n}$$

$$+ \underbrace{\|u_{g_{N+1}}(y_n - x_n) u_{g_{N+1}}^* - (y_n - x_n)\|_2}_{\to 0}$$

since $\pi_{g_{N+1}}$ str-cts, $\|x_n - y_n\|_2 \to 0$

and thus

$$\text{II} = \|y_n - E_{LK_N}(y_n)\|_2 \to 0$$

II

$\boxed{\text{III}}$: $\| E_{LK_N}(y_n) - \tau(y_n)\cdot 1 \|_2$

We know $y_n = E_{LG \cap (LA)'}(x_n) \in (LG)_1$.

we will show that, since $\bigcap_{k=0}^{\infty} K_k = \{e\}$

and $E_{LK_N}(y_n) \in (LK_N)_1$, $\boxed{\text{III}} \to 0$

"$E_{LK_N}(y_n)$ is in the unit ball of a vN algebra that, for N large, is approximately trivial"

To make this rigorous, we use the following lemma which describes the structure of $LG \cap (LA)'$ and its subalgebras:

<u>Lemma 2:</u> Set (A_n, τ) to be the tracial vN algebra

$$A_n := \mathbb{C}e_n \oplus \mathbb{C}(1-e_n) \cong \mathbb{C}^2$$

with $e_n, 1-e_n$ minimal projections satisfying $\tau(e_n) = p_n^{-3}$.

Set $$(A, \tau) := \bigotimes_{n=1}^{\infty} (A_n, \tau)$$

Then there is a unique trace-preserving bijective isomorphism

$$\alpha : A \longrightarrow LG \cap (L\Lambda)'$$

Satisfying $\alpha(e_n) = p_n^{-3} \sum_{h \in H_n} u_h$

In particular, α restricts to a bijection

$$\bigotimes_{n=N}^{\infty} (A_n, \tau) \longrightarrow LK_N \cap (L\Lambda)'$$

for each N.

<u>Proof of Lemma 2</u>: Recall from our argument in II that $L G \cap (L\Lambda)' = LK \cap (L\Lambda)'$.

Set $B_n := \ell^\infty(H_n)$ and define τ on B_n to just be the normalized counting measure.

Since $e_n \in A_n$ has $\tau(e_n) = p_n^{-3}$, we may view $A_n \subseteq B_n$ by $e_n \longleftrightarrow \chi_{\{0\}}$.

Let θ denote the action of Λ on B_n:

$$(\theta_g F)(x) = F(g^{-1}x)$$

for $g \in \Lambda = SL(3,\mathbb{Z})$, $x \in H_n = (\mathbb{Z}/p_n\mathbb{Z})^3$

Also set $\sigma : \Lambda \curvearrowright LK$ denote the action

$$\sigma_g(u_x) = u_{g \cdot x} \qquad \begin{matrix} g \in \Lambda \\ x \in K \end{matrix}$$

The Pontrjagin dual $\widehat{H_n} \cong H_n$ and so the Fourier transform yields a trace-preserving isomorphism

$$\begin{array}{c} \alpha_n : \mathcal{B}_n \longrightarrow LH_n \\ \shortparallel \\ \ell^\infty(H_n) \end{array}$$

satisfying the intertwining identity

$$\alpha_n \circ \theta_g = \sigma_{(g^{-1})^T} \circ \alpha_n$$

"Fourier transform turns translation into modulation"

Finally, put $(\mathcal{B}, \tau) := \bigotimes_{n=1}^{\infty} (\mathcal{B}_n, \tau)$ and let $\Lambda \overset{\theta}{\curvearrowright} \mathcal{B}$ diagonally.

The isomorphisms α_n combine to give

$$\alpha : B \to LK$$

trace-preserving isom. satisfying

$$\alpha \circ \theta_g = \sigma_{(g^{-1})^T} \circ \alpha \qquad \forall g \in \Lambda$$

Notice that, in $G_0 = K \rtimes \Lambda$,

$$(0 \rtimes (g^{-1})^T) \cdot (y \rtimes I_3) = (g^{-1})^T \cdot y \rtimes (g^{-1})^T$$

$$(y \rtimes I_3) \cdot (0 \rtimes (g^{-1})^T) = y \rtimes (g^{-1})^T$$

so $y \in LK$ commutes with $L\Lambda$

if and only if $\sigma_{(g^{-1})^T}(y) = y \quad \forall g \in \Lambda$

Pulling back by α, we see that

$$LK \cap (L\Lambda)' = \alpha(B^\Lambda)$$

fixed-point subset of B

Since $A_n \subseteq B_n$ for each n,

$$A = \overline{\bigotimes} A_n \subseteq \overline{\bigotimes} B_n = B$$

a vN subalgebra. We wish to show that

$\alpha|_A : A \to LK$ is an isom. onto

$LK \cap (L\Lambda)'$; by the above, we need to

show

$$B^\Lambda = A,$$

For $h \in (H_1 \times \cdots \times H_n)$,

$$\Lambda h = \Lambda h_1 \times \cdots \times \Lambda h_n$$

$$= U_1 \times \cdots \times U_n$$

with $U_j = \{0\}$ if $h_j = 0$

and $U_j = H_j \setminus \{0\}$ otherwise

[$SL(3,\mathbb{Z})$ acts transitively on $\mathbb{Z}/p\mathbb{Z}$]

Consequently,

$$\left(\bigotimes_{n=1}^{N} B_n\right)^{\wedge} \overset{\substack{\text{const. on orbits,} \\ \text{so breaks into} \\ X_{\{0\}} \,\Delta\, X_{\{0\}^c}}}{=} \bigotimes_{n=1}^{N} A_n$$

Taking $N \to \infty$, we have

$$B^{\wedge} = \left(\bigotimes_{n=1}^{\infty} B_n\right)^{\wedge} = \bigotimes_{n=1}^{\infty} A_n = A$$

So we're done.

We now show that, for N large & fixed,

$$\mathrm{III} = \| E_{LK_N}(y_n) - \tau(y_n)\cdot 1 \|_2 \quad \text{small.}$$

Since $y_n \in (L\Lambda)'$, also $E_{LK_N}(y_n) \in (L\Lambda)'$.

So $E_{LK_N}(y_n) \in \alpha\left(\bigotimes_{k=N}^{\infty} A_k\right)$ by Lemma 2

and so there exist a sequence

$$a_n \in \left(\bigotimes_{n=N}^{\infty}(A_n, \tau)\right), \text{ with}$$

$$\alpha(a_n) = E_{L\kappa N}(y_n) .$$

Now, since $p_n^{-3} = \tau(e_n) = \tau(1-(1-e_n))$,

and each $1-e_n \geq 0$, and since

$$\sum_{n=N}^{\infty} p_n^{-3} < \infty \quad \text{so that}$$

$$\prod_{n=N}^{\infty}(1-p_n^{-3}) \quad \text{converges,}$$

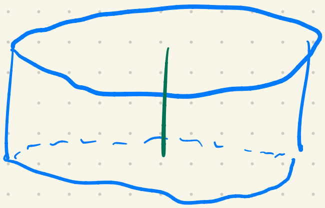
So too does $\lim_{k\to\infty} \prod_{n=N}^{k}(1-e_n)$

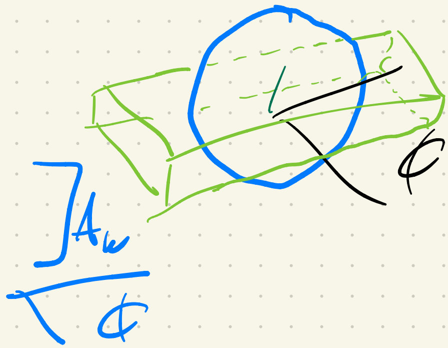
converge to a (minimal) projection f_N

with $\tau(f_N) = \prod_{n=N}^{\infty}(1-p_n^{-3})$

Set $\varepsilon_N := 1-\tau(f_N)$. For $a \in \left(\bigotimes_{n=1}^{\infty} A_n\right), \cap\left(\bigotimes_{n=N}^{\infty} A_n\right)$

Take $b = \frac{a - \tau(a)\cdot 1}{2}$, binary expansion of b^*b $\rightsquigarrow \|a - \tau(a)\cdot 1\|_2 \le 4\sqrt{\varepsilon_N}$





Recall that $a_n \in \left(\bigotimes_{w=N}^{\infty} A_w\right) \cap \left(\bigotimes_{w=1}^{\infty} A_w\right)_1$,

$$\tau(a_n) = \tau(E_{LK_N}(y_n)) = \tau(y_n),$$

so for any N, n we have

$$\|E_{LK_N}(y_n) - \tau(y_n)\cdot 1\|_2 \le 4\sqrt{\varepsilon_N}$$

which is III, since $\varepsilon_N \to 0$ as $N \to \infty$.

III

Thus $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV} \to 0$

and hence

$$\| x_n - \tau(x_n) \cdot 1 \|_2 \to 0$$

implying that LG does not have Gamma,

which is 3 as desired.

So this G is ICC, inner amenable, while

LG does not have Gamma.