High-low analysis and small cap decoupling over non-Archimedean fields

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September 13, 2024

Abstract

We prove a small cap decoupling theorem for the parabola over a general non-Archimedean local field for which $2 \neq 0$. We obtain polylogarithmic dependence on the scale parameter R and polynomial dependence in the residue prime, except for the prime 2 for which the polynomial depends on degree. Our constants are fully explicit.

1 Introduction

In this note, we prove that the small cap decoupling theorem for the parabola may be extended to non-Archimedean local fields $\mathbb{K}$ of characteristic different from 2. We do so by first adapting the "high-amplitude wave envelope estimate" of [16]. In addition to recovering the desired power law in the scale parameter, we also obtain a fully explicit subpolynomial factor whose scale dependence is of the form $(\log R)^{O(1)}$ with $O(1)$ explicit and not too large, and whose dependence on $\mathbb{K}$ is polynomial in the order of the residue field.

We recall the standard formalism of small cap decouplings, adapted to the non-Archimedean context. Let $R > 1$ be in the range of $|\cdot|_{\mathbb{K}}$ (the absolute value on $\mathbb{K}$), and write $\mathcal{N}_{\mathbb{R}^{-1}}(\mathbb{P}^1)$ for the set

$$\{(x, y) \in \mathbb{K}^2 : |x| \leq 1,\, |y - x^2| \leq R^{-1}\}.$$

Here we write $\mathbb{P}^1$ for the truncated parabola $\{(x, x^2) \in \mathbb{K}^2 : |x| \leq 1\}$. If $\mathcal{O} = \{x \in \mathbb{K} : |x| \leq 1\}$ is the closed unit ball, and $\beta \in [\frac{1}{2}, 1]$ is such that R^β also belongs to the range of $|\cdot|_{\mathbb{K}}$, then we write $\mathcal{P}(\mathcal{O}, R^{-\beta})$ for the partition of $\mathcal{O}$ into closed balls of radius $R^{-\beta}$, and for each $I \in \mathcal{P}(\mathcal{O}, R^{-\beta})$ we write γ_I for the set

$$\{(x, y) \in \mathbb{K}^2 : x \in I,\, |y - x^2| \leq R^{-1}\}.$$

We will write

$$\Gamma_\beta(R^{-1}) = \{\gamma_I : I \in \mathcal{P}(\mathcal{O}, R^{-\beta})\};$$

thus, $\Gamma_\beta(R^{-1})$ is the partition of $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ into small caps of dimensions $R^{-\beta} \times R^{-1}$.

For $p, q \geq 1$ and R, β as above, we will write $D^{\mathbb{K}}_{p,q}(R; \beta)$ for the infimal constant such that, for any Schwartz-Bruhat function[1] $f : \mathbb{K}^2 \to \mathbb{C}$ with Fourier support contained in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$, we have

$$\|f\|^p_{L^p(\mathbb{K}^2)} \leq D^{\mathbb{K}}_{p,q}(R; \beta) \left(\sum_{\gamma \in \Gamma_\beta(R^{-1})} \|f_\gamma\|^q_{L^p(\mathbb{K}^2)} \right)^{p/q}.$$

Here and elsewhere we will write f_γ for the Fourier projection of f onto γ.

Our primary goal will be to show the following.

[1] i.e. a finite linear combination of indicators of metric balls.

Theorem 1.1 (Small cap decoupling over $\mathbb{K}$). *Let $p, q \geq 1$ satisfy $\frac{3}{p} + \frac{1}{q} \leq 1$, $R \geq \mathbf{p}^{32}$, and $\beta \in [\frac{1}{2}, 1]$. Then the small cap decoupling constant satisfies*

$$D_{p,q}^{\mathbb{K}}(R;\beta) \leq \frac{10^6}{(\log \mathbf{p})^{16}} \mathbf{p}^{12} (\log R)^{16+6\beta^{-1}} \left(R^{\beta(p-\frac{p}{q}-1)-1} + R^{p\beta(\frac{1}{2}-\frac{1}{q})} \right). \tag{1.1}$$

Here $\mathbf{p} = \mathbf{p}_{\mathbb{K}}$ is defined as:

$$\mathbf{p} = \begin{cases} p & \mathbb{K} \text{ extends } \mathbb{Q}_p \text{ or } \mathbb{F}_p((t)),\ p > 2, \\ 2^d & \mathbb{K} \text{ extends } \mathbb{Q}_2,\ [\mathbb{K} : \mathbb{Q}_2] = d. \end{cases}$$

The study of non-Archimedean decouplings was initiated in [6], where a bilinear variant of an inequality of the form $D_{p,2}^{\mathbb{Q}_q}(R; \frac{1}{2}) \lesssim_{q,\varepsilon} (\log R)^{2p+\varepsilon}$ was established for $q > 2$ to achieve good discrete restriction estimates for the parabola. It was continued in [15], where that author observed many of the features of non-Archimedean analysis that made it particularly appropriate for the setting of decoupling. In [13], the current author and Lin generalized the decoupling theorem for the moment curve to the q-adic setting for each $q > n$ (with n the ambient dimension); the result there may be extended to any local field of characteristic either 0 or greater than n.

Aside from decoupling, there has been a recent flurry of non-Archimedean Fourier and harmonic analysis in general; see e.g. [1, 4, 9–11, 14, 17, 19] for a short sampling.

Over the real numbers, small cap decouplings were introduced in [3], and the estimate $D_{p,p}^{\mathbb{R}}(R;\beta) \lesssim_\varepsilon R^{\beta(\frac{p}{2}-1)+\varepsilon}$ was proved (with the reasonable interpretation of $D_{p,q}^{\mathbb{R}}$). The work of [5] established the real version of the estimate we will show, which is a superlevel set estimate implying sharp bounds on the $D_{p,q}^{-}(R;\beta)$ in the regime $\frac{1}{q} + \frac{3}{p} \leq 1$. Substantial work has also generalized the idea of small cap decouplings to other manifolds, such as that contained in [7] and [8].

Theorem 1.1 will proved by the following auxiliary estimate.

Theorem 1.2 (Wave envelope estimate). *Let $f : \mathbb{K}^2 \to \mathbb{C}$ be Schwartz-Bruhat with Fourier support in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$. Then, for any $\alpha > 0$,*

$$\alpha^4 \mu\big(\{x : |f(x)| > \alpha\}\big) \leq 13\mathbf{p}^{12}(\log R)^{10} \sum_{\substack{s \in \mathbf{p}^{\mathbb{Z}} \\ R^{-1/2} \leq s \leq 1}} \sum_{\substack{\tau \\ \mathrm{diam}(\tau) = s}} \sum_{U \in \mathcal{G}_\tau} \mu(U) \left(\fint_U \sum_{\theta \subseteq \tau} |f_\theta|^2 \right)^2.$$

Here we use the following notation: each $U \in \mathcal{G}_\tau$ is a rectangle of dimensions $R \times sR$, with long edge in the direction of the normal vector to $\mathbb{P}^1$ at the center of τ, centered at 0; the set $\mathcal{G}_\tau$ is the subset of the standard tiling of $\mathbb{K}^2$ by such rectangles for which the following holds:

$$\frac{e^2}{2} \frac{(\log R)^2}{(\log \mathbf{p})^2} \fint_U \sum_{\theta \subseteq \tau} |f_\theta|^2 \geq \frac{\alpha^2}{(\#\tau)^2}. \tag{1.2}$$

Here $\#\tau$ denotes the number of τ of a particular length for which $f_\tau \not\equiv 0$. We also write θ for a generic cap on $\mathbb{P}^1$ of dimensions $R^{-1/2} \times R^{-1}$.

In [16], these wave envelope estimates were refined to include only those envelopes corresponding to "high-amplitude" components of the various square functions. The latter paper demonstrated that the wave envelope estimate could also be used to derive the small cap results of [5]. Our argument closely follows our earlier paper [12], which in turn was an adaptation of the method in [16].

We briefly mention the classification of local fields, to supply examples of the fields we will be working over. Any nondiscrete locally compact topological field $\mathbb{K}$ whose topology is induced by an absolute value (i.e. a multiplicative norm), is necessarily one of the following:

(a) $\mathbb{K} = \mathbb{R}, \mathbb{C}$ (i.e. the Archimedean cases);

(b) $\mathbb{K} = \mathbb{Q}_p$, for some prime p, or a finite extension thereof; or

(c) $\mathbb{K} = \mathbb{F}_{p^n}((t))$, for some prime p and natural n.

We will restrict our attention to cases (b), (c), i.e. the non-Archimedean local fields of characteristic 0, resp. $p > 2$. The reader will generally benefit from imagining the cases $\mathbb{Q}_p, \mathbb{F}_p((t))$ with $p > 2$.

We also mention the utility in tracking dependence on $\mathbf{p}$ in the above theorems. Over the reals, Fourier-analytic methods for counting solutions to Diophantine equations frequently entail a subpolynomial loss in the diameter of the variable set. For instance, the Bourgain-Demeter-Guth resolution [2] of the main conjecture of Vinogradov's mean value theorem shows the particular result

$$J_{s,k}(\mathcal{A}) \lesssim_\varepsilon \mathrm{diam}(\mathcal{A})^\varepsilon (A^s + A^{2s - \frac{1}{2}k(k+1)}),$$

whenever $\mathcal{A} \subseteq \mathbb{Z}$ has $\#\mathcal{A} = A$. Here we use the usual notation of

$$J_{s,k}(\mathcal{A}) = \#\Big\{(\mathbf{n}, \mathbf{m}) \in \mathcal{A}^{2s} : \sum_{\iota=1}^{s} n_\iota^j - m_\iota^j = 0,\ \forall 1 \leq j \leq k\Big\},$$

for each $s, k \in \mathbb{N}$. If we instead use the p-adic decoupling theorem for the moment curve (Theorem 6.1 of [13]), we obtain the alternate estimate

$$J_{s,k}(\mathcal{A}) \lesssim_{p,\varepsilon} \delta_p(\mathcal{A})^{-\varepsilon} (A^s + A^{2s - \frac{1}{2}k(k+1)}).$$

Here $\delta_p(\mathcal{A}) = \min\{|n - m|_p : n \neq m \in \mathcal{A}\}$, where $|\cdot|_p$ is the usual p-adic norm. In particular, if p does not divide any of the differences $n - m$ (say, if $p > \max(|n| : n \in \mathcal{A})$), then the subpolynomial factor trivializes. However, there are corresponding losses in the choice of prime. Thus, judicious tracking of the dependence on p, together with number-theoretic considerations, may reduce the dependence on features of $\mathcal{A}$ other than its cardinality. See [18] for another approach to the same problem.

Next, we record the non-Archimedean version of the "block example" of [5], which demonstrates that the estimate in 1.1 cannot be extended to any (p, q) with $\frac{1}{q} + \frac{3}{p} > 1$ and $p > 2 + 2\beta^{-1}$, in the small cap regime $\beta > \frac{1}{2}$. Let $f = f_\theta = \sum_{\gamma \prec \theta} f_\gamma$, where θ is above $B(0, R^{-1/2})$ and each $f_\gamma = e(c_\gamma \cdot x) 1_{B(0,R^\beta) \times B(0,R)}$; here each c_γ is an arbitrary point chosen from γ. It is quick to see that $\hat{f}_\gamma$ is supported in γ, for each γ. Then we have

$$f 1_{B(0,R^\beta) \times B(0,R^{2\beta})} = R^{\beta - \frac{1}{2}} 1_{B(0,R^\beta) \times B(0,R^{2\beta})},$$

so that

$$\|f\|_{L^p} \geq R^{\beta - \frac{1}{2} + \frac{3\beta}{p}}, \quad \left(\sum_\gamma \|f_\gamma\|_{L^p}^q\right)^{\frac{1}{q}} = R^{\frac{3\beta}{p} + \frac{1}{q}(\beta - \frac{1}{2})}.$$

If $\frac{1}{q} + \frac{3}{p} > 1$ and $\beta > \frac{1}{2}$, then the corresponding ratio exceeds $R^{\beta(1 - \frac{1}{q} - \frac{1}{p}) - \frac{1}{p}}$.

Lastly, we discuss the special role of the prime $p = 2$ in the above; particularly, why we have excluded the fields $\mathbb{F}_{2^d}((t))$, and why $\mathbf{p}$ is much larger for extensions of $\mathbb{Q}_2$. In the case of characteristic 2, say $\mathbb{K} = \mathbb{F}_{2^d}((t))$, by the Frobenius identity $(x+y)^2 = x^2 + y^2$ we see that $\mathbb{P}^1$ is contained in a linear subspace of $\mathbb{K}^2$, when the latter is regarded a vector space over $\mathbb{F}_{p^d}$. In particular, linear equations among the first powers of frequency variables imply corresponding equations among the second powers. A straightforward computation (working first over even integer exponents, then interpolating and comparing with the trivial Cauchy-Schwarz bounds) supplies the identity $D_{p,2}^{\mathbb{F}_{2^d}((t))}(R; \frac{1}{2}) = R^{\frac{1}{2}(\frac{p}{2} - 1)}$, for any $R \in 2^{d\mathbb{N}}$ and $p \geq 2$. One may compare with the proof of the local bilinear restriction estimate Theorem 2.4 below to see the effect of the size of 2 in $\mathbb{K}$, when $2 \neq 0$ is small.

1.1 Brief overview of method

We discuss the method for proving Theorem 1.2. We will adopt several temporary notations for the sake of an intuitive sketch, which will later be abandoned in favor of the technical approach.

The basic decomposition used in the proof of Theorem 1.2 is the two-part decomposition

$$\mathcal{N}(\mathbb{P}^1) = \bigsqcup \theta, \quad \theta - \theta =: \delta\theta = \bigsqcup \delta\theta^{(k)}.$$

Here the sets $\delta\theta^{(k)}$ are understood as follows: if θ is the cap about 0 for simplicity, that is, $\theta = B(0, R^{-1/2}) \times B(0, R^{-1})$, such that in particular $\theta = \delta\theta$ (in our non-Archimedean setting), then for $0 \le k \le N$ we write $\delta\theta^{(k)}$ for

$$\begin{aligned} \delta\theta^{(k)} &:= \Big\{(x, y) \in \delta\theta : |x| \in \big(R^{\frac{k-1}{2N}-1}, R^{\frac{k}{2N}-1}\big]\Big\}, \quad 1 \le k \le N, \\ \delta\theta^{(0)} &:= \Big\{(x, y) \in \delta\theta : |x| \le R^{-1}\Big\}. \end{aligned}$$

Here $N \sim \log R$ is an integer. Other $\delta\theta$ will be decomposed similarly; so too for caps of different sizes, e.g. the τ of shape $R^{-1/3} \times R^{-2/3}$. Thus, the parameter k measures the distance from the center of the cap. The raison d'être of this decomposition is the pair of estimates

$$\int \left|\sum_{\theta} \mathcal{P}_{\delta\theta^{(k)}}\left[|f|^2\right]\right|^2 \lesssim \sum_{\tau} \int \left|\sum_{\theta\subseteq\tau} \mathcal{P}_{\delta\theta^{(k)}}\left[|f|^2\right]\right|^2, \tag{1.3}$$

(writing of course $\mathcal{P}_A$ for the Fourier projection onto a set A), valid whenever $k > 0$ and the τ have diameter $d(\tau) \ge R^{\frac{k}{2N}-1}$; and the pointwise estimate

$$\left|\sum_{\theta} \mathcal{P}_{\delta\theta^{(k)}}[f]\right|^2 = \sum_{\tau} \left|\sum_{\theta\subseteq\tau} \mathcal{P}_{\delta\theta^{(k)}}[f]\right|^2, \tag{1.4}$$

valid whenever $k > 0$ and the τ have diameter $d(\tau) \ge R^{-\frac{|k|}{N}}$. (1.3) and (1.4) are known as the high and low lemmas, respectively; see Lemmas 2.2 and 2.1 below. By repeatedly applying estimates of the form (1.3) and (1.4), together with the usual L^4 Córdoba-Fefferman square function estimate, we find that for each $k \ge \ell$ we have that

$$\int \Big| \sum_{\substack{\tau \\ \operatorname{diam}(\tau)=R^{-\frac{\ell}{N}}}} \mathcal{P}_{\delta\tau^{(k)}}[f]\Big|^4 \lessapprox \sum_{m=\ell}^{N} \sum_{\substack{\tau \\ \operatorname{diam}(\tau)=R^{-\frac{m}{N}}}} \int \Big|\sum_{\theta\subseteq\tau} \mathcal{P}_{\delta\theta^{(m)}}\left[|f|^2\right]\Big|^2.$$

See the proof of Prop. 2.3 below. It remains to analyze the right-hand side. One may observe that each $\left|\sum_{\theta\subseteq\tau} \mathcal{P}_{\delta\theta^{(k)}}\left[|f|^2\right]\right|^2$ is constant on rectangles U of dimension $R \times R^{1-\frac{m}{N}}$, oriented as stated in Theorem 1.2. Thus, for each U,

$$\int_U \Big|\sum_{\theta\subseteq\tau} \mathcal{P}_{\delta\theta^{(k)}}\left[|f|^2\right]\Big|^2 \le \mu(U) \left(\fint_U \sum_{\theta\subseteq\tau} |\mathcal{P}_\theta[f]|^2\right)^2.$$

Thus, we are done in the special case that (a) there is some ℓ and $k \ge \ell$ such that $f = \sum_{\tau:\operatorname{diam}(\tau)=R^{-\ell/N}} \mathcal{P}_{\tau^{(k)}}[f]$, and that (b) for each $m \ge \ell$ and each τ with $\operatorname{diam}(\tau) = R^{-m/N}$ we have that $\sum_{\theta\subseteq\tau} \mathcal{P}_{\delta\theta^{(k)}}\left[|f|^2\right]$ is supported on the rectangles in $\mathcal{G}_\tau$. It happens that this special case may be achieved from the general one by a pruning procedure on f; that is, an arbitrary input function f is a sum of functions $f_m^{\mathcal{B}}$ satisfying (a) and (b), plus an inanity f_0 which is small for trivial reasons.

The argument we follow will put in front the pruned functions $f_m^{\mathcal{B}}$, and the Fourier decomposition we sketched above will be expressed in somewhat different language (i.e. the high/low analysis of square functions). In particular, the decompositions $\delta\theta = \bigsqcup \delta\theta^{(k)}$ discussed above are not referenced past this point.

The principal thrust of the argument is equivalent to that of [16] and [12], though translated to the non-Archimedean setting. This latter stipulation provides many technical advantages, particularly related to the absence of Schwartz tails. The most visible consequence of this is the removal of many technical weights that were present in earlier papers. As a corollary, our analysis is relatively simple, and may be used as a comparative document for those wishing to study the earlier works.

1.2 Initial notation-setting

Let $\mathbb{K}$ be a non-Archimedean local field of characteristic not 2, i.e. a nondiscrete totally disconnected locally compact topological field which is equipped with a complete absolute value $|\cdot|$ inducing the topology, for which $2 \neq 0$. We normalize $|\cdot|$ by insisting that it is the modular function for $\mathbb{K}$, regarded as a LCA group. Let $\mathcal{V} \subseteq (0, \infty)$ be the range of $|\cdot|$ on the nonzero members of $\mathbb{K}$. Let $\mathbf{m} \subseteq \mathcal{O}$ be the maximal ideal, and $\varpi \in \mathbf{m}$ be a uniformizer. Let $\eta \in \mathbb{N}$ be minimal such that $\mathbf{p} = |\varpi|^{-\eta}$ satisfies $\mathbf{p} \geq |2|^{-1}$. Fix some $R \in \mathbf{p}^{2\mathbb{N}}$, and write $N = \frac{1}{2}\log_{\mathbf{p}}(R)$. For $0 \leq k \leq N$, we write $R_k = \mathbf{p}^k$. Let $\alpha \in (0, R)$ and $U_\alpha = \{x \in B_R : |f(x)| > \alpha\}$.

Write $B_T = \{x \in \mathbb{K}^2 : |x| \leq T\}$ for each $T \in \mathbb{R}_{>0}$. For each k, write $\mathcal{P}(\mathcal{O}, R_k^{-1})$ for the partition of the unit ball B_1 into metric balls of radius R_k^{-1}. If $I \in \mathcal{P}(\mathcal{O}, R_k^{-1})$, we will write τ_I for the set

$$\tau_I = (a, a^2) + M_{a,\varpi^{\eta k}}[\mathcal{O}^2],$$

where $a \in I$ is chosen arbitrarily and, for each $\lambda \in \mathbb{K}^\times$,

$$M_{a,\lambda} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}.\mathrm{diag}(\lambda, \lambda^2).$$

It is quick to see that τ_I is independent of $a \in I$. We will write $A_{a,\lambda}(x) = (a, a^2) + M_{a,\lambda}(x)$. We will also lift the subset partial order $\subseteq$ on the set of metric balls I to the associated caps τ_I, which we will write as $\prec$. Thus,

$$I \subseteq J \iff \tau_I \prec \tau_J.$$

We introduce this special notation $\prec$ to help clarify at various points that the two objects on either side of the relation symbol are objects of a special type (i.e. caps), rather than arbitrary subsets of $\mathbb{K}^2$.

Let μ denote Haar measure on $\mathbb{K}^1$ and $\mathbb{K}^2$; the choice will always be clear from context. We will let $e : \mathbb{K} \to \mathbb{C}$ be some choice of character such that $e(\mathcal{O}) = \{1\} \neq e(\varpi^{-1}\mathcal{O})$. A convenient choice for $\mathbb{Q}_p$ and $\mathbb{F}_p(\!(t)\!)$ would be

$$e\left(\sum_{n=N}^{\infty} a_n p^n\right) = \exp(2\pi i a_{-1} p^{-1}), \quad N \in \mathbb{Z}, \{a_n\}_n \in \{0, \dots, p-1\}^{\mathbb{Z}}$$

for $\mathbb{Q}_p$, and

$$e\left(\sum_{n=N}^{\infty} a_n t^n\right) = \exp(2\pi i a_{-1} p^{-1}), \quad N \in \mathbb{Z}, \{a_n\}_n \in \{0, \dots, p-1\}^{\mathbb{Z}}$$

for $\mathbb{F}_p(\!(t)\!)$.

With respect to e and μ, we understand the Fourier transform to be

$$\hat{f}(\xi) = \int_{\mathbb{K}} e(x\xi) f(x) d\mu(x),$$

for any Schwartz-Bruhat function $f : \mathbb{K} \to \mathbb{C}$. Functions on $\mathbb{K}^n$ will be handled similarly.

For $I \in \mathcal{P}(\mathcal{O}, R_k^{-1})$, we will write

$$N_{\tau_I} = M_{a,1}.\mathrm{diag}(\varpi^{-\eta(N-k)}, \varpi^{-\eta N}),$$

where we make arbitrary choices of $a \in I$. Let $\mathcal{U}_{\tau_I}$ be the image of the set of translates of $\mathcal{O}^2$ under N_{τ_I}; $\mathcal{U}$ does not depend on the choice of a. For $U \in \mathcal{U}_{\tau_I}$, define the averaging operator $\mathcal{A}_U$ by

$$\mathcal{A}_U[f] = \mu(U)^{-1} \int_U f d\mu.$$

We will use the symbol χ for the indicator functions of annuli, and put the radius bounds in the denominator. Thus,

$$\chi_{\leq r} = 1_{B_r}, \quad \chi_{>r} = 1_{\mathbb{K}^2 \setminus B_r}, \quad \chi_{(r_1, r_2]} = 1_{B_{r_2} \setminus B_{r_1}}.$$

We will use subscripts to denote Fourier projections, e.g. $\mathcal{P}_\theta[f] = f_\theta$, and hereafter omit the projection operators $\mathcal{P}_A$ discussed exclusively in subseection 1.1. We will write $g_\tau = \sum_{\theta \prec \tau} |f_\theta|^2$.

1.3 Overview of remainder of paper

In Section 2, we prove Theorem 1.2. In the first subsection 2.1, we state and prove the basic lemmas that apply to general functions of the prescribed spectral support, which will be the foundation of our analysis. In the second subsection 2.2, we fix the particular function f and define a pruning of that function, to define a decomposition $f = \sum_m f_m^{\mathcal{B}} + (f - f_N) + f_0$ into pieces over which the preceding lemmas may be applied fruitfully. The upshot of that subsection is a set of estimates that will resolve Theorem 1.2 in the special case of "broad" domination. In the next subsection 2.3, we run a broad/narrow analysis to conclude a local version of Theorem 1.2. In the terminal subsection 2.4, we prove the full theorem by removing the local assumption. In Section 3, we prove Theorem 1.1 by an essentially elementary manipulation of the conclusion of Theorem 1.2.

2 Proof of Theorem 1.2

We begin by establishing a number of technical high/low decomposition results, applicable to a general Schwartz-Bruhat function f with Fourier support in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$. Later, we will fix a single f and perform a pruning procedure, in order to obtain appropriate functions for which the preceding results are useful.

2.1 Technical lemmas

Lemma 2.1 (Low lemma)**.** *Let f have Fourier support in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$. For any $1 \leq m \leq k \leq N$, and $0 \leq s \leq k$,*

$$|f_{m,\tau_s}^{\mathcal{B}}|^2 * \chi_{\leq R_k^{-1}}^{\vee} = \sum_{\tau_k \prec \tau_s} |f_{m,\tau_k}^{\mathcal{B}}|^2 * \chi_{\leq R_k^{-1}}^{\vee}$$

for any τ_s.

Proof. Indeed, if $\tau_k \neq \tau_k'$, then for each $x \in \tau_k$ and $y \in \tau_k'$, $|x - y| > R_k^{-1}$. Thus, $\tau_k - \tau_k'$ is disjoint from $B_{R_k^{-1}}$. □

Lemma 2.2 (High Lemmas)**.** *Let f have Fourier support in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$. For any $m, k,$ and l such that $1 \leq m \leq N$, $l \leq k$, and $k + l \leq N$,*

(a)
$$\int \Big| \sum_{\theta} |f_\theta|^2 * \chi^\vee_{>R_k/R} \Big|^2 \leq \sum_{\tau_k} \int \Big| \sum_{\theta \prec \tau_k} |f_\theta|^2 * \chi^\vee_{>R_k/R} \Big|^2,$$

(b)
$$\int \Big| \sum_{\tau_k} |f_{\tau_k}|^2 * \chi^\vee_{>R^{-1}_{k+l}} \Big|^2 \leq |2|^{-1} \mathbf{p}^l \sum_{\tau_k} \int \Big| |f_{\tau_k}|^2 * \chi^\vee_{>R^{-1}_{k+l}} \Big|^2.$$

Proof. (a): By Plancherel,

$$\int \Big| \sum_{\theta} |f_\theta|^2 * \chi^\vee_{>R_k/R} \Big|^2 = \int_{|\xi|>R_k/R} \Big| \sum_{\tau_k} \sum_{\theta \prec \tau_k} \widehat{|f_\theta|^2} \Big|^2.$$

The supports of the summands $\sum_{\theta \prec \tau_k} \widehat{|f^{\mathcal{B}}_{m,\theta}|^2}$, ranging over distinct τ_k, are disjoint outside of the ball $B_{R_k/R}$. Applying Plancherel, we conclude.

(b): Note that $|f_{\tau_k}|^2$ has Fourier support in the set $\tau_k - \tau_k$. Suppose that τ_k is centered at $\gamma(t_1)$ and τ'_k is centered at $\gamma(t_2)$, for some $|t_1 - t_2| \geq |2|^{-1}\mathbf{p}^{l-k}$. Then, if $(\tau_k - \tau_k) \cap (\tau'_k - \tau'_k) \setminus B_{R^{-1}_{k+l}}$ is nontrivial, then we may find a solution $a_1, a_2, b_1, b_2 \in \mathcal{O}$ to the system

$$|a_1|, |a_2| > \mathbf{p}^{k-(k+l)},$$
$$(\varpi^{\eta k}(a_1 - a_2), \varpi^{2\eta k}(b_1 - b_2) + 2\varpi^{\eta k}(t_1 a_1 - t_2 a_2)) = (0, 0).$$

By the first condition on the second display, $a_1 = a_2$. But then

$$|\varpi^{2\eta k}(b_1 - b_2) + 2\varpi^{\eta k}(t_1 a_1 - t_2 a_2)| \geq |2|\mathbf{p}^{-k}|a_1||t_1 - t_2| - \mathbf{p}^{-2k}|b_1 - b_2| > \mathbf{p}^{-(-k+k-k-l+l-k)} - \mathbf{p}^{-2k} = 0.$$

By the strict inequality, the leftmost expression is nonzero, contradicting the previous assummption.

Thus, we know that the $\tau_k - \tau_k$ can only overlap on $\mathbb{Q}_p^2 \setminus B_{R^{-1}_{k+l}}$ with those $\tau'_k - \tau'_k$ corresponding to time parameters within a distance of $|2|^{-1}\mathbf{p}^{l-k}$. Thus, appealing to Plancherel,

$$\begin{aligned}\int \Big| \sum_{\tau_k} |f_{\tau_k}|^2 * \chi^\vee_{>R^{-1}_{k+l}} \Big|^2 &= \int_{\mathbb{Q}_p^2 \setminus B_{R^{-1}_{k+l}}} \Big| \sum_{\tau_k} \widehat{|f_{\tau_k}|^2} \Big|^2 \\ &\leq |2|^{-1}\mathbf{p}^l \sum_{\tau_k} \int \Big| |f_{\tau_k}|^2 * \chi^\vee_{>R^{-1}_{k+l}} \Big|^2\end{aligned}$$

by the Schur test.

□

Proposition 2.3 (Wave envelope bound of high parts)**.** Let f have Fourier support in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ and $1 \leq \ell < N$. Then

$$\int \Big| \sum_{\tau_\ell} |f_{\tau_\ell}|^2 * \chi^\vee_{>R_\ell/R} \Big|^2 \leq 3\mathbf{p}^4 N \sum_{k=\ell}^{N} \sum_{\tau_k} \int \Big| \sum_{\theta \prec \tau_k} |f_\theta|^2 * \chi_{\leq R_k/R} \Big|^2.$$

Proof. We decompose

$$\int \Big|\sum_{\tau_\ell} |f_{\tau_\ell}|^2 * \chi^\vee_{>R_\ell/R}\Big|^2 = \sum_{k=\ell+1}^{N} \int \Big|\sum_{\tau_\ell} |f_{\tau_\ell}|^2 * \chi^\vee_{(R_{k-1}/R, R_k/R]}\Big|^2 + \sum_{k=\ell+1}^{N} \int \Big|\sum_{\tau_\ell} |f_{\tau_\ell}|^2 * \chi^\vee_{(R_k^{-1}, R_{k-1}^{-1}]}\Big|^2 .$$

Consider the first summand. By the low lemma 2.1,

$$\int \Big|\sum_{\tau_\ell} |f_{\tau_\ell}|^2 * \chi^\vee_{(R_{k-1}/R, R_k/R]}\Big|^2 = \int \Big|\sum_{\theta} |f_\theta|^2 * \chi^\vee_{(R_{k-1}/R, R_k/R]}\Big|^2 .$$

By the high lemma (a) and Plancherel, we have

$$\int \Big|\sum_{\theta} |f_\theta|^2 * \chi^\vee_{(R_{k-1}/R, R_k/R]}\Big|^2 \le \mathbf{p} \sum_{\tau_k} \int \Big|\sum_{\theta \prec \tau_k} |f_\theta|^2 * \chi^\vee_{\le R_k/R}\Big|^2 .$$

Next, for each $\ell + 1 \le k \le N$, by the low lemma 2.1,

$$\int \Big|\sum_{\tau_\ell} |f_{\tau_\ell}|^2 * \chi^\vee_{(R_k^{-1}, R_{k-1}^{-1}]}\Big|^2 = \int \Big|\sum_{\tau_{k-1}} |f_{\tau_{k-1}}|^2 * \chi^\vee_{(R_k^{-1}, R_{k-1}^{-1}]}\Big|^2 .$$

By part (b) of the high lemma 2.2,

$$\int \Big|\sum_{\tau_{k-1} \prec \tau_s} |f_{\tau_{k-1}}|^2 * \chi^\vee_{(R_k^{-1}, R_{k-1}^{-1}]}\Big|^2 \le \mathbf{p} \sum_{\tau_{k-1}} \int |f_{\tau_{k-1}}|^4 .$$

By the reverse square function estimate for $\mathbb{P}^1$, Prop. 2.5 below,

$$\int |f_{\tau_{k-1}}|^4 \le 2 \int \Big|\sum_{\theta \prec \tau_{k-1}} |f_\theta|^2\Big|^2 .$$

We decompose the right-hand side of the resulting inequality:

$$\begin{aligned}\int \Big|\sum_{\tau_{k-1} \prec \tau_s} |f_{\tau_{k-1}}|^2 * \chi^\vee_{(R_k^{-1}, R_{k-1}^{-1}]}\Big|^2 &\le 2\mathbf{p} \sum_{\tau_{k-1}} \int \Big|\sum_{\theta \prec \tau_{k-1}} |f_\theta|^2\Big|^2 \\ &= 2\mathbf{p} \sum_{\tau_{k-1}} \int \Big|\sum_{\theta \prec \tau_{k-1}} |f_\theta|^2 * \chi^\vee_{\le R_{k-1}/R}\Big|^2 \\ &\quad + 2\mathbf{p} \sum_{k \le t \le N} \sum_{\tau_{k-1}} \int \Big|\sum_{\theta \prec \tau_{k-1}} |f_\theta|^2 * \chi^\vee_{(R_{t-1}/R, R_t/R]}\Big|^2 .\end{aligned}$$

The first summand is of the desired form. Take now $k \le t \le N$. By the high lemma,

$$\sum_{\tau_{k-1}} \int \Big|\sum_{\theta \prec \tau_{k-1}} |f_\theta|^2 * \chi^\vee_{(R_{t-1}/R, R_t/R]}\Big|^2 \le \sum_{\tau_{t-1}} \int \Big|\sum_{\theta \prec \tau_{t-1}} |f_\theta|^2 * \chi^\vee_{(R_{t-1}/R, R_t/R]}\Big|^2 .$$

By Cauchy-Schwarz and Plancherel, we have

$$\sum_{\tau_{t-1}} \int \Big|\sum_{\theta \prec \tau_{t-1}} |f_\theta|^2 * \chi^\vee_{(R_{t-1}/R, R_t/R]}\Big|^2 \le \mathbf{p}^3 \sum_{\tau_t} \int \Big|\sum_{\theta \prec \tau_t} |f_\theta|^2 * \chi^\vee_{\le R_t/R}\Big|^2 .$$

Summing the above bounds, we achieve the desired result.

□

We conclude this subsection by recording the Córdoba-Fefferman argument over $\mathbb{K}$, which is here represented by two results: Theorem 2.4 (the local form) and Prop. 2.5 (the global form).

Theorem 2.4 (Local bilinear restriction)**.** *Suppose* $|2| = \epsilon \in (0,1]$ *in* $\mathbb{K}$*. Let* $S \le D \le \epsilon\Gamma \le \epsilon$*, and* $R > 0$*. Let* $J \in \mathcal{P}(\mathcal{O}, \Gamma)$ *and* $L, L' \in \mathcal{P}(J, \epsilon^{-1}D)$ *be distinct,* $I, I' \in \mathcal{P}(L, D), \mathcal{P}(L', D)$*, respectively, and write* $\tau = \tau_I, \tau' = \tau_{I'}$ *for the caps above* I, I' *on* $\mathbb{P}^1$*. Then we have the estimate*

$$\int_{B_R} |f_\tau|^2 |f_{\tau'}|^2 \le \max(1, (\Gamma/D)^2 (SR)^{-2}) \int_{B_R} \left(\sum_{K \in \mathcal{P}(J,S)} |f_{\tau_K}|^2 \right)^2 .$$

Proof. We may assume that $J = B(0, \Gamma)$. We will use θ, ϑ for various τ_K with $K \in \mathcal{P}(J, S)$. Computing directly,

$$\int_{B_R} |f_\tau|^2 |f_{\tau'}|^2 = \sum_{\substack{\theta_1, \theta_2 \prec \tau \\ \vartheta_1, \vartheta_2 \prec \tau'}} \int_{B_R} f_{\theta_1} \overline{f}_{\theta_2} f_{\vartheta_1} \overline{f}_{\vartheta_2},$$

where a given tuple $(\theta_1, \theta_2, \vartheta_1, \vartheta_2)$ produces a nontrivial summand only if $(\theta_1 - \theta_2 + \vartheta_1 - \vartheta_2) \cap B_{R^{-1}} \neq \emptyset$. Suppose this is the case. Let $\xi_1, \xi_2, \eta_1, \eta_2$ be such that

$$(\xi_1, \xi_1^2) \in \theta_1, (\xi_2, \xi_2^2) \in \theta_2, \quad (\eta_1, \eta_1^2) \in \vartheta_1, (\eta_2, \eta_2^2) \in \vartheta_2.$$

Note that

$$|(\xi_1 + \xi_2) - (\eta_1 + \eta_2)| > D;$$

indeed, the two bracketed expressions belong to distinct members of $\mathcal{P}(\mathcal{O}, |2|\epsilon^{-1}D)$. By the assumption on the caps, $|(\xi_1 - \xi_2) + (\eta_1 - \eta_2)| \le R^{-1}$. Thus,

$$\xi_1^2 - \xi_2^2 + \eta_1^2 - \eta_2^2 = (\xi_1 - \xi_2)((\xi_1 + \xi_2) - (\eta_1 + \eta_2)) + \mathcal{O}(\Gamma R^{-1}).$$

The first summand has size $> D|\xi_1 - \xi_2|$, so the vanishing condition may hold only if $|\xi_1 - \xi_2| < D^{-1}\Gamma R^{-1}$. Symmetrically, it is necessary for $|\eta_1 - \eta_2| < D^{-1}\Gamma R^{-1}$.

If $D^{-1}\Gamma R^{-1} \le S$, we are done. Assume now $D^{-1}\Gamma R^{-1} \ge S$. For each $\theta_1 \prec \tau$ and $\vartheta_1 \prec \tau'$ we may find $\le (D^{-1}\Gamma S^{-1} R^{-1})^2$ pairs $\theta_2 \prec \tau, \vartheta_2 \prec \tau'$ for which the corresponding integral is nonzero. Thus, by the Schur test,

$$\int_{B_R} |f_\tau|^2 |f_{\tau'}|^2 \le (\Gamma/D)^2 (SR)^{-2} \int_{B_R} \sum_{\theta \prec \tau, \vartheta \prec \tau'} |f_\theta|^2 |f_\vartheta|^2.$$

The result follows immediately.

□

Similarly, we may prove:

Proposition 2.5 (Córdoba-Fefferman square function estimate)**.** Suppose $\operatorname{char}(\mathbb{K}) \neq 2$. If f has Fourier support in $\mathcal{N}_\delta(\mathbb{P}^1) \subseteq \mathbb{Q}_p^2$, then

$$\int |f|^4 \le 2 \int \left(\sum_\theta |f_\theta|^2 \right)^2 ,$$

where the caps θ have size $\delta \times \delta^2$.

Proof. We expand out

$$\int |f|^4 = \sum_{\theta_1,\theta_2,\vartheta_1,\vartheta_2} \int f_{\theta_1}\overline{f}_{\theta_2} f_{\vartheta_1}\overline{f}_{\vartheta_2}.$$

By essentially the same analysis above, each tuple corresponds to a nontrivial integral only if $\theta_1 = \theta_2, \vartheta_1 = \vartheta_2$ or $\theta_1 = \vartheta_2, \theta_2 = \vartheta_1$. Thus,

$$\sum_{\theta_1,\theta_2,\vartheta_1,\vartheta_2} \int f_{\theta_1}\overline{f}_{\theta_2} f_{\vartheta_1}\overline{f}_{\vartheta_2} \le 2\sum_{\theta,\vartheta} \int |f_\theta|^2|f_\vartheta|^2.$$

The result follows immediately. □

2.2 Bounding the broad sets

In this section, we fix a single function f satisfying the hypotheses of Theorem 1.2. We apply the results of the previous subsection to pruned functions arising from f.

We will first need to define the decomposition. For each $I \in \mathcal{P}(\mathcal{O}, R^{-1/2})$ and associated θ, write

$$\mathcal{G}_\theta = \left\{ U \in \mathcal{U}_\theta : \fint_U |f_\theta|^2 \ge \frac{\alpha^2}{2e^2(\#\theta)^2} \right\},$$

and

$$f_{N,\theta} = f_\theta \sum_{U\in\mathcal{G}_\theta} 1_U, \quad f_N = \sum_\theta f_{N,\theta}.$$

Inductively, set

$$\mathcal{G}_{\tau_k} := \left\{ U \in \mathcal{U}_{\tau_k} : \fint_U \sum_{\theta\prec\tau_k} |f_{k+1,\theta}|^2 \ge \frac{\alpha^2}{2e^2(\#\tau_k)^2} \right\},$$

and

$$f_{k,\theta} := f_{k+1,\theta} \sum_{U\in\mathcal{G}_{\tau_k}} 1_U, \quad f_k = \sum_\theta f_{k+1,\theta}.$$

We will also write

$$f^{\mathcal{B}}_{k,\theta} = f_{k,\theta} - f_{k-1,\theta}, \quad f^{\mathcal{B}}_k = \sum_\theta f^{\mathcal{B}}_{k,\theta}$$

We will write $g_{k,\tau} = \sum_{\theta\prec\tau} |f_{k,\theta}|^2$, $g^{\mathcal{B}}_{k,\tau} = \sum_{\theta\prec\tau} |f^{\mathcal{B}}_{k,\theta}|^2$.

Remark 2.6. Each $f_{k,\theta}, f^{\mathcal{B}}_{k,\theta}$ is Fourier-supported in θ.

An immediate consequence of the pruning is the following lemma, which demonstrates that the terms on the right-hand side of Prop. 2.3 are appropriate for Theorem 1.2 when applied to the $f^{\mathcal{B}}_m$.

Lemma 2.7 (Wave envelope expansion)**.** *(a) For each k and τ_k, we have*

$$\int \Big| \sum_{\theta\prec\tau_k} |f^{\mathcal{B}}_{m,\theta}|^2 * \chi^\vee_{\le R_k/R} \Big|^2 \le \int \Big| \mathcal{A}_{\mathcal{U}_{\tau_k}}[g^{\mathcal{B}}_{m,\tau_k}] \Big|^2.$$

(b) If $k \ge m$, then

$$\int \Big| \mathcal{A}_{\mathcal{U}_{\tau_k}}[g^{\mathcal{B}}_{m,\tau_k}] \Big|^2 \le \sum_{U\in\mathcal{G}_{\tau_k}} \mu(U) \left(\fint_U \sum_{\theta\prec\tau_k} |f_\theta|^2 \right)^2.$$

Proof. (a): The Fourier support of $\sum_{\theta\prec\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2*\chi^\vee_{\le R_k/R}$ is contained in the set

$$\bigcup_{\theta\prec\tau_k}(\theta-\theta)\cap B_{R_k/R}\subseteq U^*_{\tau_k,R},$$

where $U^*_{\tau_k,R}$ is a rectangle of dimensions $R_k/R\times R^{-1}$ with long edge parallel to $\mathbf{t}_{c_{\tau_k}}$. Thus

$$\begin{aligned}\int\Big|\sum_{\theta\prec\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2*\chi^\vee_{\le R_k/R}\Big|^2&=\int\Big|\sum_{\theta\prec\tau_k}\widehat{|f^{\mathcal{B}}_{m,\theta}|^2}\chi_{\le R_k/R}\Big|^2\\&\le\int\Big|\sum_{\theta\prec\tau_k}\widehat{|f^{\mathcal{B}}_{m,\theta}|^2}1_{U^*_{\tau_k,R}}\Big|^2\\&=\int\Big|\mathcal{A}_{\mathcal{U}_{\tau_k}}\Big[\sum_{\theta\prec\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2\Big]\Big|^2\end{aligned}$$

as claimed.

(b): Since $k\ge m$, $|f^{\mathcal{B}}_{m,\theta}|\le|f_{k,\theta}|\le|f_{k+1,\theta}|\le|f_\theta|$ by the pruning, so

$$\int\Big|\mathcal{A}_{\mathcal{U}_{\tau_k}}\Big[\sum_{\theta\prec\tau_k}|f^{\mathcal{B}}_{m,\theta}|^2\Big]|\Big|^2\le\int\Big|\mathcal{A}_{\mathcal{U}_{\tau_k}}[g_{k,\tau_k}]\Big|^2.$$

By the definition of the pruning, g_{k,τ_k} is supported on the union over $\mathcal{G}_{\tau_k}$, so

$$\int\Big|\mathcal{A}_{\mathcal{U}_{\tau_k}}[g_{k,\tau_k}]\Big|^2=\sum_{U\in\mathcal{G}_{\tau_k}}\int_U\Big|\mathcal{A}_{\mathcal{U}_{\tau_k}}[g_{k,\tau_k}]\Big|^2.$$

On the other hand, $\mathcal{A}_{\mathcal{U}_{\tau_k}}[g_{k,\tau_k}]$ is constant on each such box, so that

$$\sum_{U\in\mathcal{G}_{\tau_k}}\int_U\Big|\mathcal{A}_{\mathcal{U}_{\tau_k}}[g_{k,\tau_k}]\Big|^2=\sum_{U\in\mathcal{G}_{\tau_k}}\mu(U)\left(⨍_U\sum_{\theta\prec\tau_k}|f_{k,\theta}|^2\right)^2.$$

□

We now initiate a decomposition of the left-hand side of Theorem 1.2, which has the effect of isolating the contributions of the various $f^{\mathcal{B}}_m$. This portion of the argument follows closely the approach of [16], Section 3. Recall that U_α is defined as the set

$$U_\alpha=\Big\{x\in B_R:|f(x)|>\alpha\Big\}.$$

We consider also the auxiliary set

$$V_\alpha=\Big\{x\in B_R:|f_N(x)|>(1-\frac{1}{2^{1/2}eN})\alpha\Big\}.$$

We note the relation

$$U_\alpha\subseteq V_\alpha;\tag{2.1}$$

indeed, for any x, from the difference

$$|f(x)-f_N(x)|\le\sum_\theta\sum_{U\notin\mathcal{G}_\theta}1_U(x)|f_\theta(x)|,$$

together with the fact that each $|f_\theta|$ is constant on the sets U, we see that for any $U \notin \mathcal{G}_\theta$ and $x \in U$, we have the bound

$$|f_\theta(x)| = \left(\fint_U |f_\theta|^2 \right)^{1/2} \leq \frac{\alpha}{2^{1/2} e N (\#\theta)}.$$

Summing over the θ, we obtain (2.1).

By the definition of the prunings,

$$\begin{aligned} V_\alpha \subseteq \left\{ x \in V_\alpha : |f_0(x)| \geq (N^5+1)^{-1} |f_N(x)| \right\} \cup \bigcup_{m=1}^N \left\{ x \in V_\alpha : |f_m^{\mathcal{B}}|(x) \geq N^{-1}(1+N^{-5})^{-1} |f_N(x)| \right\} \\ =: U_\alpha^0 \cup \bigcup_{m=1}^N U_\alpha^m. \end{aligned} \tag{2.2}$$

We may immediately verify that the set U_α^0 is acceptable from the point of view of Theorem 1.2.

Proposition 2.8 (Case $m = 0$)**.**

$$\alpha^4 \mu(U_\alpha^0) \leq 4N^{10} \sum_{U \in \mathcal{G}_{\tau_0}} \mu(U) \left(\fint_U \sum_\theta |f_\theta|^2 \right)^2.$$

Proof. Take any $x \in U_\alpha^0$. Then $\mathcal{G}_{\tau_0} \neq \emptyset$, in particular is composed of the set B_R. Hence we have

$$|f_0(x)| = |\sum_\theta 1_{B_R}(x) f_{1,\theta}(x)|.$$

Thus, we have an inequality

$$\int_{U_\alpha^0} |f_0|^2 \leq \sum_\theta \int_{B_R} |f_\theta|^2.$$

Since $B_R \in \mathcal{G}_{\tau_0}$ we have

$$\fint_{B_R} \sum_\theta |f_\theta|^2 \geq \alpha^2.$$

Multiplying the two inequalities together,

$$\alpha^2 \int_{U_\alpha^0} |f_0|^2 \leq \mu(U) \left(\fint_{B_R} \sum_\theta |f_\theta|^2 \right)^2.$$

Finally, on U_α^0 we have $|f_0(x)| \geq (1 - \frac{1}{2^{1/2} e N}) \frac{\alpha}{N^5+1}$; the result follows. □

In order to bound the contributions of the U_α^m, we will need to assert an extra "broad" hypothesis. It will happen that this entails a high-frequency dominance property, which will allow us to apply Prop. 2.3. The basic result is in the following lemma.

Lemma 2.9 (Weak high-domination of bad parts)**.** *Let $1 \leq m \leq \ell \leq N$ and $0 \leq k < m$. Let τ_k be arbitrary.*

(a) For each x, we have

$$\left| \sum_{\tau_\ell \prec \tau_k} |f_{m,\tau_\ell}^{\mathcal{B}}|^2 * \chi_{\leq R_\ell / R}^\vee(x) \right| \leq \frac{\alpha^2 (\#\tau_\ell \prec \tau_k)}{2 e^2 N^2 (\#\tau_\ell)^2}.$$

(b) If $\frac{C\alpha}{2^{1/2}eN} \leq |f^{\mathcal{B}}_{m,\tau_k}(x)|$*, then*

$$\sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2(x) \leq \frac{C^2}{C^2-1} \left| \sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2 * \chi^{\vee}_{>R_\ell/R}(x) \right|.$$

Proof. (a): By the low lemma,

$$\sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2 * \chi^{\vee}_{\leq R_\ell/R} = \sum_{\tau_\ell \prec \tau_k} \sum_{\theta \prec \tau_\ell} |f^{\mathcal{B}}_{m,\theta}|^2 * \chi^{\vee}_{\leq R_\ell/R}.$$

If $x \in U \in \mathcal{U}_{\tau_\ell} \setminus \mathcal{G}_{\tau_\ell}$, then a straightforward calculation supplies

$$\sum_{\theta \prec \tau_\ell} |f^{\mathcal{B}}_{m,\theta}|^2 * \chi^{\vee}_{\leq R_\ell/R}(x) = \fint_U \sum_{\theta \prec \tau_\ell} |f^{\mathcal{B}}_{m,\theta}|^2 * \chi^{\vee}_{\leq R_\ell/R}.$$

By the pruning inequality,

$$\fint_U \sum_{\theta \prec \tau_\ell} |f^{\mathcal{B}}_{m,\theta}|^2 * \chi^{\vee}_{\leq R_\ell/R} \leq \fint_U \sum_{\theta \prec \tau_\ell} |f^{\mathcal{B}}_{\ell,\theta}|^2 * \chi^{\vee}_{\leq R_\ell/R}.$$

By the definition of $\mathcal{G}_{\tau_\ell}$,

$$\fint_U \sum_{\theta \prec \tau_\ell} |f^{\mathcal{B}}_{\ell,\theta}|^2 * \chi^{\vee}_{\leq R_\ell/R} \leq \frac{\alpha^2}{2e^2(\#\tau_\ell)^2}.$$

Thus, summing over $\tau_\ell \prec \tau_k$, we arrive at the estimate

$$\sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2 * \chi^{\vee}_{\leq R_\ell/R} \leq \frac{\alpha^2(\#\tau_\ell \prec \tau_k)}{2e^2N^2(\#\tau_\ell)^2}.$$

(b): The assumption and Cauchy-Schwarz give

$$\frac{C^2\alpha^2}{2e^2N^2} \leq (\#\tau_\ell \prec \tau_k) \sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2.$$

Assuming to the contrary that

$$\sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2(x) < C^2 \left| \sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2 * \chi^{\vee}_{\leq R_\ell/R}(x) \right|,$$

we conclude from (a) that

$$\frac{C^2\alpha^2}{2e^2N^2} < \frac{C^2\alpha^2(\#\tau_\ell \prec \tau_k)^2}{2e^2(\#\tau_\ell)^2},$$

a contradiction. Thus we must have

$$\sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2(x) \leq \frac{C^2}{C^2-1} \left| \sum_{\tau_\ell \prec \tau_k} |f^{\mathcal{B}}_{m,\tau_\ell}|^2 * \chi^{\vee}_{>R_\ell/R}(x) \right|.$$

$\square$

Finally, we use the above estimates to control the integrals of the $f_m^{\mathcal{B}}$ on broad sets by high-frequency integrals of the square functions. Define the mth $(1 \leq m \leq N)$ broad sets in U_α to be as follows. Fix any $\tau_k, \tau_k' \prec \tau_{k-1}$ distinct, and write $\ell = \max(m-1, k)$. We define

$$\mathrm{Br}_\alpha^m(\tau_k, \tau_k') = \Big\{ x \in B_R : (1 - 2^{-1/2} e^{-1} N^{-1}) e^{-1} N^{-1} \alpha \leq |f_{m,\tau_{k-1}}^{\mathcal{B}}(x)| \leq \mathbf{p} N |f_{m,\tau_k}^{\mathcal{B}}(x) f_{m,\tau_k'}^{\mathcal{B}}(x)|^{1/2} \Big\}. \tag{2.3}$$

Proposition 2.10 (High domination of broad integrals)**.** Let $1 \leq k \leq m \leq N$. Suppose $\tau_k, \tau_k' \prec \tau_{k-1}$ are distinct. Write $\ell = \max(m-1, k)$. Then we have

$$\int_{\mathrm{Br}_\alpha^m(\tau_k, \tau_k')} |f_{m,\tau_k}^{\mathcal{B}} f_{m,\tau_k'}^{\mathcal{B}}|^2 \leq 4 \mathbf{p}^2 \int_{\mathbb{Q}_p^2} \Big| \sum_{\tau_\ell \prec \tau_{k-1}} |f_{m,\tau_\ell}^{\mathcal{B}}|^2 * \chi_{>R_\ell/R}^\vee \Big|^2.$$

Proof. By bilinear restriction, using the fact that $\mathbf{p} \geq |2|^{-1}$, in either case $k \leq m-1$ or $k \geq m$, we achieve the bound

$$\int_{\mathrm{Br}_\alpha^m(\tau_k, \tau_k')} |f_{m,\tau_k}^{\mathcal{B}} f_{m,\tau_k'}^{\mathcal{B}}|^2 \leq \mathbf{p}^2 \int_{\mathrm{Br}_\alpha^m(\tau_k, \tau_k') + B(0, R_\ell)} \Big| \sum_{\tau_\ell \prec \tau_{k-1}} |f_{m,\tau_\ell}^{\mathcal{B}}|^2 \Big|^2.$$

For each $x \in \mathrm{Br}_\alpha^m(\tau_k, \tau_k')$, we have that $|f_{m,\tau_{k-1}}^{\mathcal{B}}(x)| \geq 2^{1/2} e^{-1} N^{-1} \alpha$. Thus, by the weak high-domination lemma 2.9,

$$\sum_{\tau_\ell \prec \tau_{k-1}} |f_{m,\tau_\ell}^{\mathcal{B}}|^2(x) \leq 2 \Big| \sum_{\tau_\ell \prec \tau_{k-1}} |f_{m,\tau_\ell}^{\mathcal{B}}|^2 * \chi_{>R_\ell/R}^\vee(x) \Big|.$$

On the other hand, both sides are constant at scale R_ℓ, so

$$\begin{aligned} &\int_{\mathrm{Br}_\alpha^m(\tau_k, \tau_k') + B(0, R_\ell)} \Big| \sum_{\tau_\ell \prec \tau_{k-1}} |f_{m,\tau_\ell}^{\mathcal{B}}|^2 \Big|^2 \\ &\qquad \leq 4 \int_{\mathrm{Br}_\alpha^m(\tau_k, \tau_k') + B(0, R_\ell)} \Big| \sum_{\tau_\ell \prec \tau_{k-1}} |f_{m,\tau_\ell}^{\mathcal{B}}|^2 * \chi_{>R_\ell/R}^\vee \Big|^2. \end{aligned}$$

□

Remark 2.11. In the above proposition, we see the appearance of the particular constant used in the choice of pruning. It follows that, if the assumptions on the broad sets are weakened, one may correspondingly strengthen the assumption on the good envelopes $\mathcal{G}_{\tau_k}$, and improve the strengths of Theorems 1.2 and 1.1.

2.3 Broad/narrow analysis

Combining Prop.'s 2.8 and 2.10, we produced the desired bounds on the subset of the superlevel set for which f is sufficiently broad at some scale. In this subsection, we perform a broad/narrow analysis to produced the desired wave envelope estimate in each cube of sidelength R.

More precisely, this section is dedicated to establishing the following proposition.

Proposition 2.12 (Local wave envelope estimate)**.** For each cube B_R of sidelength R and each $\alpha > 0$,

$$\alpha^4 \mu\Big(\{ x \in B_R : |f(x)| > \alpha \} \Big) \leq 12 (1 + N^{-5})^{10} \mathbf{p}^{12} N^{10} \sum_{0 \leq s \leq N} \sum_{\tau_s} \sum_{U \in \mathcal{G}_{\tau_s}} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_s} |f_\theta|^2 \right)^2.$$

Lemma 2.13 (Narrow lemma)**.** *Suppose $1 \le k \le N$ and τ_{k-1} is arbitrary. Then, for each x, either*

$$|f^{\mathcal{B}}_{m,\tau_{k-1}}(x)| \le \mathbf{p}N \max_{\substack{\tau_k \ne \tau_k' \\ \tau_k, \tau_k' \prec \tau_{k-1}}} |f^{\mathcal{B}}_{m,\tau_k}(x) f^{\mathcal{B}}_{m,\tau_k'}(x)|^{1/2} \tag{2.4}$$

or

$$|f^{\mathcal{B}}_{m,\tau_{k-1}}(x)| \le \left(1 + \frac{1}{N-1}\right) \max_{\tau_k \prec \tau_{k-1}} |f^{\mathcal{B}}_{m,\tau_k}(x)|. \tag{2.5}$$

Proof. Fix $\tau_k' \prec \tau_{k-1}$ which realizes the maximum

$$|f^{\mathcal{B}}_{m,\tau_k'}(x)| = \max_{\tau_k \prec \tau_{k-1}} |f^{\mathcal{B}}_{m,\tau_k}(x)|.$$

Suppose (2.5) fails. Then, since $f^{\mathcal{B}}_{m,\tau_{k-1}}(x) = \sum_{\tau_k \prec \tau_{k-1}} f^{\mathcal{B}}_{m,\tau_k}(x)$, we have the inequality

$$|f^{\mathcal{B}}_{m,\tau_{k-1}}(x) - \sum_{\tau_k \ne \tau_k'} f^{\mathcal{B}}_{m,\tau_k}(x)| < \left(1 + \frac{1}{N-1}\right)^{-1} |f^{\mathcal{B}}_{m,\tau_{k-1}}(x)|.$$

On the other hand,

$$|f^{\mathcal{B}}_{m,\tau_{k-1}}(x) - \sum_{\tau_k \ne \tau_k'} f^{\mathcal{B}}_{m,\tau_k}(x)| \ge |f^{\mathcal{B}}_{m,\tau_{k-1}}(x)| - (\#\tau_k \prec \tau_{k-1}) \max_{\tau_k \ne \tau_k'} |f^{\mathcal{B}}_{m,\tau_k}(x)|;$$

thus

$$(\#\tau_k \prec \tau_{k-1}) \max_{\tau_k \ne \tau_k'} |f^{\mathcal{B}}_{m,\tau_k}(x)| > \left(1 - \left(1 + \frac{1}{N-1}\right)^{-1}\right) |f^{\mathcal{B}}_{m,\tau_{k-1}}(x)|.$$

Relating the above to (2.4), for each $\tau_k \prec \tau_{k-1}$,

$$|f^{\mathcal{B}}_{m,\tau_k}(x)| \le |f^{\mathcal{B}}_{m,\tau_k}(x) f^{\mathcal{B}}_{m,\tau_k'}(x)|^{1/2},$$

and thus

$$|f^{\mathcal{B}}_{m,\tau_{k-1}}(x)| < (\#\tau_k) \left(1 - \left(1 + \frac{1}{N-1}\right)^{-1}\right)^{-1} \max_{\tau_k \ne \tau_k'} |f^{\mathcal{B}}_{m,\tau_k}(x) f^{\mathcal{B}}_{m,\tau_k'}(x)|^{1/2}.$$

The conclusion follows from the identities

$$\left(1 - \left(1 + \frac{1}{N-1}\right)^{-1}\right)^{-1} = N$$

and

$$(\#\tau_k \prec \tau_{k-1}) \le \mathbf{p}.$$

□

We wish to use this to divide the integral of $|f^{\mathcal{B}}_m|^4$ into broad and narrow parts, with a small constant on narrow parts. For the narrow component, we wish to relate $\int |f^{\mathcal{B}}_m|^4$ to $\sum_\tau \int |f^{\mathcal{B}}_{m,\tau}|^4$, so that we may further decompose each $f^{\mathcal{B}}_{m,\tau}$ into broad and narrow components and proceed inductively.

Definition 2.14. We define $\mathrm{Broad}_{1,m}$ to be the set

$$\mathrm{Broad}_{1,m} = \left\{ x \in U^m_\alpha : |f^{\mathcal{B}}_m(x)| \le \mathbf{p}N \max_{\tau_1 \ne \tau_1'} |f^{\mathcal{B}}_{m,\tau_1}(x) f^{\mathcal{B}}_{m,\tau_1'}(x)|^{1/2} \right\}.$$

The complementary set $\mathrm{Narrow}_{1,m}$ is defined as $U^m_\alpha \setminus \mathrm{Broad}_{1,m}$.

Lemma 2.15 (Decoupling the narrow part ($k=1$))**.** *It holds that*

$$\int_{\mathrm{Narrow}_{1,m}} |f_m^{\mathcal{B}}|^4 \leq \left(1+\frac{1}{N-1}\right)^4 \sum_{\tau_1} \int_{\mathrm{Narrow}_{1,m}} |f_{m,\tau_1}^{\mathcal{B}}|^4.$$

Proof. If $x \in \mathrm{Narrow}_{1,m}$, then by the narrow lemma 2.13

$$|f_m^{\mathcal{B}}(x)| \leq \left(1+\frac{1}{N-1}\right) |f_{m,\tau_1}^{\mathcal{B}}(x)|$$

for a suitable τ_1. Thus,

$$|f_m^{\mathcal{B}}(x)| \leq \left(1+\frac{1}{N-1}\right) \left(\sum_{\tau_1} |f_{m,\tau_1}^{\mathcal{B}}(x)|^4\right)^{1/4},$$

and hence

$$\int_{\mathrm{Narrow}_{1,m}} |f_m^{\mathcal{B}}|^4 \leq \left(1+\frac{1}{N-1}\right)^4 \sum_{\tau_1} \int_{\mathrm{Narrow}_{1,m}} |f_{m,\tau_1}^{\mathcal{B}}|^4.$$

□

Definition 2.16. Write, for each τ_1,

$$\mathrm{Broad}_{2,m}(\tau_1) := \left\{ x \in \mathrm{Narrow}_{1,m} : |f_{m,\tau_1}^{\mathcal{B}}(x)| \leq \mathbf{p}N \max_{\substack{\tau_2 \neq \tau_2' \\ \tau_2,\tau_2' \prec \tau_1}} |f_{m,\tau_2}^{\mathcal{B}}(x) f_{m,\tau_2'}^{\mathcal{B}}(x)|^{1/2} \right\}.$$

Write also $\mathrm{Narrow}_{2,m}(\tau_1) := \mathrm{Narrow}_{1,m} \setminus \mathrm{Broad}_{2,m}(\tau_1)$.

Definition 2.17. Let $2 \leq k < N$. Suppose $\tau_k \prec \tau_{k-1}$. We inductively write

$$\mathrm{Broad}_{k+1,m}(\tau_k) := \left\{ x \in \mathrm{Narrow}_{k,m}(\tau_{k-1}) : |f_{m,\tau_k}^{\mathcal{B}}(x)| \leq \mathbf{p}N \max_{\substack{\tau_{k+1} \neq \tau_{k+1}' \\ \tau_{k+1},\tau_{k+1}' \prec \tau_k}} |f_{m,\tau_{k+1}}^{\mathcal{B}}(x) f_{m,\tau_{k+1}'}^{\mathcal{B}}(x)|^{1/2} \right\}$$

and $\mathrm{Narrow}_{k+1,m}(\tau_k) := \mathrm{Narrow}_{k,m}(\tau_{k-1}) \setminus \mathrm{Broad}_{k+1,m}(\tau_k)$.

Lemma 2.18 (Decoupling the narrow part ($k \geq 2$))**.** *Fix any $2 \leq k \leq N$. Then, for each τ_{k-1},*

$$\int_{\mathrm{Narrow}_{k,m}(\tau_{k-1})} |f_{m,\tau_{k-1}}^{\mathcal{B}}|^4 \leq \left(1+\frac{1}{N-1}\right)^4 \sum_{\tau_k \prec \tau_{k-1}} \int_{\mathrm{Narrow}_{k,m}(\tau_{k-1})} |f_{m,\tau_k}^{\mathcal{B}}|^4.$$

Proof. The argument is identical to 2.15. □

Combining Lemmas 2.15 and 2.18, we conclude

$$\begin{aligned}\int_{U_\alpha^m} |f_m^{\mathcal{B}}|^4 \leq{}& \left(1+\frac{1}{N-1}\right)^{4N} \sum_{\tau_{N-1}} \int_{\mathrm{Narrow}_{N,m}(\tau_{N-1})} \sum_{\tau_N \prec \tau_{N-1}} |f_{m,\tau_N}^{\mathcal{B}}|^4 \\ &+ \sum_{k=1}^{N} \left(1+\frac{1}{N-1}\right)^{4(k-1)} \sum_{\tau_{k-1}} \int_{\mathrm{Broad}_{k,m}(\tau_{k-1})} |f_{m,\tau_{k-1}}^{\mathcal{B}}|^4.\end{aligned}$$

Our next steps are bounding each of the summands in turn.

Lemma 2.19 (Narrow bound)**.** *We have*

$$\sum_{\tau_{N-1}} \int_{B_R} \sum_{\theta \prec \tau_{N-1}} |f^{\mathcal{B}}_{m,\theta}|^4 \le \sum_{\theta} \sum_{U \in \mathcal{G}_\theta} \mu(U) \left(\fint_U |f_\theta|^2 \right)^2 .$$

Proof. By the pruning inequalities, for each θ we may take $|f^{\mathcal{B}}_{m,\theta}| \le |f_{N,\theta}|$. By the definition of the pruning, for each θ,

$$\int_{B_R} |f_{N,\theta}|^4 = \int_{B_R} \Big| \sum_{U \in \mathcal{G}_\theta} 1_U f_\theta \Big|^4 \le \sum_{U \in \mathcal{G}_\theta} \int_U |f_\theta|^4.$$

Each $|f_\theta|$ is constant on the blocks U, so for each θ and $U \in \mathcal{G}_\theta$ we have

$$\int_U |f_\theta|^4 = \mu(U) \left(\fint_U |f_\theta|^2 \right)^2 .$$

□

Lemma 2.20 (Broad bound)**.** *We have, for each* $1 \le k \le N$*,*

$$\sum_{\tau_{k-1}} \int_{\mathrm{Broad}_{k,m}(\tau_{k-1})} |f^{\mathcal{B}}_{m,\tau_{k-1}}|^4 \le 12\mathbf{p}^{12} N^5 \sum_{m \le s \le N} \sum_{\tau_s} \sum_{U \in \mathcal{G}_{\tau_s}} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_s} |f_\theta|^2 \right)^2$$

Proof. By the definition of the broad set, for each τ_{k-1} and each $x \in \mathrm{Broad}_{k,m}(\tau_{k-1})$ there is some pair $\tau_k \ne \tau'_k \prec \tau_{k-1}$ such that $|f^{\mathcal{B}}_{m,\tau_{k-1}}(x)| \le \mathbf{p}N |f^{\mathcal{B}}_{m,\tau_k}(x) f^{\mathcal{B}}_{m,\tau'_k}(x)|^{1/2}$, as well as

$$N^{-1}(1 - (2^{1/2} e N)^{-1})(1 + (N-1)^{-1})^{-(k-1)} \alpha \le |f^{\mathcal{B}}_{m,\tau_{k-1}}(x)|,$$

i.e.

$$\mathrm{Broad}_{k,m}(\tau_{k-1}) \subseteq \bigcup_{\substack{\tau_k, \tau'_k \prec \tau_{k-1} \\ \tau_k \ne \tau'_k}} \mathrm{Br}^m_\alpha(\tau_k, \tau'_k).$$

Thus,

$$\sum_{\tau_{k-1}} \int_{\mathrm{Broad}_{k,m}(\tau_{k-1})} |f^{\mathcal{B}}_{m,\tau_{k-1}}|^4 \le \mathbf{p}^4 N^4 \sum_{\tau_{k-1}} \sum_{\substack{\tau_k, \tau'_k \prec \tau_{k-1} \\ \tau_k \ne \tau'_k}} \int_{\mathrm{Br}^m_\alpha(\tau_k, \tau'_k)} |f^{\mathcal{B}}_{m,\tau_k} f^{\mathcal{B}}_{m,\tau'_k}|^2.$$

By Prop. 2.10, for each τ_{k-1} and each distinct $\tau_k, \tau'_k \prec \tau_{k-1}$, then

$$\int_{\mathrm{Br}^m_\alpha(\tau_k, \tau'_k)} |f^{\mathcal{B}}_{m,\tau_k} f^{\mathcal{B}}_{m,\tau'_k}|^2 \le 4\mathbf{p}^2 \int_{\mathbb{Q}_p^2} \left| \sum_{\tau_\ell \prec \tau_{k-1}} |f^{\mathcal{B}}_{m,\tau_\ell}|^2 * \chi^\vee_{>R_\ell/R} \right|^2 ,$$

where $\ell = \max(m-1, k)$. By Prop. 2.3,

$$\int_{\mathbb{Q}_p^2} \left| \sum_{\tau_\ell \prec \tau_{k-1}} |f^{\mathcal{B}}_{m,\tau_\ell}|^2 * \chi^\vee_{>R_\ell/R} \right|^2 \le 3\mathbf{p}^4 N \sum_{m \le s \le N} \sum_{\tau_s \prec \tau_{k-1}} \sum_{U \in \mathcal{G}_{\tau_s}} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_s} |f_\theta|^2 \right)^2 .$$

Combining the above, we obtain

$$\sum_{\tau_{k-1}} \int_{\mathrm{Broad}_{k,m}(\tau_{k-1})} |f^{\mathcal{B}}_{m,\tau_{k-1}}|^4 \le 12\mathbf{p}^{12} N^5 \sum_{m \le s \le N} \sum_{\tau_s} \sum_{U \in \mathcal{G}_{\tau_s}} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_s} |f_\theta|^2 \right)^2 .$$

□

Proof of Prop. 2.12. By (2.1),

$$\alpha^4\mu(U_\alpha) \leq \alpha^4\mu(V_\alpha).$$

Write

$$\alpha^4\mu(V_\alpha) \leq \sum_{m=0}^{N} \alpha^4\mu(U_\alpha^m)$$

where the sets in the right-hand side are as defined in (2.2). By Prop. 2.8,

$$\alpha^4\mu(U_\alpha^0) \leq 4N^{10} \sum_{U\in\mathcal{G}_{\tau_0}} \mu(U) \left(\fint_U \sum_\theta |f_\theta|^2 \right)^2.$$

We now fix some $1 \leq m \leq N$. Over U_α^m, we have a lower bound on $f_m^{\mathcal{B}}$ implying

$$\alpha^4\mu(U_\alpha^m) \leq N^4(1+N^{-5})^{10}(1-(2^{1/2}eN)^{-1})^{-1} \int_{U_\alpha^m} |f_m^{\mathcal{B}}|^4.$$

By the definition of the broad/narrow sets above, we may bound

$$\begin{aligned} \int_{U_\alpha^m} |f_m^{\mathcal{B}}|^4 \leq \left(1+\frac{1}{N}\right)^{4N} \sum_{\tau_{N-1}} \int_{\mathrm{Narrow}_{N,m}(\tau_{N-1})} \sum_{\theta\prec\tau_{N-1}} |f_{\theta,m}^{\mathcal{B}}|^4 \\ + \sum_{k=1}^{N} \left(1+\frac{1}{N}\right)^{4(k-1)} \sum_{\tau_{k-1}} \int_{\mathrm{Broad}_{k,m}(\tau_{k-1})} |f_{\tau_{k-1},m}^{\mathcal{B}}|^4. \end{aligned}$$

By the narrow bound 2.19,

$$\sum_{\tau_{N-1}} \int_{B_R} \sum_{\theta\prec\tau_{N-1}} |f_{m,\theta}^{\mathcal{B}}|^4 \leq \sum_\theta \sum_{U\in\mathcal{G}_\theta} \mu(U) \left(\fint_U |f_\theta|^2 \right)^2.$$

By the broad bound 2.20,

$$\sum_{\tau_{k-1}} \int_{\mathrm{Broad}_{k,m}(\tau_{k-1})} |f_{m,\tau_{k-1}}^{\mathcal{B}}|^4 \leq 12\mathbf{p}^{12}N^5 \sum_{m\leq s\leq N} \sum_{\tau_s} \sum_{U\in\mathcal{G}_\tau} \mu(U) \left(\fint_U \sum_{\theta\prec\tau_s} |f_\theta|^2 \right)^2.$$

Thus, for each $1 \leq m \leq N$,

$$\alpha^4\mu(U_\alpha^m) \leq 12(1+N^{-5})^{10}(2^{1/2}eN)^{-1})^{-1}\mathbf{p}^{12}N^9 \sum_{m\leq s\leq N} \sum_{\tau_s} \sum_{U\in\mathcal{G}_\tau} \mu(U) \left(\fint_U \sum_{\theta\prec\tau_s} |f_\theta|^2 \right)^2,$$

and hence

$$\alpha^4\mu(U_\alpha) \leq 13(1+N^{-5})^{10}(1-(2^{1/2}eN)^{-1})^{-1}\mathbf{p}^{12}N^{10} \sum_{0\leq s\leq N} \sum_{\tau_s} \sum_{U\in\mathcal{G}_{\tau_s}} \mu(U) \left(\fint_U \sum_{\theta\prec\tau_s} |f_\theta|^2 \right)^2, \quad (2.6)$$

as claimed.

□

2.4 Reduction to local estimates

In the above subsections we produced bounds on the measure of the set $U_\alpha = \{x \in B_R : |f(x)| > \alpha\}$. In this subsection we note that, if we can prove Theorem 1.2 in the special case that $\{x \in \mathbb{R}^2 : |f(x)| > \alpha\} \subseteq Q_R$ for a suitable cube Q_R of radius R, then we can conclude that Theorem 1.2 is true in the general case.

Proof that Prop. 2.12 implies Theorem 1.2. Let f be as in the hypothesis of Theorem 1.2. Then, for each metric ball $Q_R \subseteq \mathbb{Q}_p^2$ of radius R, the function $1_{Q_R} f$ satisfies the hypothesis of Prop. 2.12. Thus, we have

$$\mu(\{x \in \mathbb{Q}_p^2 : |1_{Q_R} f| > \alpha\}) \leq 13 \frac{(1+N^{-5/2})^{10}}{1-(2^{1/2}eN)^{-1}} N^{10} \mathbf{p}^{12} \sum_{R^{-1/2} \leq s \leq 1} \sum_{\tau_s} \sum_{U \in \mathcal{G}_{\tau_s}} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_s} |(1_{Q_R} f)_\theta|^2 \right)^2 .$$

Summing over the Q_R, we obtain that

$$\mu(\{x \in \mathbb{Q}_p^2 : |f(x)| > \alpha\}) \leq 13 \frac{(1+N^{-5/2})^{10}}{1-(2^{1/2}eN)^{-1}} N^{10} \mathbf{p}^{12} \sum_{R^{-1/2} \leq s \leq 1} \sum_{\tau_s} \sum_{U \in \mathcal{G}_{\tau_s}} \mu(U) \sum_{Q_R} \left(\fint_U \sum_{\theta \prec \tau_s} |(1_{Q_R} f)_\theta|^2 \right)^2 .$$

By Minkowski, for each τ_s and $U \in \mathcal{G}_{\tau_s}$,

$$\sum_{Q_R} \left(\fint_U \sum_{\theta \prec \tau_s} |(1_{Q_R} f)_\theta|^2 \right)^2 \leq \left(\fint_U \sum_{\theta \prec \tau_s} \left(\sum_{Q_R} |(1_{Q_R} f)_\theta|^2 \right)^2 \right)^2 .$$

By elementary properties of the ultrametric, for each Q_R and each θ we have

$$(1_{Q_R} f)_\theta = 1_{Q_R} f_\theta,$$

so that in fact

$$\left(\fint_U \sum_{\theta \prec \tau_s} \left(\sum_{Q_R} |(1_{Q_R} f)_\theta|^2 \right)^2 \right)^2 = \left(\fint_U \sum_{\theta \prec \tau_s} |f_\theta|^2 \right)^2 .$$

The result follows immediately. $\square$

3 Proof of Theorem 1.1

Theorem 1.1 will be proved via first establishing the critical case $(p,q) = (2+\frac{2}{\beta}, \frac{2+2\beta^{-1}}{2\beta^{-1}-1})$, and then interpolating with the easier endpoints $(\infty, 1)$ and $(3, \infty)$, together with Hölder inequalities. We will repeatedly cite the result of Theorem 1.2, and we will abbreviate the constants as

$$\alpha^4 |\{x : |f(x)| > \alpha\}| \leq C N^{E_2} \sum_{\substack{k \in \mathbf{p}^{\mathbb{Z}} \\ R^{-1/2} \leq s \leq 1}} \sum_{\tau_k} \sum_{U \in \mathcal{G}_{\tau_k}} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_k} |f_\theta|^2 \right)^2 .$$

We will in fact make use of the more refined quantities in (2.6).

We begin with the partial decoupling statement, using the right-hand side of Theorem 1.2.

Proposition 3.1. Suppose $p \geq 4$, $\lambda > 0$, and $\mathfrak{C}_p > 1$. Let $0 \leq k \leq N$ be arbitrary, and fix a canonical scale cap τ_k. Suppose as before that $\Gamma_\beta(R^{-1})$ is a partition of $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$ into approximate $R^{-\beta} \times R^{-1}$ boxes γ. Assume $f = \sum_\gamma f_\gamma$ satisfies the following regularity properties.

(a) $\|f_\gamma\|_\infty \leq \lambda$ for each γ.

(b) $\|f_\gamma\|_2^2 \leq \lambda^{2-p}\mathfrak{C}_p\|f_\gamma\|_p^p$ for each γ and each $p \geq 1$.

Write γ_k for boxes of dimensions $\max(R^{-\beta}, R_k/R) \times R^{-1}$. Then

$$\begin{aligned}\sum_{U\in\mathcal{G}_{\tau_k}} \mu(U)\left(\fint_U \sum_{\theta\prec\tau_k}|f_\theta|^2\right)^2 &\leq \mathfrak{C}_p\big[2^{\frac{1}{2}}eN(\#\tau_k)\alpha^{-1}\big]^{p-4} \\ &\quad\times\left(\max_{\gamma_k\prec\tau_k}\#(\gamma\prec\gamma_k)\times\#(\gamma\prec\tau_k)\right)^{\frac{p}{2}-1}\sum_{\gamma\prec\tau_k}\|f_\gamma\|_p^p.\end{aligned} \tag{3.1}$$

Proof. For each $\theta \prec \tau_k$, the small caps $\gamma_k \prec \theta$ are $\max(R^{-\beta}, R_k/R) \geq R_k/R$-separated. Fix any $U \in \mathcal{G}_{\tau_k}$. Since $U\|U_{\tau_k,R}$ has dimensions $R/R_k \times R$, we conclude that the f_{γ_k} are locally orthogonal on U. Thus

$$\int_U \sum_{\theta\prec\tau_k}|f_\theta|^2 = \int_U \sum_{\gamma_k\prec\tau_k}|f_{\gamma_k}|^2,$$

and so, appealing to the definition of $\mathcal{G}_{\tau_k}$,

$$\frac{\alpha^2}{2e^2N^2(\#\tau_k)^2} \leq \fint_U \sum_{\gamma_k\prec\tau_k}|f_{\gamma_k}|^2.$$

Multiplying the left-hand side of (3.1) by the $(\frac{p}{2}-2)$-power of the latter display, we obtain the estimate

$$\sum_{U\in\mathcal{G}_{\tau_k}} \mu(U)\left(\fint_U \sum_{\theta\prec\tau_k}|f_\theta|^2\right)^2 \leq \big[2^{1/2}eN(\#\tau_k)\alpha^{-1}\big]^{p-4}\sum_{U\in\mathcal{G}_{\tau_k}}\mu(U)\left(\fint_U\sum_{\gamma_k\prec\tau_k}|f_{\gamma_k}|^2\right)^{\frac{p}{2}}. \tag{3.2}$$

Uniformity assumption (a) implies

$$\Big\|\sum_{\gamma_k\prec\tau_k}|f_{\gamma_k}|^2\Big\|_\infty \leq \lambda^2\big[\max_{\gamma_k\prec\tau_k}\#(\gamma\prec\gamma_k)\big]\times\#(\gamma\prec\tau_k).$$

By removing factors of $\big\|\sum_{\gamma_k\prec\tau_k}|f_{\gamma_k}|^2\big\|_\infty$ from (3.2), we obtain

$$\sum_{U\in\mathcal{G}_{\tau_k}} \mu(U)\left(\fint_U \sum_{\theta\prec\tau_k}|f_\theta|^2\right)^2 \leq (\#\tau_k)^{p-4}\lambda^{p-2}\alpha^{4-p}\left(\max_{\gamma_k\prec\tau_k}\#(\gamma\prec\gamma_k)\times\#(\gamma\prec\tau_k)\right)^{\frac{p}{2}-1}\sum_{U\in\mathcal{G}_{\tau_k}}\int_U\sum_{\gamma_k\prec\tau_k}|f_{\gamma_k}|^2,$$

and by local orthogonality and uniformity assumption (b)

$$\sum_{U\in\mathcal{G}_{\tau_k}}\int_U\sum_{\gamma_k\prec\tau_k}|f_{\gamma_k}|^2 = \int\sum_{\gamma\prec\tau_k}|f_\gamma|^2 \leq \lambda^{2-p}\mathfrak{C}_p\sum_{\gamma\prec\tau_k}\|f_\gamma\|_p^p.$$

Together we get the estimate

$$\sum_{U\in\mathcal{G}_{\tau_k}}\mu(U)\left(\fint_U\sum_{\theta\prec\tau_k}|f_\theta|^2\right)^2\le\mathfrak{C}_p\big[2^{1/2}eN(\#\tau_k)\alpha^{-1}\big]^{p-4}$$
$$\times\left(\max_{\gamma_k\prec\tau_k}\#(\gamma\prec\gamma_k)\times\#(\gamma\prec\tau_k)\right)^{\frac{p}{2}-1}\sum_{\gamma\prec\tau_k}\|f_\gamma\|_p^p,$$

as claimed. □

The above amounts to a proof of Theorem 1.1, in the special case that f satisfies the assumed regularity properties. It remains to remove those assumptions, which we do now.

We find it convenient to separate out a proof of a superlevel set estimate for small cap decoupling at the critical exponent pair. The full result, Theorem 1.1, will follow by an elementary argument and interpolation.

Theorem 3.2 (Main estimate for critical exponents)**.** *Let f be Schwartz-Bruhat with Fourier support in $\mathcal{N}_{R^{-1}}(\mathbb{P}^1)$, such that $\max_\theta\|f_\theta\|_\infty=1$. Let $(p,q)=(2+\frac{2}{\beta},\frac{2+2\beta^{-1}}{2\beta^{-1}-1})$. Then, for each $R^{-1/2}\le\alpha\le R^{1/2}$, we have the estimate*

$$\alpha^p\mu\big(\{x:|f(x)|>\alpha\}\big)|\lesssim\mathbf{p}^{12}N^{2p-2}(\log R)^{p+2}\max\big(R^{\beta(p-\frac{p}{q}-1)-1}R^{\beta(\frac{p}{2}-\frac{p}{q})}\big)\Big(\sum_\gamma\|f_\gamma\|_p^q\Big)^{\frac{p}{q}}.\tag{3.3}$$

Proof of Theorem 3.2. We may write

$$f=\sum_{N^{-2}R^{-1/2}<\lambda\le1}\ \sum_{\substack{\gamma\in\Gamma_\beta(R^{-1})\\ \|f_\gamma\|_\infty\in(e^{-1}\lambda,\lambda]}}f_\gamma+N^{-2}R^{-1/2}\eta,$$

where the λ range over values of the form $\lambda_{k+1}=\lfloor e^{-1}\lambda_k\rfloor$, $\lambda_0=1$, and η is Schwartz-Bruhat, supported on B_R, and uniformly bounded by 1. We abbreviate

$$\Gamma_\beta^\lambda(R^{-1})=\big\{\gamma\in\Gamma_\beta(R^{-1}):\|f_\gamma\|_\infty\in(e^{-1}\lambda,\lambda]\big\}.$$

Then, for each λ, consider the wave envelope expansion

$$\sum_{\gamma\in\Gamma_\beta^\lambda(R^{-1})}f_\gamma=\sum_{\gamma\in\Gamma_\beta^\lambda(R^{-1})}\sum_U 1_Uf_\gamma,$$

where each U has dimensions $\sim R^\beta\times R$ and has long edge parallel to $\mathbf{n}_{c_\gamma}$. Since $\gamma\in\Gamma_\beta^\lambda(R^{-1})$, there is some U such that $\|1_Uf\|_\infty\in(e^{-1}\lambda,\lambda]$. If we write $\mathcal{U}_\lambda=\mathcal{U}_\lambda^\gamma$ for the set of U for which $\|1_Uf_\gamma\|_\infty\in(e^{-1}\lambda,\lambda]$, then for all $\gamma\in\Gamma_\beta^\lambda(R^{-1})$

$$e^{-p}(\#\mathcal{U}_\lambda)R^{1+\beta}\lambda^p\le\Big\|\sum_{U\in\mathcal{U}_\lambda}1_Uf_\gamma\Big\|_p^p\le(\#\mathcal{U}_\lambda)R^{1+\beta}\lambda^p,$$

and so

$$\Big\|\frac{1}{\lambda}\sum_{U\in\mathcal{U}_\lambda}1_Uf_\gamma\Big\|_2^2\le(\#\mathcal{U}_\lambda)R^{1+\beta}\le e^p\Big\|\frac{1}{\lambda}\sum_{U\in\mathcal{U}_\lambda}1_Uf_\gamma\Big\|_p^p\tag{3.4}$$

and

$$\Big\|\frac{1}{\lambda}\sum_{U\in\mathcal{U}_\lambda}1_Uf_\gamma\Big\|_p^p\le(\#\mathcal{U}_\lambda^\gamma)R^{1+\beta}\le e^2\Big\|\frac{1}{\lambda}\sum_{U\in\mathcal{U}_\lambda}1_Uf_\gamma\Big\|_2^2.\tag{3.5}$$

For each $1 \le \mathfrak{r} \le R$ and each λ, write $\Gamma_\beta^{\lambda;\mathfrak{r}}(R^{-1})$ to be the collection of $\gamma \in \Gamma_\beta^\lambda(R^{-1})$ such that $\#\mathcal{U}_\lambda^\gamma \in (e^{-1}\mathfrak{r}, \mathfrak{r}]$. Define for $\gamma \in \Gamma_\beta^\lambda(R^{-1})$

$$g_\gamma^{(\lambda)} = \frac{1}{\lambda} \sum_{U \in \mathcal{U}_\lambda} 1_U f_\gamma$$

and

$$g^{(\lambda,\mathfrak{r})} = \sum_{\gamma \in \Gamma_\beta^{\lambda;\mathfrak{r}}(R^{-1})} g_\gamma^{(\lambda)}.$$

We may observe that $g^{(\lambda,\mathfrak{r})}$ is a Schwartz-Bruhat function with Fourier support in the R^{-1}-neighborhood of the truncated parabola. Thus, for each $\lambda, \mathfrak{r}$, and $\mathfrak{a} > 0$ we have

$$\mathfrak{a}^4 \mu\big(\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}\big) \le C N^{E_2} \sum_{0 \le k \le N} \sum_{\tau_k} \sum_{U \in \mathcal{G}_{\tau_k}[\lambda g^{(\lambda,\mathfrak{r})};\mathfrak{a}]} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_k} |\lambda g_\theta^{(\lambda,\mathfrak{r})}|^2 \right)^2,$$

where we have written $\mathcal{G}_{\tau_k}[\lambda g^{(\lambda,\mathfrak{r})};\mathfrak{a}]$ to record that the pruning is with respect to the function $\lambda g^{(\lambda,\mathfrak{r})}$ with the amplitude parameter $\mathfrak{a}$. For each $0 \le k \le N$ and each $1 \le \mathfrak{s} \le R^{1/2}$, let $\mathcal{T}_k(\mathfrak{s})$ denote the collection of τ_k such that $\#\{\gamma \prec \tau_k : g_\gamma^{(\lambda,\mathfrak{r})} \ne 0\} \in (e^{-1}\mathfrak{s}, \mathfrak{s}]$. By pigeonholing, for each k we may find $\mathfrak{s}^k$ such that

$$\mathfrak{a}^4 \mu\big(\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}\big) \le 2^{-1} C N^{E_2} (\log R) \sum_{0 \le k \le N} \sum_{\tau_k \in \mathcal{T}_k(\mathfrak{s}^k)} \sum_{U \in \mathcal{G}_{\tau_k}[\lambda g^{(\lambda,\mathfrak{r})};\mathfrak{a}]} \mu(U) \left(\fint_U \sum_{\theta \prec \tau_k} |\lambda g_\theta^{(\lambda,\mathfrak{r})}|^2 \right)^2.$$

By Prop. 3.1 and (3.4), we have

$$\begin{aligned} \mathfrak{a}^p \mu\big(\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}\big) &\le C 2^{\frac{p}{2}-3} e^{2p-4} N^{p+E_2-4} (\log R) \sum_{0 \le k \le N} (\#\mathcal{T}_k(\mathfrak{s}^k))^{p-4} \\ &\quad \times \sum_{\tau_k \in \mathcal{T}_k(\mathfrak{s}^k)} \left(\mathfrak{s}^k \max_{\gamma_k \prec \tau_k} \#(\gamma \prec \gamma_k) \right)^{\frac{p}{2}-1} \sum_{\gamma \prec \tau_k} \|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^p, \end{aligned}$$

and by pigeonholing to a single $0 \le k_* \le N$ we have

$$\begin{aligned} &\mathfrak{a}^p \mu\big(\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}\big) \\ &\quad \le C 2^{\frac{p}{2}-3} e^{2p-4} (N+1) N^{p+E_2-4} (\log R) (\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-4} \sum_{\tau_{k_*} \in \mathcal{T}_{k_*}(\mathfrak{s}^{k_*})} \left(\mathfrak{s}^{k_*} \max_{\gamma_{k_*} \prec \tau_{k_*}} \#(\gamma \prec \gamma_{k_*}) \right)^{\frac{p}{2}-1} \sum_{\gamma \prec \tau_{k_*}} \|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^p \\ &\quad \le C 2^{\frac{p}{2}-3} e^{2p-4} (N+1) N^{p+E_2-4} (\log R) (\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-3} \left(\mathfrak{s}^{k_*} \max_{\gamma_{k_*} \prec \tau_{k_*}} \#(\gamma \prec \gamma_{k_*}) \right)^{\frac{p}{2}-1} \mathfrak{s}^{k_*} \lambda^p \mathfrak{r} \cdot R^{1+\beta}. \end{aligned}$$

We claim that this amounts to a decoupling inequality for the $\lambda g^{(\lambda,\mathfrak{r})}$. That is,

Lemma 3.3. *For each $\lambda, \mathfrak{r}$, and $\mathfrak{a}$, and each $p \ge 4$, we have the estimate*

$$\mathfrak{a}^p \mu\big(\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}\big) \le C 2^{\frac{p}{2}-3} e^{4p-3} (N+1) N^{p+E_2-4} (\log R) \max\big(R^{\beta(p-\frac{p}{q}-1)-1} R^{\beta(\frac{p}{2}-\frac{p}{q})}\big) \Big(\sum_\gamma \|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^q \Big)^{\frac{p}{q}}. \tag{3.6}$$

We postpone the proof until the end of the argument, to preserve the flow of our calculation. Recalling the identity

$$f = \sum_{N^{-2}R^{-1/2} < \lambda \le 1} \lambda g^{(\lambda)} + N^{-2} R^{-1/2} \eta,$$

and consequently, for a suitable λ_*,

$$\begin{aligned}\alpha^p \mu\big(\{x : |f(x)| > \alpha\}\big) &\le \alpha^p \sum_{N^{-2}R^{-1/2} < \lambda \le 1} \mu\Big(\Big\{x : |\lambda g^{(\lambda)}(x)| \ge \frac{\alpha}{\mathcal{Z}}\Big\}\Big) \\ &\le \mathcal{Z}^{p+1} \Big(\frac{\alpha}{\mathcal{Z}}\Big)^p \mu\Big(\Big\{x : |\lambda_* g^{(\lambda_*)}(x)| \ge \frac{\alpha}{\mathcal{Z}}\Big\}\Big),\end{aligned}$$

(where we have abbreviated $\mathcal{Z} = 2\log N + \frac{1}{2}\log R$) and hence, for a suitable $\mathfrak{r}$,

$$\alpha^p \mu\big(\{x : |f(x)| > \alpha\}\big) \le [\mathcal{Z} \log R]^{p+1} \Big(\frac{\alpha}{\mathcal{Z}\log R}\Big)^p \mu\Big(\Big\{x : |\lambda_* g^{(\lambda_*,\mathfrak{r})}(x)| \ge \frac{\alpha}{\mathcal{Z}\log R}\Big\}\Big),$$

which by (3.6), applied to $\mathfrak{a} = \frac{\alpha}{\mathcal{Z}\log R}$, implies

$$\begin{aligned}\alpha^p \mu\big(\{x : |f(x)| > \alpha\}\big)| &\le C 2^{\frac{p}{2}-3} e^{4p-3} \mathcal{Z}^{p+1} (N+1) N^{p+E_2-4} (\log R)^{p+2} \\ &\quad \times \max\big(R^{\beta(p-\frac{p}{q}-1)-1} R^{\beta(\frac{p}{2}-\frac{p}{q})}\big) \Big(\sum_\gamma \|\lambda_* g_\gamma^{(\lambda_*,\mathfrak{r})}\|_p^q\Big)^{\frac{p}{q}}.\end{aligned}$$

Finally, we note that each $\lambda_* g_\gamma^{(\lambda_*,\mathfrak{r})}$ is obtained by taking a subsum of a partition of unity applied to f_γ, so we conclude that

$$\begin{aligned}\alpha^p \mu\big(\{x : |f(x)| > \alpha\}\big)| &\le C 2^{\frac{p}{2}-3} e^{4p-3} \mathcal{Z}^{p+1} (N+1) N^{p+E_2-4} (\log R)^{p+2} \\ &\quad \times \max\big(R^{\beta(p-\frac{p}{q}-1)-1} R^{\beta(\frac{p}{2}-\frac{p}{q})}\big) \Big(\sum_\gamma \|f_\gamma\|_p^q\Big)^{\frac{p}{q}},\end{aligned} \tag{3.7}$$

as claimed.

$\square$

Proof of Theorem 1.1. By scaling we may assume that $\max_\theta \|f_\theta\|_\infty = 1$. By [13], Lemma 5.4, we may assume that f is supported in B_R. By the layer-cake integral

$$\int |f|^p = p \int_0^{R^{1/2}} \alpha^{p-1} \mu(U_\alpha) d\alpha,$$

and the inequalities

$$R^{-\frac{p}{2}+2} \le R^{\frac{p}{2}+1-\beta(p-1)} \le \max_\gamma \|f_\gamma\|_p^p, \tag{3.8}$$

$$\int_{R^{-1/2}}^{R^{1/2}} p\alpha^{p-1} \mu(U_\alpha) d\alpha \le p(\log R) \sup_{\alpha \in [R^{-1/2}, R^{1/2}]} \alpha^p \mu(U_\alpha), \tag{3.9}$$

we transform (3.7) into the decoupling estimate

$$D^{\mathbb{K}}_{p_\beta, q_\beta}(R;\beta) \le 10^6 (\log \mathbf{p})^{-16} \mathbf{p}^{12} (\log R)^{17+6\beta^{-1}} \max\Big(R^{\beta(\frac{p_\beta}{2}-\frac{p_\beta}{q_\beta})}, R^{\beta(p_\beta - \frac{p_\beta}{q_\beta}-1)-1}\Big) \Big(\sum_\gamma \|f_\gamma\|_{p_\beta}^{q_\beta}\Big)^{\frac{p_\beta}{q_\beta}}.$$

Remark 3.4. The condition $R \ge \mathbf{p}^{32}$, equivalent to $N \ge 16$, is arranged to make the universal constant at most 10^6. Asymptotically, one can arrange for the constant to be ≤ 4885.

It remains to use interpolation to cover the remaining exponents. The two quantities we will compare against are

$$D^{\mathbb{K}}_{\infty,1}(R;\beta) \le 1$$

and

$$D^{\mathbb{K}}_{4,4}(R;\beta) \le 2^{-5}e^4(\log \mathbf{p})^{-8}\mathbf{p}^5(\log R)^8 R^\beta.$$

The latter estimate is established in Prop. 3.5 below; there a different small cap decoupling argument is used than the one expounded so far, which is more efficient in the subcritical regime $2 \le p \le 4$.

For each $3 \le p \le \infty$, write $\frac{1}{q_p} = 1 - \frac{3}{p}$. Write $p_\beta = 2 + \frac{2}{\beta}$ and $q_\beta = q_{p_\beta}$. Let $\frac{3}{p} + \frac{1}{q} \le 1$. Suppose that $2 + \frac{2}{\beta} \le p$ and $\beta < 1$. Then, from the two inequalities

$$D^{\mathbb{K}}_{p_\beta,q_\beta}(R;\beta) \le 10^6(\log \mathbf{p})^{-16}\mathbf{p}^{12}(\log R)^{17+6\beta^{-1}} R^{\beta(\frac{p_\beta}{2}-\frac{p_\beta}{q_\beta})},$$

$$D^{\mathbb{K}}_{\infty,1}(R;\beta) \le 1,$$

we interpolate to obtain

$$D^{\mathbb{K}}_{p,q_p}(R;\beta) \le 10^6(\log \mathbf{p})^{-16}\mathbf{p}^{12}(\log R)^{17+6\beta^{-1}} R^{\beta(p-\frac{p}{q_p}-1)-1}.$$

By Hölder, we conclude the desired

$$D^{\mathbb{K}}_{p,q}(R;\beta) \le 10^6(\log \mathbf{p})^{-16}\mathbf{p}^{12}(\log R)^{17+6\beta^{-1}} R^{\beta(p-\frac{p}{q}-1)-1}$$

Next, suppose that $4 \le p \le 2 + \frac{2}{\beta}$. By Lemma 3.5, we may assume $\beta < 1$. Then, from the inequalities

$$D^{\mathbb{K}}_{p_\beta,q_\beta}(R;\beta) \le 10^6(\log \mathbf{p})^{-16}\mathbf{p}^{12}(\log R)^{17+6\beta^{-1}},$$

$$D^{\mathbb{K}}_{4,4}(R;\beta) \le 2^{-5}e^4(\log \mathbf{p})^{-8}\mathbf{p}^5(\log R)^8 R^\beta$$

we interpolate to obtain

$$D^{\mathbb{K}}_{p,q_p}(R;\beta) \le 10^6(\log \mathbf{p})^{-8}\mathbf{p}^{5+\frac{7\beta}{2-2\beta}(p-4)}(\log R)^{8+\frac{9\beta+6}{2-2\beta}(p-4)} R^{\beta(\frac{p}{2}-\frac{p}{q_p})}.$$

By Hölder, we obtain the desired

$$D^{\mathbb{K}}_{p,q}(R;\beta) \le 10^6(\log \mathbf{p})^{-8}\mathbf{p}^{5+\frac{7\beta}{2-2\beta}(p-4)}(\log R)^{8+\frac{3\beta+2}{1-\beta}(p-4)} R^{\beta(\frac{p}{2}-\frac{p}{q})}.$$

It remains to consider the regime $3 \le p \le 4$. Then, from the estimates

$$D^{\mathbb{K}}_{4,4}(R;\beta) \le 2(\log \mathbf{p})^{-8}\mathbf{p}^5(\log R)^8 R^\beta,$$

$$D^{\mathbb{K}}_{2,2}(R;\beta) = 1,$$

we obtain the estimate

$$D^{\mathbb{K}}_{3,3}(R;\beta) \le 2(\log \mathbf{p})^{-4}\mathbf{p}^{5/2}(\log R)^4 R^{\frac{\beta}{2}},$$

whence

$$D^{\mathbb{K}}_{3,\infty}(R;\beta) \le 2(\log \mathbf{p})^{-4}\mathbf{p}^{5/2}(\log R)^4 R^{\frac{3\beta}{2}}.$$

By interpolation and Hölder, we conclude

$$D^{\mathbb{K}}_{p,q}(R;\beta) \le 2(\log \mathbf{p})^{-4(p-2)}\mathbf{p}^{5/2(p-2)}(\log R)^{4(p-2)} R^{\beta(\frac{p}{2}-\frac{p}{q})}.$$

□

To close the argument, it remains only to prove Lemmas 3.3 and 3.5.

Proof of Lemma 3.3. The proof is by casework on p and k. Suppose $R_k \geq R^{1-\beta}$. By a straightforward calculation (see Case 2 of the proof of Theorem 5 in [16]),

$$(\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-4}\left(\mathfrak{s}^{k_*}\max_{\substack{\gamma_{k_*}\prec\tau_{k_*}\\ \tau_{k_*}\in\mathcal{T}_{k_*}(\mathfrak{s}^{k_*})}}\#(\gamma\prec\gamma_{k_*})\right)^{\frac{p}{2}-1}\leq R^{\beta(p-\frac{p}{q}-1)-1}\big(\mathfrak{s}^{k_*}\times\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*})\big)^{\frac{p}{q}-1}.$$

Consequently,

$$\begin{aligned}
&\mathfrak{a}^p\mu\big(\{x:|\lambda g^{(\lambda,\mathfrak{r})}(x)|>\mathfrak{a}\}\big)\\
&\quad\leq C2^{\frac{p}{2}-3}e^{2p-4}(N+1)N^{p+E_2-4}(\log R)R^{\beta(p-\frac{p}{q}-1)-1}\big(\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*})\big)^{\frac{p}{q}}(\mathfrak{s}^{k_*})^{\frac{p}{q}}\lambda^p\mathfrak{r}R^{1+\beta}\\
&\quad\leq C2^{\frac{p}{2}-3}e^{4p-3}(N+1)N^{p+E_2-4}(\log R)R^{\beta(p-\frac{p}{q}-1)-1}\Big(\sum_\gamma\|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^q\Big)^{\frac{p}{q}}.
\end{aligned}\tag{3.10}$$

Suppose instead that $R_k \leq R^{1-\beta}$, and $2+\frac{2}{\beta}\leq p\leq 6$. Then, from the inequality

$$1\leq R^{\beta(\frac{p}{2}-1)}R^{-1},$$

it follows that for any γ_k we have

$$[\#(\gamma\prec\gamma_k)]^{\frac{p}{2}-1}\leq R^{-1}R^{\beta(\frac{p}{2}-1)}R_k^{3-\frac{p}{2}}.$$

Rearranging, we have

$$R^{\beta(p-\frac{p}{q}-3)}[\#(\gamma\prec\gamma_k)]^{\frac{p}{2}-1}\leq R^{\beta(p-\frac{p}{q}-1)-1}\Big(R^{-\beta}R_k\Big)^{3-\frac{p}{2}},$$

which implies (using $\frac{1}{q}+\frac{3}{p}\leq 1$)

$$(\mathfrak{s}^{k_*}\times\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-3}[\#(\gamma\prec\gamma_k)]^{\frac{p}{2}-1}\leq R^{\beta(p-\frac{p}{q}-1)-1}\Big(R^{-\beta}R_k\Big)^{3-\frac{p}{2}}(\#\gamma)^{\frac{p}{q}}.$$

Note that, for each $\tau_k\in\mathcal{T}_{k_*}(\mathfrak{s}^{k_*})$,

$$(\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-4}(\#\gamma\prec\tau_k)^{\frac{p}{2}-1}\leq(\mathfrak{s}^{k_*})^{3-\frac{p}{2}}(\mathfrak{s}^{k_*}\times\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-4},$$

so that

$$\mathfrak{s}^{k_*}(\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-3}(\#\gamma\prec\tau_k)^{\frac{p}{2}-1}[\#(\gamma\prec\gamma_k)]^{\frac{p}{2}-1}\leq R^{\beta(p-\frac{p}{q}-1)-1}(\#\gamma)^{\frac{p}{q}}.$$

Thus, we achieve the estimate

$$\begin{aligned}
\mathfrak{a}^p\mu\big(\{x:|\lambda g^{(\lambda,\mathfrak{r})}(x)|>\mathfrak{a}\}\big)&\leq C2^{\frac{p}{2}-3}e^{2p-4}(N+1)N^{p+E_2-4}(\log R)R^{\beta(p-\frac{p}{q}-1)-1}(\#\gamma)^{\frac{p}{q}}\lambda^p\mathfrak{r}\cdot R^{1+\beta}\\
&\leq C2^{\frac{p}{2}-3}e^{4p-3}(N+1)N^{p+E_2-4}(\log R)R^{\beta(p-\frac{p}{q}-1)-1}\Big(\sum_\gamma\|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^q\Big)^{\frac{p}{q}}.
\end{aligned}$$

It remains to consider the case $R_k\leq R^{1-\beta}$, $4\leq p\leq 2+\frac{2}{\beta}$. In this case, from the inequalities

$$R_k^{\frac{p}{2}-3}\leq 1,\quad \mathfrak{s}^{k_*}\leq RR_k^{-1},\quad \#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*})\leq R_k,$$

we conclude the inequality

$$(\mathfrak{s}^{k_*})^{\frac{p}{2}-\frac{p}{q}}(\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-\frac{p}{q}-3} \leq R^{\beta(\frac{p}{2}-\frac{p}{q})}.$$

Thus,

$$(\mathfrak{s}^{k_*})^{\frac{p}{2}}(\#\mathcal{T}_{k_*}(\mathfrak{s}^{k_*}))^{p-3} \leq R^{\beta(\frac{p}{2}-\frac{p}{q})}(\#\gamma)^{\frac{p}{q}},$$

and we conclude that

$$\begin{aligned}\mathfrak{a}^p\mu\big(\{x : |\lambda g^{(\lambda,\mathfrak{r})}(x)| > \mathfrak{a}\}\big) &\leq C2^{\frac{p}{2}-3}e^{2p-4}(N+1)N^{p+E_2-4}(\log R)R^{\beta(\frac{p}{2}-\frac{p}{q})}(\#\gamma)^{\frac{p}{q}}\lambda^p\mathfrak{r}\cdot R^{1+\beta}\\ &\leq C2^{\frac{p}{2}-3}e^{4p-3}(N+1)N^{p+E_2-4}(\log R)R^{\beta(\frac{p}{2}-\frac{p}{q})}\Big(\sum_\gamma \|\lambda g_\gamma^{(\lambda,\mathfrak{r})}\|_p^q\Big)^{\frac{p}{q}}.\end{aligned}$$

□

Proposition 3.5. $D_{4,4}^{\mathbb{K}}(R;\beta) \leq 8e^4\mathbf{p}^5N^8R^\beta$.

Proof. By repeated applications of the narrow lemma,

$$|f(x)| \leq \left(1+\frac{1}{N-1}\right)^N \max_\theta |f_\theta(x)| + \mathbf{p}N\sum_{k=1}^N \left(1+\frac{1}{N-1}\right)^{k-1} \max_{\tau_{k-1}} \max_{\tau_k\neq\tau_k'\prec\tau_{k-1}} |f_{\tau_k}(x)f_{\tau_k'}(x)|^{1/2},$$

so that

$$|f(x)|^4 \leq 2^3\left(1+\frac{1}{N-1}\right)^{4N} \max_\theta |f_\theta(x)|^4 + \mathbf{p}^4N^4\sum_{k=1}^N 2^3N^3\left(1+\frac{1}{N-1}\right)^{4(k-1)} \max_{\tau_{k-1}} \max_{\tau_k\neq\tau_k'\prec\tau_{k-1}} |f_{\tau_k}(x)f_{\tau_k'}(x)|^2.$$

For each θ,

$$\|f_\theta\|_4^4 \leq R^{2(\beta-\frac{1}{2})}\sum_{\gamma\prec\theta}\|f_\gamma\|_4^4.$$

Choose now some τ_{k-1}. Let $(c,c^2)\in\tau_{k-1}$, and write

$$g(x,y) = f_{\tau_{k-1}}(\varpi^{-\eta(k-1)}(x-c),\varpi^{-2\eta(k-1)}(y-c^2)),$$

$$g_1(x,y) = f_{\tau_k}(\varpi^{-\eta(k-1)}(x-c),\varpi^{-2\eta(k-1)}(y-c^2)),\quad g_2(x,y) = f_{\tau_k'}(\varpi^{-\eta(k-1)}(x-c),\varpi^{-2\eta(k-1)}(y-c^2))$$

Then g has spectral support in $\mathcal{N}_{R_{k-1}^2R^{-1}}(\mathbb{P}^1)$, so by Córdoba-Fefferman we have

$$\int |g_1g_2|^2 \leq 2R^\beta R_{k-1}^{-2\beta}\sum_{\tilde\theta}\|g_{\tilde\theta}\|_4^4,$$

where the $\tilde\theta$ have dimensions $R^{-\beta}R_{k-1}^\beta \times R^{-1}R_{k-1}$. By flat decoupling,

$$\|g_{\tilde\theta}\|_4^4 \leq R_{k-1}^{2(2\beta-1)}\sum_{\tilde\gamma}\|g_{\tilde\gamma}\|_4^4$$

Thus,

$$\int |f_{\tau_k}f_{\tau_k'}|^2 \leq 2R^\beta R_{k-1}^{2\beta-2}\sum_{\gamma\prec\tau_k\sqcup\tau_k'}\int|f_\gamma|^4.$$

We have reached the estimate

$$\int |f|^4 \leq 2^3\left(1+\frac{1}{N-1}\right)^{4N} R^{2\beta-1}\sum_\gamma\|f_\gamma\|_4^4 + 16\mathbf{p}^5N^7R^\beta\sum_{k=1}^N\left(1+\frac{1}{N-1}\right)^{4(k-1)} R_{k-1}^{2\beta-2}\sum_\gamma\|f_\gamma\|_4^4,$$

i.e.

$$D_{4,4}^{\mathbb{K}}(R;\beta) \leq 8e^4\mathbf{p}^5N^8R^\beta.$$

□

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