UCLA 245B discussion notes

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These are a complete set of documents I wrote in the course of TAing Math 245B at UCLA in Winter 2024. The main focus of that discussion was preparation for the UCLA Analysis qualifying exam (real half). The problems each week were chosen based on the material being covered in lecture.

Prior to each discussion, I sent out the selection of problems to be covered, together with some hints and remarks. After the discussion, I would send out the version of the document seen here, with the solutions filled in.

1 : Week 1

Fall 2004, Problem 2 Let $f : [0,1] \to [0,\infty)$ be a nonnegative, L^1 function (with respect to Lebesgue measure). Prove that the following two statements are equivalent:

(a) There exists a constant $0 < C < \infty$ such that

$$\left(\int_0^1 f(x)^p dx\right)^{1/p} \leq Cp \quad \forall p \geq 1$$

(b) There exists a constant $0 < c < \infty$ such that

$$\int_0^1 e^{cf(x)} dx < \infty$$

You are free to make use of the following version of Stirling's approximation:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(ne^{-1})^n} = 1$$

Proof. Assuming (a), we prove (b). For each choice of c and $x \in [0,1]$, we have

$$e^{cf(x)} = \sum_{n=0}^{\infty} \frac{c^n}{n!}(f(x))^n$$

Since f is nonnegative and measurable, by Tonelli's theorem we have

$$\int_0^1 e^{cf(x)} dx = \sum_{n=0}^{\infty} \frac{c^n}{n!} \int_0^1 f(x)^n dx$$

By the assumption,

$$\int_0^1 f(x)^n dx \leq C^n n^n$$

and hence

$$\int_0^1 e^{cf(x)} dx \leq \sum_{n=0}^{\infty} (cC)^n \frac{n^n}{n!}$$

Assume now that c is chosen so that $cC \leq e/2$. Recall by Stirling's approximation that

$$\frac{n^n}{n!} \lesssim n^{1/2} e^n$$

Thus we have the estimate

$$\int_0^1 e^{cf(x)} dx \lesssim \sum_{n=0}^{\infty} (cCe)^n n^{1/2} \leq \sum_{n=0}^{\infty} 2^{-n} n^{1/2} < \infty$$

as was to be shown.

Now assume (b). Again by Tonelli,

$$\int_0^1 e^{cf(x)}dx = \sum_{n=0}^{\infty} \frac{c^n}{n!} \int_0^1 f(x)^n dx$$

Recall that Stirling provides

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(ne^{-1})^n} = 1$$

In particular, we may find a constant $C_0 > 0$ such that

$$n! \leq C_0 n^{n+\frac{1}{2}} e^{-n} \quad \forall n \in \mathbb{N}$$

Thus

$$\sum_{n=0}^{\infty} \frac{c^n}{n!} \int_0^1 f(x)^n dx \geq C_0^{-1} \sum_{n=0}^{\infty} (ce)^n n^{-n-\frac{1}{2}} \int_0^1 f(x)^n dx$$

We claim that $\int_0^1 f(x)^n dx \leq (\frac{2}{ce})^n n^n$ for all n. If this doesn't hold, then we may find an increasing sequence $k \mapsto n_k$ such that $\int_0^1 f(x)^{n_k} dx > (\frac{2}{ce})^{n_k} n_k^{n_k}$ for all k; but then

$$\begin{aligned}\sum_{n=0}^{\infty} (ce)^n n^{-n-\frac{1}{2}} \int_0^1 f(x)^n dx &\geq \sum_{k=0}^{\infty} (ce)^{n_k} n_k^{-n_k-\frac{1}{2}} \int_0^1 f(x)^{n_k} dx \\ &\geq \sum_{k=0}^{\infty} (2)^{n_k} n_k^{-\frac{1}{2}} = +\infty\end{aligned}$$

which violates the convergence from (b). Thus $\int_0^1 f(x)^n dx \leq C^n n^n$ for a suitable $C > 0$, for all n.

Finally, if $n < p < n+1$, we have by Hölder

$$\left(\int_0^1 f(x)^p dx\right)^{1/p} \leq \left(\int_0^1 f(x)^{n+1} dx\right)^{\frac{1}{n+1}} \leq C(n+1) \leq 2Cp$$

and we are done.

$\square$

Necessity of measurable hypothesis in Fubini-Tonelli, part 1. Observe the following easy consequence of Fubini-Tonelli:

> Suppose $A \subseteq \mathbb{R}^2$ is Lebesgue-measurable, and suppose that every intersection A_v of A with the line $x = v$ is null. Then A is null.

Of course, any nullset is Lebesgue measurable, so there's a sense that the assumption that A is measurable might not be necessary. The point of this problem is to demonstrate that the framed statement fails if this assumption is removed.[1]

To this end, do the following:

[1] Method taken from van Douwen, "Fubini's theorem for null sets." *The American Mathematical Monthly* 96.8 (1989): 718-721.

(a) Let $\mathscr{P}$ be the family of sets $K \subseteq \mathbb{R}^2$ that are compact with positive measure. Write $\mathfrak{c} = |\mathbb{R}|$ for the cardinality of the continuum. Show that $|\mathscr{P}| = \mathfrak{c}$.

(b) Let K be a compact uncountable subset of $\mathbb{R}$. Show that $|K| = \mathfrak{c}$.

(c) Show that, for each $P \in \mathscr{P}$, the set $\{v \in \mathbb{R} : P_v \neq \emptyset\}$ also has cardinality $\mathfrak{c}$. Here, as before, the subscript v denotes intersection with the line $x = v$.

(d) By standard set theory arguments, by (a) it follows that there is a well-order $\prec$ on $\mathscr{P}$ such that every set of the form $\{P \in \mathscr{P} : P < Q\}$ has cardinality $< \mathfrak{c}$, where Q ranges over $\mathscr{P}$.

   Using this, do the following. Let $Q \in \mathscr{P}$ and write $L_Q = \{P \in \mathscr{P} : P \prec Q\}$. Suppose $\{(x_P, y_P)\}_{P \in L_Q}$ is a collection of points $(x, y) \in \mathbb{R}^2$ such that $(x_P, y_P) \in P$, and $x_P \neq x_{P'}$ for $P \neq P' \in L_Q$. Show then that there is some $(x_Q, y_Q) \in Q$ such that x_Q is distinct from all x_P with $P \prec Q$.

(e) By (d) and transfinite induction, there exists a family $\{(x_P, y_P)\}_{P \in \mathscr{P}}$ of points in $\mathbb{R}^2$ such that $(x_P, y_P) \in P$ for each $P \in \mathscr{P}$. Write $A = \{(x_P, y_P) : P \in \mathscr{P}\}$. Show that A is not null, and that each intersection A_v is finite.

Proof. (a): Let τ be the collection of Euclidean open sets in $\mathbb{R}^2$. We show that $|\tau| = \mathfrak{c}$. Let τ_0 be the collection of open balls of rational radii, whose centers are pairs of rational points. Clearly τ_0 is a base for τ, and is countable. It follows that

$$2^{\tau_0} \xrightarrow{\phi} \tau, \quad \mathcal{U} \mapsto \bigcup_{U \in \mathcal{U}} U$$

is a surjection, so $|\tau| \leq 2^{|\tau_0|} = \mathfrak{c}$. Thus the set $\mathcal{C}$ of closed sets has $|\mathcal{C} \leq \mathfrak{c}$ as well. Since $\mathscr{P} \subseteq \mathcal{C}$, it follows that $|\mathscr{P}| \leq \mathfrak{c}$.

On the other hand, each box $[v, v+1]^2$ belongs to $\mathscr{P}$ for $v \in \mathbb{R}$, so $|\mathscr{P}| \geq \mathfrak{c}$ as well.

(b): Let $a < b$ be any two real numbers such that $K_0 = (-\infty, a] \cap K$ and $K_1 = [b, \infty) \cap K$ are both uncountable. Then K_0, K_1 are also compact uncountable subsets of $\mathbb{R}$. Iterating this, we obtain a family of compact subsets $\{K_w\}_w$ indexed by finite words in the alphabet $0, 1$ such that $K_w \supseteq K_{w'}$ if w' extends w (i.e. is given by adding extra digits onto the end of w). By taking intersections over all truncations $w_{|n}$ of a given word w, we obtain (by the compact intersection theorem) a nonempty compact set K_w. Any two infinite words w, w' that are distinct have $K_w \cap K_{w'} = \emptyset$. Consequently, we may find $|2^{\mathbb{N}}| = \mathfrak{c}$ many distinct points in K. Since $|K| \leq \mathfrak{c}$ trivially, we conclude that $|K| = \mathfrak{c}$.

(c): Denote the set in question as C. Note that C is the image of P under the (continuous) orthogonal projection onto the x-axis. In particular, C is compact. Also, $P \subseteq C \times \mathbb{R}$, so (since P is positive measure), C has positive measure. Thus C is uncountable, and since C is a compact metric space we conclude that $|C| = \mathfrak{c}$.

(d): Note that L_Q has cardinality $< \mathfrak{c}$. Thus $\pi_1(L_Q)$, the collection of x_P with $P \prec Q$, has cardinality $< \mathfrak{c}$. Since $\pi_1(Q)$ has cardinality $\mathfrak{c}$, we may find $x_Q \in \pi_1(Q) \setminus \pi_1(L_Q)$, and hence $(x_Q, y_Q) \in Q$ with x_Q distinct from all x_P with $P \prec Q$.

(e): First we demonstrate that A is not null. In fact, we demonstrate that any $U \supseteq A$ open has full measure. Indeed, suppose U were a neighborhood of A such that $C = \mathbb{R}^2 \setminus U$ has positive measure. In particular, $K = C \cap B_{\leq R}(0)$ has positive measure for large enough R. But then K is a positive measure compact set, hence intersects nontrivially with A, violating the fact that $K \cap U = \emptyset$.

Finally, observe that the $\{x_P\}_{P \in \mathscr{P}}$ are pairwise distinct, so A_v is either empty or a singleton for each $v \in \mathbb{R}$. The result follows.

□

Necessity of measurable hypothesis in Fubini-Tonelli, part 2.[2] Write A_2 for the following proposition:

> Suppose that:
>
> 1. $f : \mathbb{R}^2 \to \mathbb{R}$ is nonnegative,
> 2. for every $x \in \mathbb{R}$, the function $y \mapsto f(x, y)$ is Borel measurable and the integral $\int f(x, y)dy$ converges to a Borel measurable function,
> 3. for every $y \in \mathbb{R}$, the function $x \mapsto f(x, y)$ is Borel measurable and the integral $\int f(x, y)dx$ converges to a Borel measurable function, and
> 4. the iterated integrals $\int \int f(x, y)dxdy$ and $\int \int f(x, y)dydx$ converge.
>
> Then $\int \int f(x, y)dxdy = \int \int f(x, y)dydx$.

A_2 is called a *strong Fubini theorem*. The point of this problem is to demonstrate one-half of the statement "ZFC does not prove nor disprove A_2 (unless ZFC is inconsistent)." To this end, do the following:

Assume CH, that is, assume that any uncountable cardinal κ satisfies $\kappa \geq \mathfrak{c} := |\mathbb{R}| = |[0, 1]|$. By standard set theoretic arguments, this implies that there is a well-order $\prec$ on $[0, 1]$ such that every half-line $L_x := \{y \in [0, 1] : y \prec x\}$ is countable.

Taking this for granted, $f(x, y) = 1_E(x, y)$, where $E = \{(x, y) : x \prec y\}$, violates A_2.

Proof. We validate that f satisfies all the hypotheses of A_2, but violates the conclusion. Clearly f is nonnegative. For each fixed $x \in [0, 1]$, the function $y \mapsto f(x, y)$ is just the function 1_{L_x}, which is the indicator of a countable set, hence is Borel measurable. Additionally, since countable sets have measure zero, we conclude that $\int f(x, y)dy = 0$ for all x, and hence $\int \int f(x, y)dydx = 0$.

On the other hand, for each $y \in [0, 1]$, the function $x \mapsto f(x, y)$ is the indicator of a co-countable set in $[0, 1]$, hence is Borel measurable and satisfies $\int f(x, y)dx = 1$. Since this holds for each y, we conclude $\int \int f(x, y)dxdy = 1$, and we are done.

□

[2] The content of this problem is borrowed from Shipman, "Cardinal Conditions for Strong Fubini Theorems."

Hints and remarks regarding the preceding problems.

First problem:

Hint (a): consider the power series for $\exp(x)$*.*

Hint (b): Tonelli; to apply Stirling's in either direction, you need a "uniform" version which applies for all n*. By the limit, there is a large* N *such that, beyond* N*, the fraction is within* ε *of* 1*. On the other hand, we only ignored finitely many, so at a constant cost you have a uniform statement.*

You could view (a) as being a quantitative version of the finiteness statement (b).

Second problem:

Hint for (a): How many open sets are there?

Hint for (b): Try to find a Cantor set in K.

Hint for (c): What can the projection of P onto $\mathbb{R}$ be?

Hint for (d): How many points do you need to avoid?

Hint for (e): Use outer regularity. A would need to avoid some sets.

Third problem:

Hint: How often is $y \mapsto f(x, y)$ equal to 1? What about $x \mapsto f(x, y)$? Remember, this is Lebesgue measure.

Remark. We have shown that ZFC does not prove A_2, because then ZFC would refute CH (which it does not, unless it is inconsistent). The other direction is over my head, so I won't discuss it here; the point is that "there are cardinals κ_1, κ_2 with the property that there is a non-Lebesgue measurable subset of $\mathbb{R}$ with cardinality κ_1, and there is a subset B of the real numbers of cardinality κ_2 such that B is not the union of κ_1 measure-0 sets" is consistent with ZFC and implies A_2. See the linked paper.

2 : Week 2

Fall 2022 Problem 2: Let $f \in L^p(\mathbb{R})$, for some $1 \le p < 2$. Show that the series

$$\sum_{n=1}^{\infty} \frac{f(x+n)}{\sqrt{n}}$$

converges absolutely for almost all $x \in \mathbb{R}$. For each $2 \le p \le \infty$, give an example of a function $f \in L^p(\mathbb{R})$ for which the series diverges for every $x \in \mathbb{R}$.

Proof. We first demonstrate that the integral

$$\int_c^{c+1} \left(\sum_{n=1}^{\infty} \frac{|f(x+n)|}{\sqrt{n}} \right)^p dx$$

converges for each $c \in \mathbb{R}$. Indeed, applying Hölder to the sum, writing $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\left(\sum_{n=1}^{\infty} \frac{|f(x+n)|}{\sqrt{n}} \right)^p \le \left(\sum_{n=1}^{\infty} |f(x+n)|^p \right) \left(\sum_{n=1}^{\infty} n^{-\frac{p'}{2}} \right)^{\frac{p}{p'}}$$

Note that $p' > 2$, so the rightmost factor is a convergent series. Thus

$$\int_c^{c+1} \left(\sum_{n=1}^{\infty} \frac{|f(x+n)|}{\sqrt{n}} \right)^p dx \le \left(\sum_{n=1}^{\infty} n^{-\frac{p'}{2}} \right)^{\frac{p}{p'}} \int_c^{c+1} \sum_{n=1}^{\infty} |f(x+n)|^p dx = \int_{-\infty}^{\infty} |f(x)|^p dx < \infty$$

as was to be shown.

Write now $N = \{x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|f(x+n)|}{\sqrt{n}} = +\infty\}$; one may verify that this set is Lebesgue measurable. If $\lambda(N) > 0$, then $\lambda(N \cap [c, c+1]) > 0$ for some $c \in \mathbb{R}$, and hence

$$\int_c^{c+1} \left(\sum_{n=1}^{\infty} \frac{|f(x+n)|}{\sqrt{n}} \right)^p dx \ge \int_{N \cap [c,c+1]} (+\infty) dx = +\infty$$

violating the previous finiteness conclusion. Thus N is null, so the series converges absolutely a.e., as was to be shown.

Now we consider the second half of the problem. Observe that $f \equiv 1$ suffices in the case $p = \infty$, so we assume $p \in [2, \infty)$. Define then

$$f(x) = \frac{x^{-1/p}}{\log x} 1_{x>e}$$

We claim that $f \in L^p(\mathbb{R})$. Indeed, $|f(x)|^p \le \frac{x^{-1}}{(\log x)^2}$, and so

$$\begin{aligned} \int |f(x)|^p dx &\le \int_e^{\infty} \frac{x^{-1}}{(\log x)^2} dx \\ &= \int_1^{\infty} \frac{du}{u^2} < \infty \end{aligned}$$

by a change-of-variable. Note that $|f(x)| \geq \frac{x^{-1/2}}{\log x}$ for each $x > e$, so we may consider the $p = 2$ case. For each $a, b \in \mathbb{R}$ with $|b| \geq 1$, we have the elementary inequality

$$|a + b| \leq |a||b| + |b| = (|a| + 1)|b|$$

from which we evaluate

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|f(x+n)|}{\sqrt{n}} &\geq \sum_{n > \max(2, e-x)} \frac{(x+n)^{-1/2}}{n^{1/2}\log(x+n)} \\ &\geq \frac{1}{(|x|+1)^{1/2}(\log(|x|+1)+1)} \sum_{n>\max(2,e-x)} \frac{1}{n \log n} = +\infty\end{aligned}$$

where we have appealed to the divergence calculation

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} = \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n \log n} \gtrsim \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

or, written another way,

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \gtrsim \int_2^{\infty} \frac{1}{t \log t} dt = \int_{\log 2}^{\infty} \frac{1}{u} du = +\infty$$

$\square$

Spring 2010 Problem 5 (with added scaffolding). Do the following:

(a) Let f be a real-valued continuous compactly-supported function on $\mathbb{R}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers tending to 0. Show that the sequence of functions

$$f_n(x) := f(x_n + x)$$

converges to f in the L^2 sense.

(b) Using the prior and approximation theorems, show that for any $f \in L^2(\mathbb{R})$ and any sequence $\{x_n\}_{n=1}^{\infty}$ tending to zero, the sequence of functions

$$f_n(x) := f(x_n + x)$$

converges to f in the L^2 sense.

(c) Conclude that, for each $f \in L^2$, the map $\tau.f : \mathbb{R} \to L^2(\mathbb{R})$ defined by

$$\mathbb{R} \ni r \mapsto \tau_r f \in L^2(\mathbb{R}), \quad \tau_r f(x) := f(r + x)$$

is continuous as a function from $\mathbb{R}$ to $L^2(\mathbb{R})$.

(d) Suppose $E \subseteq \mathbb{R}$ is Lebesgue measurable with positive finite measure. Show that the function

$$\mathbb{R} \ni t \mapsto \phi(t) := \int_{\mathbb{R}} \chi_E(t + y)\chi_E(y)dy$$

is continuous.

(e) Finally, show that, for E Lebesgue measurable with positive measure, the set

$$E - E = \{z \in \mathbb{R} : \exists x, y \in E \text{ s.t. } z = x - y\}$$

contains a neighborhood of the origin $(-\varepsilon, \varepsilon)$.

Proof. (a): Let $C > 0$ be a uniform upper bound for the sequence $\{x_n\}_n$, and let $J \subseteq \mathbb{R}$ be a compact interval containing $(\operatorname{supp} f) + [-C, C]$; thus $\operatorname{supp} f_n \subseteq J$ for every n. Let $\delta = \delta(\varepsilon)$ be a uniform modulus of continuity for f, i.e. $|x - y| < \delta(\varepsilon) \implies |f(x) - f(y)| < \varepsilon$ for each $\varepsilon > 0$.

Fix now $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $n \geq N$ implies $|x_n| < \delta(\frac{\varepsilon}{2\ell(J)^{1/2}})$. Then, for each $n \geq N$,

$$\int |f_n(x) - f(x)|^2 dx = \int_J |f(x + x_n) - f(x)|^2 \leq \int_J \frac{\varepsilon^2}{4\ell(J)} dx < \varepsilon^2$$

so that $\|f_n - f\|_{L^2(\mathbb{R})} < \varepsilon$. Thus we have shown that $f_n \to f$ in L^2, as was to be shown.

(b): Let $\varepsilon > 0$ be arbitrary. By the density of C_c functions in L^2, we may find $g \in C_c(\mathbb{R})$ such that $\|g - f\|_2 < \frac{\varepsilon}{3}$. By (a), we may find $N \in \mathbb{N}$ such that $n \geq N$ implies $\|g - g_n\|_2 < \frac{\varepsilon}{3}$. Then, for each such n,

$$\begin{aligned} \|f - f_n\|_2 &\leq \|f - g\|_2 + \|g - g_n\|_2 + \|g_n - f_n\|_2 \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad (*) \end{aligned}$$

where in $(*)$ we noted that by change-of-variable

$$\int |g(x + x_n) - f(x + x_n)|^2 dx = \int |g(x) - f(x)|^2 dx$$

Thus we have $f_n \to f$ in L^2.

(c): Indeed, if $u \in \mathbb{R}$ and $u_n \to u$,

$$\|\tau_{u_n} f - \tau_u f\|_2 = \|\tau_{u_n - u} f - f\|_2 \to 0$$

since $u_n - u \to 0$. Thus $\tau_{\cdot} f$ is continuous at u, hence continuous everywhere.

(d): Let $\psi : \mathbb{R} \to L^2(\mathbb{R})$ be the function $\psi(t) = \tau_t(\chi_E)$. Let $\Phi : L^2(\mathbb{R}) \to \mathbb{R}$ be the function $\Phi(f) = \int f(y)\chi_E(y)dy$. Note by Cauchy-Schwarz that Φ is a well-defined bounded linear map, and by (c) ψ is a continuous function. Thus the composition $\Phi \circ \psi$ is a continuous function $\mathbb{R} \to \mathbb{R}$. But

$$(\Phi \circ \psi)(t) = \int_{\mathbb{R}} \chi_E(y + t)\chi_E(y)dy = \phi(t)$$

so ϕ is also continuous.

(e): We assume first that E has finite measure. Then

$$\phi(0) = \int_{\mathbb{R}} \chi_E(y)dy = \lambda(E) > 0$$

so, since ϕ is continuous, there is some $\varepsilon > 0$ such that $\phi > 0$ on $(-\varepsilon, \varepsilon)$.

Next, observe that

$$\chi_E(t+y)\chi_E(y) = 1 \qquad \Longleftrightarrow \qquad y \in E \text{ and } \exists z \in E \text{ s.t. } z - y = t$$

so that

$$\exists y \text{ s.t. } \chi_E(t+y)\chi_E(y) = 1 \qquad \Longleftrightarrow \qquad \exists y, z \in E \text{ s.t. } z - y = t$$

i.e. if and only if $t \in E - E$.

Finally, for each $-\varepsilon < t < \varepsilon$, $\phi(t) > 0$, so in particular $\chi_E(t+y)\chi_E(y) = 1$ for some y, so $t \in E - E$. Thus $(-\varepsilon, \varepsilon) \subseteq E - E$, as was to be shown.

Lastly, we remove the finiteness assumption. Taking $E \subseteq \mathbb{R}$ measurable with positive measure, we may find $E_n = [-n, n] \cap E$ of positive finite measure, so for some $\varepsilon > 0$ we have $(-\varepsilon, \varepsilon) \subseteq E_n - E_n$. But of course $E_n - E_n \subseteq E - E$, and we are done.

$\square$

Fall 2022 Problem 6: Let $E = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \notin \mathbb{Q}\}$. Show that E does not contain a set of the form $A_1 \times A_2$, where $A_1 \subseteq \mathbb{R}$, $A_2 \subseteq \mathbb{R}$ are measurable, both of positive Lebesgue measure.

Proof. Consider $A_1, A_2 \subseteq \mathbb{R}$ arbitrary Lebesgue measurable sets with positive measure; we will show that E does not contain $A_1 \times A_2$. Let I_1, I_2 be any two intervals of positive finite length such that

$$\lambda(I_1 \cap A_1) > 0.99\lambda(I_1), \quad \lambda(I_2 \cap A_2) > 0.99\lambda(I_2)$$

Without loss of generality we may assume $\lambda(I_1) \leq \lambda(I_2)$. We may also assume that $\lambda(I_2) \leq 2\lambda(I_1)$, by repeatedly dividing I_2 in half and selecting the denser subinterval. Let $t \in \mathbb{R}$ be such that $I_1 + t$ has the same left endpoint as I_2; then $I_1 + t \subseteq I_2$, by the size assumption. Write $I_1' = I_1 + t$, $A_1' = A_1 + t$. Note that

$$\chi_{I_1' \cap A_1' \cap A_2} + \chi_{I_1' \cap (A_1' \cup A_2)} = \chi_{I_1' \cap A_1'} + \chi_{I_1' \cap A_2}$$

so upon integrating

$$\lambda(I_1' \cap A_1' \cap A_2) + \lambda(I_1' \cap (A_1' \cup A_2)) = \lambda(I_1' \cap A_1') + \lambda(I_1' \cap A_2)$$

Note that

$$\lambda(I_1' \cap A_2) + \lambda(I_2 \setminus I_1') \geq \lambda(I_1' \cap A_2) + \lambda(I_2 \cap A_2 \setminus I_1') = \lambda(I_2 \cap A_2) > 0.99\lambda(I_2)$$

so

$$\lambda(I_1' \cap A_2) > 0.99\lambda(I_1') - 0.01\lambda(I_2 \setminus I_1') \geq 0.98\lambda(I_1')$$

Thus

$$\lambda(I_1' \cap A_1' \cap A_2) > 1.97\lambda(I_1') - \lambda(I_1' \cap (A_1' \cap A_2)) \geq 0.97\lambda(I_1')$$

so in particular $\lambda(I_1' \cap A_1' \cap A_2) > 0$. Write $B = I_1' \cap A_1' \cap A_2$; thus B is a positive measure, measurable set, such that $B - B \subseteq A_1' - A_2$. By the previous problem, $B - B$ contains a neighborhood of 0, so $A_1 - A_2$ contains a neighborhood of t. In particular, $A_1 \times A_2$ contains a point (x_1, x_2) such that $x_1 - x_2 \in \mathbb{Q}$, so $A_1 \times A_2 \not\subseteq E$. Since A_1, A_2 were arbitrary, we are done.

$\square$

Hints and remarks regarding the preceding problems.

Fall 2022 Problem 2:

Hint: consider suitable integrals of the series, and show that they are finite. You will need Fubini and Hölder. For the second half, you'll need a function that is in L^2, but not in any L^p with $p < 2$.

Remark: many problems of this sort have appeared on the qual over the years. They usually proceed by the method indicated here.

Spring 2010 Problem 5:

Hint for (a): use uniform continuity. Hint for (d): write ϕ as the composition of two continuous functions. Hint for (e): consider $\phi(0)$.

Remark: This is essentially the "Steinhaus theorem." Several versions of this problem have appeared on the analysis qualifying exam over the years, usually without the step-by-step guidance. One version of interest (Q3 and 4, Fall 2004) uses this to demonstrate that a *measurable* additive bijection $f : \mathbb{R} \to \mathbb{R}$ is necessarily linear. Consequently, in order to find additive bijections $\mathbb{R} \to \mathbb{R}$ that are nonlinear, one needs the existence of non-Lebesgue-measurable sets.

Fall 2022 Problem 6:

Hint: use the previous problem. $A_1 - A_2$ need not contain a neighborhood of the origin, but it will contain an open interval somewhere. Translate A_1 to intersect a lot with A_2. To justify the latter, use problem 4 from the final and elementary inclusion/exclusion.

Remark: It is easy to see that E is a Borel subset of $\mathbb{R}^2$, and has positive measure (in fact, it is a dense G_δ set of full measure!). On the other hand, this shows that you cannot always extract good approximations from below using (countable unions of) measurable rectangles.

3 : Week 3

Recall the following:

- Measures μ, ν on a measurable space (X, Σ) are said to be *mutually singular* if there is some $A \in \Sigma$ such that $\mu(A) = 0$ and $\nu(X \setminus A) = 0$.
- For any (real) signed measure μ on a measurable space, there are mutually singular measures μ_+, μ_- such that $\mu(A) = \mu_+(A) - \mu_-(A)$ for all measurable sets A.
- For any complex measure μ, there are finite real signed measures μ_1, μ_2 such that $\mu(A) = \mu_1(A) + i\mu_2(A)$ for all measurable sets A.
- If $\mu = \mu_+ - \mu_-$ is the Hahn/Jordan decomposition of a real signed measure, we write $|\mu| = \mu_+ + \mu_-$. If the total underlying space is X, we abbreviate $\|\mu\| = |\mu|(X)$.

Spring 2021 Problem 2: Let μ and ν be two finite positive Borel measures on $\mathbb{R}^d$.

(a) Suppose that there exist Borel sets $A_n \subseteq X, n \in \mathbb{N}$ so that

$$\lim_{n\to\infty} \mu(A_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \nu(X \setminus A_n) = 0.$$

Show that μ and ν are mutually singular.

(b) Suppose there are non-negative Borel functions $\{f_n\}_{n\geq 1}$ so that $f_n(x) > 0$ for ν-a.e. x and

$$\lim_{n\to\infty} \int f_n(x) d\mu(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} \int \frac{1}{f_n(x)} d\nu(x) = 0.$$

Show that μ and ν are mutually singular.

Proof. (a): Using the hypothesis we may find a subsequence $n_1 < n_2 < n_3 < \ldots$ such that

$$\mu(A_{n_k}) \leq 2^{-k} \quad (\forall k \in \mathbb{N})$$

Write $A = \bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} A_{n_k}$; we claim that $\mu(A) = 0$ and $\nu(X \setminus A) = 0$. To establish the first claim, notice that for each $r \in \mathbb{N}$

$$\mu(A) \leq \mu(\bigcup_{k=r}^{\infty} A_{n_k}) \leq \sum_{k=r}^{\infty} \mu(A_{n_k}) \leq 2^{-r+1}$$

so $\mu(A) = 0$. On the other hand,

$$\nu(X \setminus A) \leq \sum_{r=1}^{\infty} \nu(\bigcap_{k=r}^{\infty} (X \setminus A_{n_k}))$$

and, for each r,

$$\nu(\bigcap_{k=r}^{\infty}(X \setminus A_{n_k})) = \lim_{N\to\infty} \nu(\bigcap_{k=r}^{N}(X \setminus A_{n_k})) \leq \lim_{N\to\infty} \nu(X \setminus A_N) = 0$$

Thus $\nu(X \setminus A) = 0$, as claimed. Since A is obviously measurable, μ and ν are mutually singular.

(b): Writing $A_n = f_n^{-1}([1, \infty))$,

$$\mu(A_n) = \int 1_{A_n} d\mu \leq \int f_n d\mu \to 0$$

and

$$\nu(X \setminus A_n) = \int 1_{X\setminus A_n} d\nu \leq \int \frac{1}{f_n} d\nu \to 0$$

so by (a) we are done.

□

Fall 2020 Problem 5: Suppose $f \in L^1([0, 1])$ has the property that

$$\int_E |f(x)|dx \leq \sqrt{\lambda(E)}, \tag{3.1}$$

for every Borel $E \subseteq [0, 1]$.

(a) Show that $f \in L^p([0, 1])$ for all $p < 2$.

(b) Give an example of an f satisfying 3.1 that is not in $L^2([0, 1])$.

Proof. (a): Recall the layer-cake decomposition

$$\begin{aligned}
\int_{[0,1]} |f(x)|^p dx &= \int_{[0,1]} \int_0^{|f(x)|} pt^{p-1} dt dx \\
&= p \int_0^{\infty} t^{p-1} \int_{x\in[0,1]:|f(x)|\geq t} dx dt \\
&= p \int_0^{\infty} t^{p-1} \lambda(\{x \in [0, 1] : |f(x)| \geq t\}) dt \\
&= p \int_0^{\infty} t^{p} \lambda(\{x \in [0, 1] : |f(x)| \geq t\}) \frac{dt}{t}
\end{aligned}$$

From now on, we abbreviate $U_t = \{x \in [0, 1] : |f(x)| \geq t\}$. For each $t \geq 0$ we have $t1_{U_t}(x) \leq |f(x)|1_{U_t}(x)$. Thus,

$$\begin{aligned}
t\lambda(U_t) &= \int t1_{U_t}(x)dx \\
&\leq \int_{U_t} |f(x)|dx \\
&\leq \lambda(U_t)^{1/2}
\end{aligned}$$

i.e.

$$t\lambda(U_t)^{1/2} \leq 1$$

If we choose some $q \in (2p-2, 2)$, then the preceding implies the inequality

$$t^q \lambda(U_t)^{\frac{q}{2}} \leq 1$$

We now use the preceding to control the L^p norm. Indeed,

$$\begin{aligned}
\int_1^\infty t^p \lambda(U_t) \frac{dt}{t} &= \int_1^\infty t^{p-1-\frac{q}{2}} \cdot t^{1-\frac{q}{2}} \lambda(U_t)^{1-\frac{q}{2}} \cdot t^q \lambda(U_t)^{\frac{q}{2}} \frac{dt}{t} \\
&\leq \int_1^\infty t^{p-1-\frac{q}{2}} \cdot t^{1-\frac{q}{2}} \lambda(U_t)^{1-\frac{q}{2}} \frac{dt}{t} \\
&\leq \left(\int_1^\infty t^{\frac{2p-2-q}{q}} \frac{dt}{t} \right)^{\frac{q}{2}} \left(\int_0^\infty t\lambda(U_t) \frac{dt}{t} \right)^{1-\frac{q}{2}}
\end{aligned}$$

where we have used the fact that $q < 2$, so $1 - \frac{q}{2} > 0$. Note that $2p - 2 - q < 0$, so

$$\left(\int_1^\infty t^{\frac{2p-2-q}{q}} \frac{dt}{t} \right)^{\frac{q}{2}} < \infty$$

Since the second factor in the preceding display was just $\|f\|_1^{1-\frac{q}{2}} < \infty$, we conclude that

$$\int_1^\infty t^p \lambda(U_t) \frac{dt}{t} < \infty$$

But of course the remaining portion of the integral is finite, viz,

$$\int_0^1 t^p \lambda(U_t) \frac{dt}{t} = \int_0^1 t^{p-1} \lambda(U_t) dt \leq 1 < \infty$$

so we conclude by the layer-cake decomposition that $\int |f|^p < \infty$, i.e. $f \in L^p([0,1])$.

We present as well a dyadic decomposition argument. For $n \in \mathbb{Z}$, write $L_n = \{x \in [0,1] : 2^n \leq |f(x)| < 2^{n+1}\}$. Then we have

$$2^n \lambda(L_n) \leq \int_{L_n} |f(x)| dx \leq \lambda(L_n)^{1/2}$$

so that $2^n \lambda(L_n)^{1/2} \leq 1$. Let $q \in (2p-2, 2)$; then $2^{qn} \lambda(L_n)^{\frac{q}{2}} \leq 1$ as well. We may also write

$$\|f\|_1 \geq \sum_{n \in \mathbb{Z}} 2^n \lambda(L_n)$$

so that the right-hand side is finite.

Finally, we compute:

$$\begin{aligned}
\int_{[0,1]} |f(x)|^p dx &\leq 2^p \sum_{n \in \mathbb{Z}} 2^{np} \lambda(L_n) \\
&\leq \frac{2^{2p}}{2^p - 1} + 2^p \sum_{n \geq 1} 2^{n(p-1-\frac{q}{2})} \cdot 2^{n(1-\frac{q}{2})} \lambda(L_n)^{1-\frac{q}{2}} \cdot 2^{qn} \lambda(L_n)^{\frac{q}{2}} \\
&\leq \frac{2^{2p}}{2^p - 1} + 2^p \left(\sum_{n \geq 1} 2^{\frac{n(2p-2-q)}{q}} \right)^{\frac{q}{2}} \left(\sum_{n \geq 1} 2^n \lambda(L_n) \right)^{1-\frac{q}{2}}
\end{aligned}$$

Observe that the first series in the latter display is a geometric series with common ratio in $(0,1)$, hence converges. The second series in the latter display is finite, by the previous comparison to $\|f\|_1$. Thus, $f \in L^p([0,1])$, as claimed.

(b): Let $f(x) = \frac{1}{4\sqrt{x}}$. Note that $f \in L^1$ but not L^2, so it remains to show that f satisfies 3.1. Suppose $E \subseteq [0,1]$ is Borel. We will write $|E| = \lambda(E)$. Then

$$\begin{aligned}
\int_E |f(x)|dx &= \int_{E\cap[0,|E|]} |f(x)|dx + \int_{E\cap(|E|,1]} |f(x)|dx \\
&\leq \int_0^{|E|} \frac{1}{4\sqrt{x}}dx + \frac{|E|}{4\sqrt{|E|}} \\
&= \frac{1}{2}\sqrt{|E|} + \frac{1}{4}\sqrt{|E|} \leq \sqrt{|E|}
\end{aligned}$$

as claimed. □

Spring 2017 Problem 5: Let $d\mu$ be a finite complex Borel measure on $[0,1]$ such that

$$\hat{\mu}(n) = \int_0^1 e^{2\pi inx} d\mu(x) \to 0 \quad \text{as} \quad n \to \infty$$

Let $f \in L^1(|\mu|)$ and $d\nu = f d\mu$. Show that

$$\hat{\nu}(n) \to 0 \quad \text{as} \quad n \to \infty$$

Proof. It suffices to assume $\|\mu\| > 0$. Suppose first $f(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx}$ for some $N \in \mathbb{N}$ and complex numbers $a_{-N}, \dots, a_N$. Then

$$\hat{\nu}(n) = \sum_{k=-N}^{N} a_k \int e^{2\pi i(k+n)x} dx = \sum_{k=-N}^{N} a_k \hat{\mu}(k+n) \to 0, \quad \text{as} \quad n \to \infty$$

We now weaken the assumption to $f \in C([0,1];\mathbb{C})$. Fix $\varepsilon > 0$ arbitrary and, by Stone-Weierstrass, let $g(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx}$ be a trigonometric polynomial such that $\|f - g\|_{L^\infty([0,1])} < \frac{\varepsilon}{\|\mu\|}$. Then

$$\limsup_{n\to\infty} |\hat{\nu}(n)| \leq \limsup_{n\to\infty} \int_0^1 |f(x) - g(x)| d|\mu|(x) + \limsup_{n\to\infty} \left| \int e^{2\pi inx} g(x) d\mu(x) \right| \leq \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we see that $\limsup_n |\hat{\nu}(n)| \leq 0$, i.e. $\hat{\nu} \to 0$. Thus we are done in this case.

Finally, note that $|\mu|$ is a Radon measure, so continuous functions of compact support are dense in $L^1(|\mu|)$. Let $f \in L^1(|\mu|)$ be arbitrary and $g \in C([0,1];\mathbb{C})$ be such that $\|f - g\|_{L^1(\mu)} < \varepsilon$. Then

$$\limsup_{n\to\infty} |\hat{\nu}(n)| \leq \limsup_{n\to\infty} \int_0^1 |f(x) - g(x)| d|\mu|(x) + \limsup_{n\to\infty} \left| \int_0^1 g(x) e^{2\pi inx} d\mu(x) \right| \leq \varepsilon$$

and since $\varepsilon > 0$ was arbitrary we are done. □

Hints and remarks regarding the preceding problems.

Spring 2021 Problem 2.

Hint for (a): you will need suitable "limits" of the sets A_n. You may find it useful to pass to a subsequence A_{n_k} for which the measures limit to zero sufficiently fast.

Hint for (b): consider suitable superlevel sets of the f_n.

Remark. If μ is Lebesgue measure on $[0,1]$ and ν is Cantor measure, then you may take A_n to be the indicators of the intervals remaining in the n-th stage of the construction of the Cantor set. In the language of the f_n's, you might take $f_n = (2+\varepsilon)^n 1_{A_n}$.

Fall 2020 Problem 5.

Hint for (a), part 1: one approach is to decompose $f(x) = \sum_{n\in\mathbb{Z}} f(x) 1_{|f(x)|\in[2^n,2^{n+1})}$. Another is to use the classical "layer-cake" decomposition of f; to derive the latter, you'll need to use Fubini in a very clever way.

Hint for (a), part 2: Apply the hypotheses to sets of the form $\{x \in [0,1] : |f(x)| \in [2^n, 2^{n+1})\}$. This will give you a useful "exponent-lowering" inequality for the measures of such sets.

Hint for (a), part 3: Raise the preceding inequality to a power of the form $2-\varepsilon$ (or $1-\varepsilon$, depending on how you formulate things) so as to overwhelm the exponent $p < 2$. You will actually need $2-\varepsilon > 2p-2$. You will need to use Hölder, together with an alternate expression for $\|f\|_1$.

Hint for (b): use the standard example of a function in L^p for all $p < 2$, that is not in L^2. For a given E, break E into the part near the singularity and the part away from it.

Remark. The inequality 3.1 establishes the statement "$f \in L^{2,\infty}([0,1])$." The latter is the so-called "weak L^2," a special case of the Lorentz spaces $L^{p,q}$. The conclusion of the problem is that $f \in L^1 \cap L^{2,\infty} \implies f \in L^p$ for all $1 < p < 2$. The latter would also hold with L^2 in place of $L^{2,\infty}$, but the former is a milder condition.

In fact, one major reason for the study of spaces like $L^{2,\infty}$ is that they do just as well as classical Lebesgue spaces for the purpose of "interpolation" (read: statements like the implication in the last paragraph), while often being easier to establish.

Spring 2017 Problem 5.

Hint: use approximation theorems to bootstrap simple f to complicated f. As your base case, try trigonometric polynomials.

Remarks. The condition "$d\nu = f d\mu$ for some $f \in L^1(|\mu|)$" is just the condition that "ν is absolutely continuous with respect to μ," which is usually defined as the implication "$\mu(N) = 0 \implies \nu(N) = 0$." The equivalence of these definition is the content of the Radon-Nikodym theorem.

Note, by a standard calculation, that the hypothesis holds for $\mu = \lambda|_{[0,1]}$. Thus, $\hat{\nu}(n) \to 0$ for all measure ν which are absolutely continuous with respect to μ. The converse does not hold; there are measures ν which are mutually singular with respect to Lebesgue measure for which the stated decay holds (indeed, this is one of your TA's research areas); however, there do exist *partial* converses. Indeed, note the following:

- If μ is a "pure-point measure" (i.e. it is a sum of δ-masses), then $\hat{\mu}$ never decays.
- If μ is our Cantor measure, then (up to a constant) $|\hat{\mu}(\xi)| = \left|\prod_{k=1}^{\infty} \cos(\frac{\pi}{3^k}\xi)\right|$ (in the sense of pointwise limit). In particular, for $\xi = 3^n$ an integer power of 3, then the first $k \leq n$ factors are all 1, and for $k > n$ the factor is approximately $1 - \frac{\pi^2}{2 \cdot 3^{2k-2n}}$, which when multiplied together results in a quantity bounded uniformly away from 0.
- Fourier decay is frequently useful to assert regularity estimates on various approximations f to μ, i.e. thinking of μ as a limit of expressions of the form $f d\lambda$; if the f are all sufficiently regular, then any limit will retain some regularity, which will imply absolute continuity. Indeed, regularity (say, C^1) implies that f varies slowly over small intervals, so f cannot look like $(3/2)^n 1_{[0,3^{-n}]}$ (as in Cantor measure).

4 : Week 4

Some non-qual material:

Zorn's lemma: Suppose $(P, \leq)$ is a nonempty partially-ordered set, such that any $C \subseteq P$ with the property that $(C, \leq)$ is totally-ordered (i.e. is a chain), has the property that there exists some $p \in P$ with $c \leq p$ for all $c \in C$. Then there is some $m \in P$ such that $p \geq m \implies p = m$ for all $p \in P$, (i.e. m is maximal).

The slogan version of Zorn's lemma is, "in a partially-ordered set, if every chain has an upper-bound, then there exists a maximal element."

In ZF, it turns out that Zorn's lemma is equivalent to the axiom of choice. So, since we take choice for granted in measure theory, we will also take Zorn's lemma for granted.

Here is the standard application of Zorn's lemma:

Theorem 4.1. Every vector space has a basis.

Proof. We will not need any assumptions on the underlying field or the vector space. Let V be an arbitrary vector space over an arbitrary field $\mathbb{K}$. Let P be the set of all linearly-independent subsets of V, ordered with respect to inclusion. We verify that P satisfies the hypothesis of Zorn's lemma.

First, observe that $\emptyset \subseteq V$ is always linearly-independent, so $\emptyset \in P$. Thus P is nonempty.

Now, let C be any chain of linearly-independent subsets of V. Take $L = \bigcup_{c \in C} c$ to be the union of elements of C, i.e. the collection of vectors $v \in V$ such that $v \in c$ for some $c \in C$. Observe that $L \subseteq V$, and that L is linearly-independent. Indeed, take $v_1, \ldots, v_n \in L$ to be arbitrary; it will suffice to show that they are linearly-independent. We may find $c_1, \ldots, c_n \in C$ such that $v_j \in c_j$ for each j. Since C is totally ordered and $\{c_1, \ldots, c_n\}$ is finite, we may find $1 \leq k \leq n$ such that $c_j \subseteq c_k$ for each $1 \leq j \leq n$. Consequently, $v_1, \ldots, v_n \in c_k$. Thus $v_1, \ldots, v_n$ are linearly-independent, as claimed; hence we conclude that $L \in P$.

Finally, we observe that L is an upper bound for C; indeed, each $c \in C$ has $c \subseteq L$ by the definition of L. Thus we have shown that every chain in P has an upper bound.

Thus Zorn's lemma implies that there is some $m \in P$ such that no other $p \in P$ has $p > m$. If $\text{span}(m) \neq V$, then we may find $v \in V \setminus \text{span}(m)$. But then, by elementary linear algebra, we conclude that v is linearly-independent of m, i.e. $\{v\} \cup m \in P$, contradicting our assumption on m. Thus m spans V, so m is a linearly-independent spanning set for V, i.e. m is a basis. $\square$

Bounded linear operators.

Recall that, for $(V, \| \|_1)$ and $(W, \| \|_2)$ normed vector spaces, we say that a linear map $T : V \to W$ is bounded if there exists some $C > 0$ such that

$$\|Tx\|_2 \leq C\|x\|_1, \quad \forall x \in V$$

The least such C is called the *operator norm* of T (with respect to the norms $\| \|_1, \| \|_2$). Recall also that $T : V \to W$ is continuous (in the norm topologies) if and only if it is bounded.

Theorem 4.2. (a) Let $\| \|_1, \| \|_2$ be two norms on $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that

$$C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \forall x \in \mathbb{R}^n$$

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear map, and equip $\mathbb{R}^n, \mathbb{R}^m$ with arbitrary norms. Then T is automatically bounded.

Proof. (a): It suffices to exhibit a particular norm, such that any other norm is equivalent to that norm. So, we change notation and let $\| \|_\infty$ be the max norm ($\|(x_1, \ldots, x_n)\|_\infty = \max_{1\leq j\leq n} |x_j|$) and let $\| \|$ be any other norm. Write $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Then, for each $x \in \mathbb{R}^n$,

$$\|x\| \leq \sum_{j=1}^{n} |x_j|\|e_j\| \leq \|x\|_\infty \sum_{j=1}^{n} \|e_j\|$$

so that $\sum_{j=1}^n \|e_j\|$ is a suitable constant for one of the inequalities.

Next, observe that in particular we have shown that $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function (since the topology on $\mathbb{R}^n$ induced by the sup-norm is clearly the usual topology). Thus, $\| \cdot \|$ achieves a minimum on the compact set $\{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$, say c. Since each element of the latter set is nonzero, we conclude that $c > 0$. Then, for any $x \in \mathbb{R}^n$ nonzero and $\lambda = \|x\|_\infty$,

$$\|x\| = |\lambda|\|\lambda^{-1}x\| \geq c|\lambda| = c\|x\|_\infty$$

so that

$$c\|x\|_\infty \leq \|x\| \leq \left(\sum_{j=1}^{n} \|e_j\|\right) \|x\|_\infty$$

so we are done if we take $C = \max(c^{-1}, \sum_{j=1}^n \|e_j\|)$.

(b): By (a) and basic topology, it suffices to show the conclusion in the case $m = 1$, and both spaces have the standard Euclidean norm. By elementary linear algebra, T is necessarily of the form $v \mapsto v \cdot w$ for a fixed $w \in \mathbb{R}^n$. By Cauchy-Schwarz,

$$|Tv| \leq \|v\| \cdot \|w\|$$

so T is bounded with constant $\|w\|$. □

Remark. One can further show that the Euclidean topology is in fact the unique Hausdorff topology on $\mathbb{R}^n$ ($n \in \mathbb{N}$), such that addition $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and scalar multiplication $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

Spring 2022 Problem 1: Given a finite (positive) Borel measure μ on $\mathbb{R}$, its support is the set

$$\text{spt}(\mu) = \{x \in \mathbb{R} : \mu((x-\varepsilon, x+\varepsilon)) > 0 \text{ for every } \varepsilon > 0\}$$

(a) Prove that $\text{spt}(\mu)$ is closed, that $\mu(\mathbb{R} \setminus \text{spt}(\mu)) = 0$, and that any other set with these two properties must contain $\text{spt}(\mu)$.

(b) Prove that there is a finite Borel measure μ on $\mathbb{R}$ such that

(i) μ has support equal to $\mathbb{R}$;

(ii) μ and Lebesgue measure are mutually singular.

Proof. (a): Let $y \in \mathbb{R} \setminus \text{spt}(\mu)$. Then for some $\varepsilon > 0$, we have $\mu((y-\varepsilon, y+\varepsilon)) = 0$. But then any other $z \in (y-\varepsilon, y+\varepsilon)$ has $z \in \mathbb{R} \setminus \text{spt}(\mu)$, so $\mathbb{R} \setminus \text{spt}(\mu)$ is open, i.e. $\text{spt}(\mu)$ is closed.

Note that μ is a Radon measure, so is inner regular. Thus, to show $\mu(\mathbb{R} \setminus \text{spt}(\mu)) = 0$, it suffices to show that $\mu(K) = 0$ for any compact $K \subseteq \mathbb{R} \setminus \text{spt}(\mu)$. So, fix some compact K as indicated. Let $\{I_x\}_{x \in K}$ be a collection of nonempty open intervals such that I_x is centered at x and $\mu(I_x) = 0$, as guaranteed by the fact that $K \cap \text{spt}(\mu) = \emptyset$. By compactness, there are $x_1, \ldots, x_n \in K$ such that $K \subseteq I_{x_1} \cup \cdots \cup I_{x_n}$. But then

$$\mu(K) \leq \sum_{j=1}^{n} \mu(I_{x_j}) = 0$$

and we are done.

Finally, we show that any closed set S with the property $\mu(\mathbb{R} \setminus S) = 0$, must contain $\text{spt}(\mu)$. It suffices to fix some $x \in \mathbb{R}$ such that all $\varepsilon > 0$ have $\mu((x-\varepsilon, x+\varepsilon)) > 0$, and show that $x \in S$. So, fix such an x. If $x \notin S$, then since S is closed we may find $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap S = \emptyset$. But then

$$0 = \mu(\mathbb{R} \setminus S) \geq \mu((x-\varepsilon, x+\varepsilon)) > 0$$

a contradiction. Thus $x \in S$, and we are done.

(b): Fix an enumeration $\{q_n\}_{n=1}^{\infty}$ of $\mathbb{Q}$. Define

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{q_n}$$

Then $\mu(U) > 0$ for every nonempty open set U, so $\text{spt}(\mu) = \mathbb{R}$. Note that $\mu(\mathbb{R} \setminus \mathbb{Q}) = 0$ and $m(\mathbb{Q}) = 0$ (where m is Lebesgue measure), so m and μ are mutually singular. Lastly, $\mu(\mathbb{R}) = \sum_n 2^{-n} = 1 < \infty$, so we are done. $\square$

Spring 2022 Problem 5 (modified): Let μ be a Borel measure on $\mathbb{R}^2$, and assume it has the following property: for every fixed $r > 0$, the quantity $\mu(B(x, r))$ is finite and independent of x, where $B(x, r)$ is the open ball of radius r around x.

(a) Prove that there is a finite constant c such that $\mu(B(x,r)) \leq cr^2$ whenever $0 < r \leq 1$.

(b) Prove that μ is absolutely continuous with respect to Lebesgue measure.

Proof. (a): We argue by geometric considerations. For each $0 < r \leq 1$, let c_r be the unique constant such that $\mu(B(x,r)) = c_r r^2$ for each $x \in \mathbb{R}^2$. We note that, for $\lambda \geq 10$, there are $\geq \frac{1}{32}\lambda^2$ disjoint disks of radius r/λ that fit in any disk of radius r; this may be seen by considering the grid of points with separation r/λ, centered at the center of the large disk, and considering a square inscribed in the large disk; by adding a small disk at every other point in the grid in the square, we get $\frac{1}{32}\lambda^2$ small disks in the large disk, as claimed.

Consequently,

$$c_r r^2 = \mu(B(x,r)) \geq \frac{1}{32}\lambda^2 c_{r/\lambda}(\frac{r}{\lambda})^2 = \frac{1}{32}c_{r/\lambda}r^2$$

so

$$c_r \geq \frac{1}{32}c_{r/\lambda} \quad \forall 0 < r \leq 1, \lambda \geq 10$$

To finish, note that $B(x,r)$ can be covered by $\leq 4\lambda^2$ disks $B(x', r/\lambda)$ for *all* $\lambda \geq 1$, so

$$c_r r^2 \leq 4\lambda^2 c_{r/\lambda}(\frac{r}{\lambda})^2$$

and

$$c_r \leq 4c_{r/\lambda}$$

for all $\lambda \geq 10$. Thus, for any $0 < r \leq 1$, since $\frac{10}{r} \geq 10$,

$$c_r \leq 4c_{\frac{r}{10}} \leq 128c_1$$

so by setting $c = 128c_1$ we get

$$\mu(B(x,r)) = c_r r^2 \leq cr^2$$

for all $0 < r \leq 1$, as desired.

(b): First, we claim that μ is absolutely continuous with respect to Lebesgue measure m. To demonstrate this, suppose N is Borel and has measure 0. By the definition of m, for each $\varepsilon > 0$ there is a sequence of open squares Q_i such that $\sum_i m(Q_i) < \frac{\varepsilon}{2}$ and $N \subseteq \bigcup_i Q_i$. Writing B_i for circumscribed ball about Q_i, notice that $m(B_i) = \frac{\pi}{2}m(Q_i) < 2m(Q_i)$, so $\sum_i m(B_i) < \varepsilon$. Thus

$$\mu(N) \leq \sum_i \mu(B_i) \leq \tilde{c}\varepsilon$$

where $\tilde{c} = \frac{c}{\pi}$. Sending $\varepsilon \to 0$, we see $\mu(N) = 0$ as claimed, so indeed μ is absolutely continuous with respect to m.

Thus we may write $d\mu = f dm$ for some nonnegative locally integrable Borel function f. By the assumption, the average of f on $B(x,r)$ is independent of x. If f is nonconstant, then there is some $\varepsilon > 0$ and positive measure sets A, B such that $\sup_{x\in A} f(x) + \varepsilon < \inf_{x\in B} f(x)$. By Lebesgue differentiation, a.e. point of A (resp. B) is a Lebesgue point for A (resp. for B). Consequently, we may find some $x \in A, y \in B$ and $r > 0$ such that $\mu(B(x,r)) < \mu(B(y,r))$, contradicting our assumption. Thus f is constant a.e., so μ is a constant multiple of Lebesgue measure. $\square$

Fall 2009 Problem 5 (with added scaffolding): Construct a Borel subset E of the real line $\mathbb{R}$ such that for all intervals $[a, b]$ we have

$$0 < \lambda(E \cap [a, b]) < |b - a|$$

where λ denotes Lebesgue measure, by the following procedure:

(a) By modifying the construction of the standard Cantor set, show that there exists a compact totally disconnected subset of $\mathbb{R}$ with positive measure.

(b) Let $\mathscr{C}$ denote the family of compact totally disconnected subsets of $\mathbb{R}$ with positive measure. Show that, if I is any nonempty open interval, there exist $A, B \in \mathscr{C}$ such that $A \cap B = \emptyset$ and $A \cup B \subseteq I$.

(c) Let $\{I_n\}_n$ be an enumeration of the nonempty open intervals in $\mathbb{R}$ with rational endpoints. Show that there exist sequences $\{A_n\}_n, \{B_n\}_n$ in $\mathscr{C}$ such that $A_n \cup B_n \subseteq I_n$, $(A_n \cup B_n) \cap \bigcup_{j<n}(A_j \cup B_j) = \emptyset$.

(d) Show that $A = \bigcup_n A_n$ has the desired property.

Proof. (a): Repeat the Cantor-set construction, but in the nth stage remove an interval of length $(3n)^{-n}$. The resulting set C will be compact and totally disconnected, whereas $[0, 1] \setminus C$ will have measure at most $\sum_n 2^n(3n)^{-n} < 1$.

(b): Let $I_1, I_2 \subseteq I$ be disjoint nonempty open intervals. By rescaling and shifting any element $C \in \mathscr{C}$, we obtain $A, B \in \mathscr{C}$ with $A \subseteq I_1, B \subseteq I_2$. This clearly suffices.

(c): Suppose we have defined $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathscr{C}$ such that $A_j \cup B_j \subseteq I_j$ and $(A_j \cup B_j) \cap \bigcup_{k<j}(A_k \cup B_k) = \emptyset$ for each $j \leq n$. Then $I_{n+1} \setminus \bigcup_{j \leq n}(A_j \cup B_j)$ is nonempty and open, hence contains a nonempty open interval J. But then we may apply (b) to J to construct A_{n+1}, B_{n+1}.

(d): A is clearly Borel. If I is any nonempty interval, then we may pick n such that $I_n \subseteq I$. Then

$$0 < \lambda(A_n) \leq \lambda(I \cap A) < \lambda(I \cap A) + \lambda(B_n) \leq \lambda(I)$$

and we are done. □

[Bonus problem] Spring 2013 Problem 1: Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, Lebesgue measurable, and

$$\lim_{h \to 0} \int_0^1 \frac{|f(x+h) - f(x)|}{h} dx = 0$$

Show that f is a.e. constant on the interval $[0, 1]$.

Proof. It suffices to demonstrate that, for all $\varepsilon > 0$,

$$\iint_{[0,1]^2} |f(y) - f(x)| dx dy < \varepsilon$$

Let h_0 be such that the given integral is $< \varepsilon$ for any $0 < h \le h_0$. Fix any $y \in [0, 1]$. Then, if $0 \le x < y$, if $h_0 \ge h > 0$ and $N = N_h = \frac{y-x}{h} \le \frac{1}{h}$,

$$|f(y) - f(x)| \le \sum_{n=1}^{N_h} |f(x + nh) - f(x + (n-1)h)|$$

so that

$$\int_0^y |f(y) - f(x)| dx \le \sum_{n=1}^{N_h} \int_0^y |f(x + nh) - f(x + (n-1)h)| dx$$

When $0 \le x < y$, $(n-1)h \le x + (n-1)h < y$, so certainly

$$\int_0^y |f(x + nh) - f(x + (n-1)h)| dx \le \int_0^1 |f(x + h) - f(x)| dx < \varepsilon h$$

which implies

$$\int_0^y |f(y) - f(x)| dx < \varepsilon h N_h \le \varepsilon$$

Since this holds for each y, we conclude

$$\int_0^1 \int_0^y |f(y) - f(x)| dx < \varepsilon$$

which is just the estimate

$$\iint_{[0,1]^2} |f(y) - f(x)| < 2\varepsilon$$

Taking $\varepsilon \to 0$, we're done. ∎

Hints and remarks about the preceding problems

Spring 2022 Problem 1:

Hint for (a): compactness and covering.

Hint for (b): use $\mathbb{Q}$.

Spring 2022 Problem 5:

Hint for (a): consider coverings of large disks by small disks, and use this to compare the different c.

Remark. Using Lebesgue differentiation, you can further show that μ is a constant multiple of Lebesgue measure. If you know the statement of Lebesgue differentiation, try to show this! This is the original formulation of (b).

Fall 2009 Problem 5:

Hint for part (a): arrange for the middle intervals being removed to go to zero sufficiently fast.

Hint for (c): What are the properties of $I_{n+1} \setminus \bigcup_{j \leq n}(A_j \cup B_j)$?

Remark. Why doesn't this violate Lebesgue differentiation?

Spring 2013 Problem 1 [Bonus problem]:

Hint, part 1: no covering lemmas are necessary.

Hint, part 2: how big is $|f(y) - f(x)|$ *on average?*

Remark: Our condition implies that, if f is differentiable a.e., then its derivative is 0 a.e. If the limit could be pushed inside the integral, then our condition would be equivalent to that statement. On the other hand, there exists a continuous function f which increases monotonically on $[0, 1]$ from $f(0) = 0$ to $f(1) = 1$ for which $f' = 0$ a.e., i.e. the Cantor function. Conclude that our condition is strictly stronger than the condition that $f' = 0$ a.e.

5 : Week 5

Fall 2019 Problem 6 Recall that $\ell^\infty(\mathbb{N}) = \{x = \{x_n\}_{n\geq 1} : \sup_{n\geq 1}|x_n| < \infty\}$ is a Banach space with respect to the norm $\|x\|_\infty = \sup_{n\geq 1}|x_n|$.

(a) Prove that there exists a continuous linear functional ϕ on $\ell^\infty(\mathbb{N})$ such that

$$\phi(x) = \lim_{n\to\infty} x_n$$

whenever the limit exists.

(b) Prove that ϕ is not unique.

Proof. (a): Let $L \subseteq \ell^\infty(\mathbb{N})$ be the subset of $\ell^\infty(\mathbb{N})$ consisting of elements x for which $\lim_n x_n$ exists. Observe that L is a linear subspace. Write ϕ_0 for the map $L \to \mathbb{R}$, $\phi_0(x) = \lim_n x_n$. One can note that ϕ_0 is certainly linear. Then, for any $x \in L$,

$$|\lim_n x_n| \leq \lim_n |x_n| \leq \limsup_n |x_n| \leq \sup_{n\geq 1}|x_n|$$

so that ϕ_0 satisfies the bound $|\phi_0(x)| \leq \|x\|_\infty$ for all $x \in L$. By Hahn-Banach, there exists a linear map $\phi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ such that $|\phi(x)| \leq \|x\|_\infty$ for all $x \in \ell^\infty$, and such that $\phi|_L = \phi_0$ Then ϕ is a continuous linear map such that

$$\phi(x) = \lim_{n\to\infty} x_n$$

whenever $x \in L$, i.e. whenever the limit exists.

(b): Write L_1 for the linear subspace of $\ell^\infty(\mathbb{N})$ spanned by L and $b = \{b_n\}_{n\geq 1}$, with $b_n = (-1)^n$. Then, for any $x \in L_1$, there by definition exists a scalar α and $y \in L$ such that

$$x = \alpha b + y$$

Then observe that, since $\{y_n\}_{n=1}^\infty$ converges, from the identity

$$x_{n+1} - x_n = 2(-1)^{n+1}\alpha + (y_{n+1} - y_n)$$

we in particular have

$$\frac{1}{2}\lim_{k\to\infty}(x_{2k+2} - x_{2k+1}) = \alpha \tag{5.1}$$

and

$$y_n = x_n - (-1)^n \cdot \frac{1}{2}\lim_{k\to\infty}(x_{2k+2} - x_{2k+1}) \tag{5.2}$$

Define linear maps $\phi_1, \phi_2 : L_1 \to \mathbb{R}$ via

$$\phi_1(\alpha b + y) = \alpha + \lim_{n\to\infty} y_n$$

$$\phi_2(\alpha b + y) = -\alpha + \lim_{n\to\infty} y_n$$

By (5.1) and (5.2), these are well-defined. They are also clearly linear. We wish to prove a bound that allows us to use Hahn-Banach again. To this end, write $y_\infty = \lim_n y_n$. Then

$$|\alpha| + |y_\infty| = \limsup_{n\to\infty}|(-1)^n\alpha + y_n| \leq \|\alpha b + y\|_\infty$$

so that, for $i = 1, 2$, we have

$$|\phi_i(\alpha b + y)| \leq |\alpha| + |y_\infty| \leq \|\alpha b + y\|_\infty$$

Thus ϕ_1, ϕ_2 both extend to bounded linear maps $\ell^\infty(\mathbb{N}) \to \mathbb{R}$ that extend the linear functional on L. Since they disagree on the element b, we conclude that the extension in part (a) is not unique.

$\square$

Spring 2019 Problem 4: Let $\mathcal{V}$ be the subspace of $L^\infty([0,1], m)$ (where m is Lebesgue measure) defined by

$$\mathcal{V} = \{f \in L^\infty([0,1], \mu) : \lim_{n\to\infty} n \int_{[0,1/n]} f dm \text{ exists}\}$$

(a) Prove that there exists $\varphi \in L^\infty([0,1], m)^*$ (i.e. a continuous linear functional $L^\infty([0,1], m) \to \mathbb{R}$) such that $\varphi(f) = \lim_{n\to\infty} n \int_{[0,1/n]} f dm$ for every $f \in \mathcal{V}$.

(b) Show that, given any $\varphi \in L^\infty([0,1], m)^*$ satisfying the condition in (a), there exists no $g \in L^1([0,1], m)$ such that $\varphi(f) = \int f g dm$ for all $f \in L^\infty([0,1], m)$.

Proof. (a): Note first that, for each n,

$$|n \int_{[0,1/n]} f dm| \leq \|f\|_\infty \times n \int_{[0,1/n]} dm = \|f\|_\infty$$

so the linear map $\varphi_0 : \mathcal{V} \to \mathbb{R}$, $\varphi_0(f) = \lim_{n\to\infty} n \int_{[0,1/n]} f dm$ satisfies the bound

$$|\varphi_0(f)| \leq \sup_n \|f\|_\infty = \|f\|_\infty$$

By Hahn-Banach, there exists $\varphi : L^\infty([0,1], m) \to \mathbb{R}$ linear with norm bounded by 1 such that $\varphi|_{\mathcal{V}} = \varphi_0$, as was to be shown.

(b): Suppose to the contrary that $g \in L^1([0,1], m)$ is such that, for any $f \in L^\infty([0,1], m)$,

$$\varphi(f) = \int f g dm$$

where φ is as in (a). In particular, testing against $f = 1 \in \mathcal{V}$,

$$1 = \varphi(1) = \int g dm$$

so certainly $\|g\|_1 \geq 1$. On the other hand, for any $\varepsilon > 0$ and any $f \in L^\infty([0,1], m)$ such that $f|_{[0,\varepsilon]} \equiv 0$, we have

$$0 = \varphi(f) = \int f g dm$$

In particular, $g|_{(\varepsilon,1]} \equiv 0$ a.e. Since $\varepsilon > 0$ was arbitrary, we conclude that g has essential support contained in $\{0\}$, i.e. $g \equiv 0$ as an element of $L^1([0,1], m)$. But this contradicts the estimate $\|g\|_1 \geq 1$ from earlier, and we're done. $\square$

Spring 2021 Problem 5: Let $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ be such that the series

$$\sum_{i=1}^{\infty} x_i y_i$$

converges for all $y \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n y_n = 0$. Show that the series $\sum_{i=1}^{\infty} |x_i|$ converges.

Proof. Write $c_0 \subseteq \ell^{\infty}(\mathbb{N})$ for the space of $y \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n y_n = 0$. We show the contrapositive: if $\mathbf{x} \notin \ell^1(\mathbb{N})$, then there exists a $y \in c_0$ such that $\sum_{i \geq 1} x_i y_i$ fails to converge.

Assuming that hypothesis on $\mathbf{x}$, we may find $1 = n_1 < n_2 < \ldots$ an infinite sequence of indices in $\mathbb{N}$ such that $\sum_{i=n_k}^{n_{k+1}-1} |x_i| \geq 1$. Define then y by

$$y_i = \operatorname{sign}(x_i) \cdot \frac{1}{k}, \quad \text{when } n_k \leq i < n_{k+1}$$

Then clearly $y_i \in c_0$, and

$$\sum_{i=1}^{\infty} y_i x_i = \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}-1} \operatorname{sign}(x_i) \cdot \frac{1}{k} \cdot x_i = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=n_k}^{n_{k+1}-1} |x_i| \geq \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

and we are done. $\square$

[Bonus problem]; taken from mathoverflow: Consider the Hilbert space $\ell^2(\mathbb{N})$, and consider a matrix $A = [a_{ij}]_{i,j}$, consisting of nonnegative entries, such that, for all $y \in \ell^2(\mathbb{N})$, the entries of the vector Ay all converge, and the vector Ay also belongs to $\ell^2(\mathbb{N})$. Show that A is a bounded linear map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$.

Important remark: we are not here claiming that every linear map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is bounded!
The statement is also true when the entries are assumed only to be real numbers.

Proof. Consider the linear maps T_N defined by the matrix given by the entries $(T_N)_{i,j} = a_{ij}$ if $1 \leq i, j \leq N$, 0 otherwise. Then each T_N is clearly a bounded linear map. Furthermore, for each $y \in \ell^2(\mathbb{N})$,

$$T_N(y) = (1_{1\leq i\leq N} \sum_{j=1}^{N} a_{ij}y_j)_{i\geq 1}$$

which satisfies

$$\|T_N(y)\|_{\ell^2}^2 = \sum_{i=1}^{N} |\sum_{j=1}^{N} a_{ij}y_j|^2 \leq \sum_{i=1}^{N} |\sum_{j=1}^{N} a_{ij}|y_j||^2 \leq \sum_{i=1}^{\infty} |\sum_{j=1}^{\infty} a_{ij}|y_j||^2$$

By assumption, writing $|y| = \{|y_n|\}_{n\geq 1}$, the vector $A|y| \in \ell^2(\mathbb{N})$, so the terminal expression above converges to a finite number. Thus, for each y,

$$\sup_N \|T_N(y)\|_{\ell^2} < \infty$$

so by the uniform boundedness principle we see that

$$\sup_N \|T_N\|_{\ell^2\to\ell^2} < \infty$$

On the other hand, for arbitrary $y \in \ell^2$,

$$\|T_N(y) - A(y)\|_{\ell^2}^2 = \sum_{i=1}^{N} |\sum_{j=N+1}^{\infty} a_{ij}y_j|^2 + \sum_{i=N+1}^{\infty} |\sum_{j=1}^{\infty} a_{ij}y_j|^2 = (I) + (II)$$

Considering (I), we have the estimate

$$\sum_{i=1}^{N} |\sum_{j=N+1}^{\infty} a_{ij}y_j|^2 \leq \sum_{i=1}^{\infty} |\sum_{j=N+1}^{\infty} a_{ij}|y_j||^2$$

Again, the latter series all converge; for each i, the quantity

$$|\sum_{j=N+1}^{\infty} a_{ij}|y_j||$$

is decreasing in N, and has limit 0 as $N \to \infty$. Thus the full sum

$$\sum_{i=1}^{\infty} |\sum_{j=N+1}^{\infty} a_{ij}|y_j||^2$$

is (a) convergent, (b) has summands over i that decrease to 0, hence (c) limits to zero by dominated convergence, i.e.

$$\lim_{N\to\infty}(I) = 0$$

Considering (II), we have assumed

$$\infty > \|Ay\|_{\ell^2}^2 = \sum_{i=1}^{\infty}|\sum_{j=1}^{\infty} a_{ij}y_j|^2$$

so by looking at the tail of the outside convergent sum we have

$$\lim_{N\to\infty}(II) = 0$$

Thus

$$\lim_{N\to\infty}\|T_N(y) - A(y)\|_{\ell^2}^2 = 0$$

and thus $T_N(y) \to A(y)$ for all $y \in \ell^2(\mathbb{N})$. Since the T_N are uniformly bounded in operator norm, we conclude that A is a bounded linear map, as was to be shown.

$\square$

Hints and remarks about the preceding problems

Fall 2019 Problem 6:

Hint for (a): Hahn-Banach.

Hint for (b): Assign ϕ competing values on some larger space.

Remark. one might describe this as a version of "generalizing the limit functional to all bounded sequences." Notice, however, that such a ϕ need not extend the properties $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ or $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$.

Spring 2019 Problem 4:

Hint for (a): Hahn-Banach.

Hint for (b): Inspect a putative g on intervals on the form $[\varepsilon, 1]$. On the other hand, try $f = 1$.

Remark. In part (b) we have verified that $L^\infty([0,1], m)^*$ is not equal to $L^1([0,1], m)$. In contrast, for any $1 \le p < \infty$, if we write $q = \frac{p}{p-1}$ then $(L^p([0,1], m))^* \simeq L^q([0,1], m)$.

Spring 2021 Problem 5:

Hint: uniform boundedness.

Remark. The conclusion of the problem is that every element of c_0^* can be regarded as an element of ℓ^1. On the other hand, each element of ℓ^1 clearly induces a bounded linear functional on c_0, and this mapping is faithful (i.e. $x \in \ell^1$ nonzero implies that the functional is nonzero). Consequently, $c_0^* \simeq \ell^1$ in the sense that c_0^*, ℓ^1 are isomorphic normed vector spaces.

By comparison, it turns out that $L^1([0,1])$ is not isomorphic to any dual of a Banach space (using some facts about convex sets in weak-$*$ topologies, e.g. Krein-Milman). As an immediate consequence, $L^1([0,1]) \not\simeq \ell^1(\mathbb{N})$.

Bonus problem:

Hint, part one: uniform boundedness.

Hint, part two: consider the various truncations of A given by zeroing out matrix entries outside of finite sub-matrices.

Remark. Observe for comparison that there do exist discontinuous linear maps between any infinite-dimensional normed vector spaces. Indeed, between any two vector spaces, there exists a linear transformation sending linearly independent elements to any prescribed values; in particular, if $\{e_n\}_{n\ge1}$ are linearly independent and w is nonzero, then there is a linear map for which $T(e_n) = nw$; this extends to a linear map on the whole space, and is clearly unbounded.

The reason we can reconcile this fact with the bonus problem is that an arbitrary linear map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ may have a matrix that does not define a bounded linear map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ (see the last paragraph).

Finally, we briefly suggest how to extend this to matrices A with signed entries. It would be tempting to attempt to demonstrate that, if A is a signed matrix defining a linear map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, then the matrix of absolute values does the same. Unfortunately, this fails; examples are a little tricky to write down, so we won't do that here. We instead need to adapt our argument directly. The most difficult part is to arrange for $A_N y$ to have uniformly bounded $\ell^2(\mathbb{N})$ norm for each fixed y. Almost the same method as above works, but instead of truncating the rows we instead keep the full rows and consider only finitely many rows (truncating the columns).

Appendix: Isomorphism problems for infinite-dimensional vector spaces

In functional analysis, there are a variety of infinite-dimensional vector spaces with metric/topological structures. In this short summary, we will use the language of category theory to efficiently communicate a variety of general, and often highly nontrivial, results about these spaces. In particular: we will consider various strengths of isomorphism between different spaces; roughly speaking, we are interested in the question "are $L^p(X)$ and $L^q(Y)$ the same space?"

To be more precise: one could spend a great deal of time studying Hilbert spaces, Banach spaces, normed vector spaces, or topological vector spaces (in order of increasing abstraction). Recall that:

- A Hilbert space is an inner-product vector space that is complete with respect to the norm induced by the inner product,
- a Banach space is a complete normed vector space,
- a normed vector space is a vector space equipped with a (1-homogeneous, faithful) norm, and
- a topological vector space is a vector space V equipped with a topology τ such that the maps $+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$ are continuous.

At this level of abstraction, we will only be considering maps which are continuous. For the sake of discussion, we will not only consider linear functions, though those will be the majority of the cases of interest. A function T between normed vector spaces is said to be:

- an isometry if $\|Tx\| = \|x\|$ for all x;
- bounded if there is a constant C such that $\|Tx - Ty\| \leq C\|x - y\|$ for all x, y;
- uniform if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that, for any x, y satisfying $\|x - y\| < \delta$, we have $\|Tx - Ty\| < \varepsilon$

If both the source and target of T are inner product spaces, then T is said to be unitary if T is a linear bijection and $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x, y.

Consider the following categories.

- Hilb, whose objects are Hilbert spaces and whose maps are unitaries
- Ban_{str}, whose objects are Banach spaces and whose maps are linear isometries
- Ban_{wk}[3], whose objects are Banach spaces and whose maps are bounded linear maps
- Ban_u,[4] whose objects are Banach spaces and whose maps are uniformly continuous functions
- TVS_{str}, whose objects are topological vector spaces and whose maps are continuous linear maps
- TVS_{wk}, whose objects are topological vector spaces and whose maps are continuous functions.

[3] This is what is usually referred to when one says "isomorphic Banach spaces"

[4] We could actually extend the adjective "uniform" to talk about arbitrary topological vector spaces, even though they don't carry metric information; the idea is that arbitrary open neighborhoods can be translated around to give a universal/uniform notion of smallness

Note carefully that the maps in Ban_u and TVS_{wk} need not be linear.

The corresponding notions of isomorphism are frequently called:

- *unitary isomorphism*, in Hilb,
- *isometric isomorphism*, in Ban_{str},
- *Banach space isomorphism*, or just *isomorphism*, in Ban_{wk},
- *uniformly homeomorphic*, in Ban_u,
- *linearly homeomorphic*, in TVS_{str},
- *homeomorphic*, in TVS_{wk}.

We now list the following facts:

Theorem 5.1. If B_1, B_2 are any two Banach spaces, then they are isomorphic in Ban_{wk} iff they are isomorphic in TVS_{str}.

Theorem 5.2. If B_1, B_2 are any two Banach spaces with the same density character[5], then they are isomorphic in TVS_{wk}.

Theorem 5.3. If $p \neq q \in [1, \infty]$ and (X, μ), (Y, ν) are two measure spaces for which $L^p(X, \mu)$ and $L^q(Y, \nu)$ are infinite-dimensional, then they are not isomorphic in Ban_u (hence neither in Ban_{wk}, Ban_{str}, nor TVS_{str}).

The preceding failure of isomorphism for distinct p, q may be realized in the following way. If Δ, Γ are nonempty sets and $1 \leq q < p < \infty$, then any continuous linear map $T : \ell^p(\Delta) \to \ell^q(\Gamma)$ takes the unit ball to a precompact set (i.e. T is a compact operator; this is known as Pitt's theorem). Observe that this is a dramatic obstruction to isomorphism in infinite-dimensional topology.

On the subject of sequence spaces, one classical example is the closed subspace $c_0 := \{\mathbf{x} \in \ell^\infty(\mathbb{N}) : \lim_n x_n = 0\}$. Since it is a closed subspace of a Banach space, it is a Banach space in its own right.

Theorem 5.4. c_0 is not isomorphic in Ban_{wk} to any $\ell^p(\mathbb{N}), 1 \leq p \leq \infty$.

Fixing a single exponent p, we have the following:

Theorem 5.5. If H_1 and H_2 are Hilbert spaces, then they are isomorphic in Hilb if and only if they have the same (Schauder/Hilbert)[6] dimension.

Theorem 5.6. $L^p(0,1)$ and $\ell^p(\mathbb{N})$ are not isomorphic in Ban_{wk} for any $p \neq 2, 1 \leq p < \infty$ (hence not isomorphic in Ban_{str} or TVS_{str}).

When $1 \leq p < 2$, we can say a little more:

Theorem 5.7. $L^p(0,1)$ and $\ell^p(\mathbb{N})$ are not isomorphic in Ban_u for any $1 \leq p < 2$.

[5] the minimal cardinality of a dense subset

[6] the cardinality of the minimal generating set, in the sense of linear span and metric closure

The analogous statement for $p > 2$ appears to be a long outstanding open problem.

The importance of the last facts comes from the fact that the spaces $(0, 1)$ and $\mathbb{N}$ (equipped with Lebesgue and counting measure, respectively) are the simplest versions of the very few ways that (sufficiently regular) measure spaces can be distinct. More precisely, as a consequence of a deep result known as *Maharam's theorem*, if $L^p(X)$ is separable, then it is isomorphic in Ban_{str} to a space of the form

$$\ell^p(D) \oplus_p L^p(0,1) \quad \text{or} \quad \ell^p(D)$$

where the p subscript indicates that the norm on the direct sum is given by taking the ℓ^p combinations of the inner norms; here D is understood to be a purely atomic measure space of cardinality $\leq \aleph_0$. As a consequence, the isomorphism problem for L^p spaces essentially reduces to comparisons between spaces like $(0, 1)$ and $\mathbb{N}$.

A special version of this arises when X is a non-discrete Polish space and μ is a σ-finite Borel measure, in which case the isometry $L^p(X, \mu) \to L^p([0, 1])$ is implemented by a Borel isomorphism[7] between the underlying measurable spaces.

The conclusion of the prior series of results is that the sort of spaces one normally considers in measure theory are usually homeomorphic, but usually not linearly homeomorphic. One slight oddity is the following:

Theorem 5.8. $L^\infty(0, 1)$ and $\ell^\infty(\mathbb{N})$ are isomorphic in Ban_{wk}.

This essentially reflects that L^∞ spaces are unable to witness the measures of subsets, which roughly reduces the problem of isomorphism to that of *distributive lattices*. Another way to look at this is that L^∞ spaces have the structure of algebras, in addition to Banach spaces.

We now consider a more delicate situation, that of the function spaces $C(X), C_p(X)$. In this context, we will generally be obtaining non-existence results, where we show that function spaces can only be isomorphic if the underlying spaces are sufficiently similar.

Given a compact Hausdorff topological space X, define the space $C(X)$ of continuous real-valued functions with the topology of uniform convergence; similarly, define $C_p(X)$ as the space of continuous functions with pointwise convergence. Note that $C(X)$ has a finer topology than $C_p(X)$, and that $C(X)$ is a Banach space.

We begin with the theory of the spaces $C(X)$, which is closer to the preceding facts.

Theorem 5.9 (Banach-Stone)**.** Let X, Y be compact Hausdorff spaces. Suppose $C(X)$ and $C(Y)$ are isomorphic in Ban_{str}. Then X, Y are homeomorphic.

It is worth remarking that this implies that Theorem 5.8 cannot be improved to Ban_{str}. More specifically, $\ell^\infty(\mathbb{N})$ is in fact isometric to $C(\beta\mathbb{N})$ with $\beta\mathbb{N}$ the Stone-Cech compactification of $\mathbb{N}$ (in fact, this isomorphism is also an algebra isomorphism); similarly, $L^\infty(0, 1)$ is isometric to $C(S(0, 1))$, where $S(0, 1)$ is the "Stone space" of $(0, 1)$. However, $\beta\mathbb{N}$ and $S(0, 1)$ are not homeomorphic (indeed, $\beta\mathbb{N}$ is separable and $S(0, 1)$ is not), so there is no isometry between $\ell^\infty(\mathbb{N})$ and $L^\infty(0, 1)$.

The other conclusion of this, of course, is that Banach-Stone can't be improved to only assuming isomorphisms in Ban_{wk}. In fact, the setting in Ban_{wk} is as different as can be:

Theorem 5.10 (Miljutin 1966)**.** Suppose X is an uncountable compact metric space. Then $C(X)$ is isomorphic to $C([0, 1])$ in Ban_{wk}.

[7] see "Kuratowski's theorem."

Thus far our only examples have been normed vector spaces. There are many more topological vector spaces of interest - e.g. weak topologies, weak-$*$ topologies, etc. One quick way to distinguish these spaces from our earlier examples is to note that, for some of these weaker topologies, *every neighborhood of* 0 *contains a line!* Indeed, as an exercise you might justify that this holds for $C_p([0,1])$. For comparison, in any normed vector space, the unit ball is a neighborhood of the origin that doesn't contain a line. The subject of general topological vector spaces is quite vast; we conclude this note by looking at only one type that goes beyond our normed vector spaces.

Let's consider the spaces $C_p(X)$ with X compact Hausdorff. In this case, we have a much more complicated theory: within a single category, there are nontrivial isomorphisms, but also nontrivial invariants.

We first note an example of the first:

Theorem 5.11. Let $X = [0,1] \cup [2,3]$ and $Y = [0,2] \cup \{3\}$. Then $C_p(X), C_p(Y)$ are isomorphic in TVS_{str}.

Proof. Define $\Phi : C_p(X) \to C_p(Y)$ by

$$\Phi(f)(y) := \begin{cases} f(y) & 0 \le y \le 1 \\ f(y+1) - (f(2) - f(1)) & 1 \le y \le 2 \\ f(2) - f(1) & y = 3 \end{cases}$$

□

On the other hand, there are "small" spaces which witness a failure of linear isomorphism:

Theorem 5.12. There are countable compact Tychonoff spaces X, Y for which $C_p(X), C_p(Y)$ are not isomorphic in TVS_{str}. $C(X), C(Y)$ are also not isomorphic in TVS_{str}.

If we are willing to extend the definition of $C_p(Y)$ to non-compact spaces Y (still using pointwise convergence as the topology), then one sees a variety of interesting examples:

Theorem 5.13. $C_p([0,1])$ and $C_p(\mathbb{R})$ are not isomorphic in TVS_{str}, but they are isomorphic in TVS_{wk}.

We conclude by noting some ways in which the topology of $C_p(X)$ witnesses the topology of X, i.e. considering the isomorphism problem of $C_p(X)$ in TVS_{wk} and TVS_{str}:

Theorem 5.14. Suppose X, Y are Tychonoff spaces such that $C_p(X)$ is isomorphic to $C_p(Y)$ in TVS_{str}. Then the following holds:

(a) X is compact iff Y is compact

(b) X is σ-compact iff Y is σ-compact

(c) X and Y have the same (topological) dimension[8]

Theorem 5.15. Suppose X is Tychonoff. TFAE:

(a) $C_p(X)$ is separable and metrizable

(b) $C_p(X)$ is metrizable

(c) $C_p(X)$ is first-countable

(d) X is countable

[8]the so-called "covering dimension;" it takes the right value on manifolds.

A few sources:

1. Lacey, "The Isometric Theory of Classical Banach Spaces"
2. Albiac and Kalton, "Topics in Banach Space Theory"
3. van Mill, "The Infinite-Dimensional Topology of Function Spaces"
4. Weston, "On the Uniform Classification of $L_p(\mu)$ Spaces"

6 : Week 6

This discussion will be focused on Hilbert spaces. We briefly detail the basics of the subject here.

- For $\mathbb{F}$ one of $\mathbb{R}, \mathbb{C}$, an $\mathbb{F}$-*inner product space* $(\mathcal{H}, \langle \cdot, \cdot\cdot \rangle)$ is a vector space over $\mathbb{F}$ equipped with an inner product $\langle \cdot, \cdot\cdot \rangle$. That is, $\langle \cdot, \cdot\cdot \rangle$ is a function $\mathcal{H} \times \mathcal{H} \to \mathbb{F}$, which satisfies the following axioms:

  1. $\langle \cdot, \cdot\cdot \rangle$ is $\mathbb{F}$-linear in the second entry and conjugate-linear in the first entry, that is,

  $$\langle v, \alpha x + y \rangle = \alpha \langle v, x \rangle + \langle v, y \rangle$$

  $$\langle \alpha x + y, v \rangle = \bar{\alpha} \langle x, v \rangle + \langle y, v \rangle$$

  2. $\langle \cdot, \cdot\cdot \rangle$ is conjugate symmetric, i.e.

  $$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

  3. If $x \neq 0$, then $\langle x, x \rangle > 0$.

  Of course, if $\mathbb{F} = \mathbb{R}$, then conjugation is just the identity.

- In any inner product space, one has the Cauchy-Schwarz inequality $|\langle x, y \rangle| \leq |\langle x, x \rangle|^{1/2} |\langle y, y \rangle|^{1/2}$

- For $\mathbb{F}$ one of $\mathbb{R}, \mathbb{C}$, a $\mathbb{F}$-*Hilbert space* $\mathcal{H}$ is a Banach space (i.e. complete normed vector space) over $\mathbb{F}$ whose norm $\|\cdot\|$ is of the form $x \mapsto \|x\| = \langle x, x \rangle^{1/2}$, where $\langle \cdot, \cdot\cdot \rangle$ is an inner product.

- In any Hilbert space $\mathcal{H}$, we have the parallelogram law

  $$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

- If $(X, \mathcal{A}, \mu)$ is any measure space, then $L^2(X, \mu)$ is a Hilbert space, with inner product defined by

  $$(f, g) \mapsto \int_X \bar{f} g \, d\mu$$

- If $\mathcal{H}$ is a Hilbert space and $f \in \mathcal{H}^*$, then there is a unique $v \in \mathcal{H}$ such that $f(x) = \langle v, x \rangle$ for every $x \in \mathcal{H}$. Thus, $\mathcal{H}^* \simeq \mathcal{H}$ canonically. This is the *Riesz representation theorem*.

Fall 2012 Problem 3. Let $\mathcal{H}$ be a Hilbert space and E a closed convex subset of $\mathcal{H}$. Prove that there exists a unique element $x \in E$ such that

$$\|x\| = \inf_{y \in E} \|y\|$$

Proof. Consider a sequence $\{x_n\}_{n=1}^\infty \subseteq E$ with the property that $\|x_n\| \to \inf_{y\in E} \|y\|$. By the parallelogram law, applied to $\frac{1}{2}x_n$ and $\frac{1}{2}x_m$,

$$\frac{1}{4}\|x_n - x_m\|^2 = \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|x_m\|^2 - \left\|\frac{x_n - x_m}{2}\right\|^2 \leq \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|x_m\|^2 - \inf_{y\in E}\|y\|^2$$

It follows immediately that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. The result follows.

□

Fall 2009 Problem 1 (modified). Find a non-empty closed set in the Hilbert space $\ell^2(\mathbb{N})$ that does not contain an element of smallest norm. Prove your assertion.

Proof. Let e_n be the element of $\ell^2(\mathbb{N})$ which is 1 in entry n and 0 otherwise. We claim that $E = \{(1 + \frac{1}{n})e_n : n \in \mathbb{N}\}$ does the job.

Clearly E is nonempty, and $\|(1+\frac{1}{n})e_n\| = 1+\frac{1}{n}$, so E does not contain an element of smallest norm. It remains to establish that E is closed. Indeed, for $n \neq m$,

$$\|(1+\frac{1}{n})e_n - (1+\frac{1}{m})e_m\| = \left((1+\frac{1}{n})^2 + (1+\frac{1}{m})^2\right)^{1/2} > \sqrt{2}$$

so E has no limit points other than elements of E itself. It follows that E is closed, and we are done.

□

Spring 2014 Problem 6. Given a (complex) Hilbert space $\mathcal{H}$, let $\{a_n\}_{n=1}^\infty \subseteq \mathcal{H}$ be a sequence with $\|a_n\| = 1$ for all $n \geq 1$. Recall that the closed convex hull of $\{a_n\}_{n=1}^\infty$ is the closure of the set of all convex combinations of elements of $\{a_n\}_{n=1}^\infty$.

(a) Show that if $\{a_n\}_{n=1}^\infty$ spans $\mathcal{H}$ linearly (i.e. any $x \in \mathcal{H}$ is of the form $\sum_{k=1}^m c_k a_{n_k}$ for some $m \in \mathbb{N}$ and $c_k \in \mathbb{C}$), then $\mathcal{H}$ is finite dimensional.

(b) Show that if $\langle a_n, \zeta\rangle \to 0$ for all $\zeta \in \mathcal{H}$, then 0 is in the closed convex hull of $\{a_n\}_n$.

Proof. (a): We first observe that there exists a maximal subset $S \subseteq \mathbb{N}$ such that $\{a_s : s \in S\}$ is linearly independent; this is completely straightforward from Zorn's lemma, repeating the argument that establishes the existence of a basis for every vector space. Since the full sequence spans, we quickly see that $\{a_s : s \in S\}$ is also spanning. Thus $\{a_s : s \in S\}$ is a basis.

It suffices to consider the case that S is infinite; relabeling, we will take $S = \mathbb{N}$ and simply refer to a_n. Since $\mathcal{H}$ is an inner product space and and each initial segment $\{a_1, \ldots, a_n\}$ is linearly independent, we may run the Gram-Schmidt procedure and assume that the $\{a_n\}_{n=1}^\infty$ are orthonormal (i.e. $\langle a_n, a_m\rangle = \delta_{nm}$). Take now

$$x = \sum_{n=1}^\infty \frac{1}{n} a_n$$

Since the a_n are orthonormal and $\mathcal{H}$ is complete, this sum converges; indeed, if $M < N$,

$$\left\| \left(\sum_{n=1}^{N} \frac{1}{n} a_n \right) - \left(\sum_{n=1}^{M} \frac{1}{n} a_n \right) \right\| = \left(\sum_{n=M+1}^{N} \frac{1}{n^2} \right)^{1/2}$$

which is uniformly small when M is large. Thus the partial sums are Cauchy, so the series converges to $x \in \mathcal{H}$.

Since $\{a_n\}_n$ is spanning, we have $x = \sum_{k=1}^{m} c_k a_{n_k}$ for some $m \in \mathbb{N}$ and $c_k \in \mathbb{C}$. In particular, $\langle x, a_{m+1} \rangle = 0$. By continuity of the inner product (from Cauchy-Schwarz),

$$\langle \sum_{n=1}^{N} \frac{1}{n} a_n, a_m \rangle \to \langle x, a_m \rangle = 0$$

as $N \to \infty$. However, the LHS is eventually $1/m$, a contradiction.

Recalling our assumption, we see that instead S has to be finite, so indeed $\mathcal{H}$ is finite dimensional, as was to be shown.

(b): Let $C = \overline{\operatorname{conv}(\{a_n\}_n)}$ be the norm-closure of the convex hull of the members of $\{a_n\}_n$. It is easy to see that C is still convex. Assume for the sake of contradiction that $0 \notin C$. By Fall 2012 Problem 3 (above), there is some $x \in C$ of minimal norm, which under our assumption is nonzero. Choose $\varepsilon = \frac{1}{2}\|x\|^2 > 0$.

Write $U = \{y \in \mathcal{H} : \operatorname{Re} \langle x, y \rangle > \varepsilon\}$. We first claim that $C \subseteq U$. Let $z \in C$ be arbitrary, and for $t \in (0,1)$ observe that $\|x\| < \|x + t(z-x)\|$. Expanding, we have

$$\begin{aligned} \|x\|^2 &< \|x + t(z-x)\|^2 \\ &= \|x\|^2 + t^2\|z-x\|^2 + 2t\operatorname{Re} \langle x, z-x \rangle \end{aligned}$$

so that

$$2\varepsilon - \frac{t}{2}\|z-x\|^2 < \operatorname{Re} \langle x, z \rangle$$

Sending $t \to 0$, we conclude that $\operatorname{Re} \langle x, z \rangle \geq 2\varepsilon > \varepsilon$, so $z \in U$. We have shown that $C \subseteq U$, as claimed. Thus $x \in \mathcal{H}$ is a vector such that $|\langle x, y \rangle| > \varepsilon$ for all $y \in C$, so in particular $|\langle x, a_n \rangle| > \varepsilon$ for each n. But this contradicts the assumption on the $\{a_n\}_n$, and we are done.

□

Bonus problem: Show that there exists a continuous function $f : [0,1] \to L^2([0,1])$ satisfying the following. For all $a < b \leq c < d$, $f(b) - f(a)$ is orthogonal to $f(d) - f(c)$. Such a curve is called *crinkled*.

Proof. Choose $f(t) = 1_{[0,t]}$. Then $\|f(t) - f(s)\|_2 = |t-s|^{1/2}$, so f is continuous. If $a < b \leq c < d$, then

$$\int_{[0,1]} \overline{[f(b) - f(a)]}[f(d) - f(c)] d\lambda = \int_{[0,1]} 1_{[a,b]} 1_{[c,d]} d\lambda = 0$$

as was to be shown.

$\square$

Hints and remarks about the preceding problems.

Fall 2012 Problem 3

Hint: use the parallelogram law to control an infimizing sequence.

Remark. Applying this to sets of the form $\{x - y : y \in U\}$ with U a closed subspace of $\mathcal{H}$, we recover a vector w such that $x - w$ is the orthogonal projection of x onto U.

Fall 2009 Problem 1

Hint: you should leverage the infinite dimensions and try to avoid 0*.*

Spring 2014 Problem 6

Hint for (a), part 1: as a warm-up, figure out why the standard vectors $\{e_n\}_n$ *are not linearly spanning.*

Hint for (a), part 2: use Zorn's lemma and a contradiction assumption to suppose that $\{a_n\}_n$ *is actually a basis. Gram-Schmidt will also be helpful.*

Hint for (a), part 3: use the inner product to reach a contradiction. Cauchy-Schwarz is your friend.

Hint for (b), part 1: assuming the result is false, show that the closed convex hull is contained in a half-space of the form $\{y \in \mathcal{H} : \mathrm{Re}\langle x, y\rangle > \varepsilon\}$ *for a suitable* $\varepsilon > 0$*. You will find the result of Fall 2012 Problem 3 helpful in more than one way.*

Hint for (b), part 2: a useful manipulation in inner product spaces is $\|a+b\|^2 = \|a\|^2 + \|b\|^2 + 2\mathrm{Re}\,\langle a, b\rangle$*.*

Remark. Part (a) shows that an infinite-dimensional Hilbert space must have uncountable Hamel dimension. Part (b) is part of a general phenomenology in functional analysis, whereby several different topologies can look very similar when restricted to convex sets.

Bonus problem

Hint: this is possible in any infinite-dimensional Hilbert spaces, but $L^2([0,1])$ *is the best model for this problem.*

Remark. Try to draw this in finite dimensions. Conclude that infinite dimensional Hilbert spaces are weird. Also, note the resemblance to Brownian motion.

Appendix: conditional expectation

The midterm contained a special case of the "conditional expectation" construction. The general version of it is as follows: given a probability measure μ on the measurable space (X, Σ) and a sub-σ-algebra $\mathcal{N} \subseteq \Sigma$, we write $\nu = \mu|_{\mathcal{N}}$. For each $f \in L^1(\Sigma, \mu)$, there is a Radon-Nikodym derivative $g = d(f \cdot \mu|_{\mathcal{N}})/d\nu$ and denote $\mathbb{E}[f|\mathcal{N}] := g$. Thus, $\mathbb{E}[f|\mathcal{N}]$ is the (a.e. class of the) $\mathcal{N}$-measurable function satisfying the relation

$$\int_A f d\mu = \int_A \mathbb{E}[f|\mathcal{N}] d\nu$$

for all $A \in \mathcal{N}$.

The purpose of this appendix is to record some of the critical properties of this operator from the perspective of analysis. For the entirety of this appendix, we will take μ to be a probability measure.

Theorem 1: $L^p(\mathcal{N})$ is a closed linear subspace of $L^p(\Sigma)$, for all $1 \leq p \leq \infty$. If $p = \infty$, then it is also a subalgebra.

Proof. It is a familiar fact from last quarter that the sum and product of $\mathcal{N}$-measurable functions is $\mathcal{N}$-measurable. Thus it remains to demonstrate norm-closure. But notice that $L^p(\nu, \mathcal{N})$ is a Banach space in its own right, hence is complete, so is certainly closed in $L^p(\mu, \Sigma)$. $\square$

Lemma 1: $\mathbb{E}[\,\cdot\,|\,\mathcal{N}]$ is a positive operator, i.e. if $0 \leq f \in L^1(\mu; \Sigma)$, then $\mathbb{E}[f\,|\,\mathcal{N}] \geq 0$ ν-a.e.

Proof. For any $A \in \mathcal{N}$,

$$\int_A \mathbb{E}[f\,|\,\mathcal{N}] d\nu = \int_A f d\mu \geq 0$$

so we conclude $\mathbb{E}[f\,|\,\mathcal{N}] \geq 0$ ν-a.e. $\square$

Theorem 2: $\mathbb{E}[\,\cdot\,|\,\mathcal{N}] : L^p(\mu, \Sigma) \to L^p(\nu, \mathcal{N})$ is a bounded linear map with operator norm 1, for all $1 \leq p \leq \infty$.

Proof. Regarding it first as a map $L^p(\mu, \Sigma) \to L^1(\nu, \mathcal{N})$ (using $L^p(\mu, \Sigma) \subseteq L^1(\mu, \Sigma)$ since $\|\mu\| = 1$), it is clear that this map is linear.

We focus on the case $p < \infty$; $p = \infty$ is left as an exercise. In our case, we claim that $\mathbb{E}[\,\cdot\,|\,\mathcal{N}]$ satisfies a version of Jensen's inequality:

$$|\mathbb{E}[f\,|\,\mathcal{N}]|^p \leq \mathbb{E}[|f|^p\,|\,\mathcal{N}]$$

It clearly suffices to consider f positive, so we'll disregard absolute value bars. Note that, for $x > 0$,

$$x^p = \sup_{c>0} pc^{p-1}x + (1-p)c^p$$

(this comes from considering all supporting hyperplanes under the function $x \mapsto x^p$). For any such c,

$$(pc^{p-1})\mathbb{E}[f\,|\,\mathcal{N}] + (1-p)c^p = \mathbb{E}[(pc^{p-1}f + (1-p)c^p)\,|\,\mathcal{N}]$$

For each x, $(pc^{p-1})f(x) + (1-p)c^p \leq f(x)^p$, so by Lemma 1

$$\mathbb{E}[(pc^{p-1}f + (1-p)c^p)\,|\,\mathcal{N}] \leq \mathbb{E}[f(x)^p\,|\,\mathcal{N}]$$

Thus we have shown
$$(pc^{p-1})\mathbb{E}[f|\mathcal{N}] + (1-p)c^p \leq \mathbb{E}[f(x)^p|\mathcal{N}]$$
for all $c > 0$. Taking a supremum, we get
$$|\mathbb{E}[f|\mathcal{N}]|^p \leq \mathbb{E}[|f|^p|\mathcal{N}]$$
as desired.

It immediately follows that ($1 \leq p < \infty$)
$$\|\mathbb{E}[f|\mathcal{N}]\|_{L^p(\nu,\mathcal{N})} \leq (\int \mathbb{E}[|f|^p|\mathcal{N}]d\nu)^{1/p} = (\int |f|^p d\mu)^{1/p} = \|f\|_{L^p(\Sigma,\mu)}$$
and, since conditional expectation fixes constants, we see that the operator norms are all 1. □

Theorem 3: If $g \in L^\infty(\nu, \mathcal{N})$ and $f \in L^p(\mu, \Sigma)$, then $\mathbb{E}[fg|\mathcal{N}] = \mathbb{E}[f|\mathcal{N}]g$.

Proof. Suppose $g = 1_A$ for some $A \in \mathcal{N}$. Then
$$\int_E fg d\mu = \int_{A\cap E} f d\mu = \int_{E\cap A} \mathbb{E}[f|\mathcal{N}]d\nu = \int_E \mathbb{E}[f|\mathcal{N}]g d\nu$$
so $\mathbb{E}[fg|\mathcal{N}] = \mathbb{E}[f|\mathcal{N}]g$ in this case. By the linearity of expectation, we have the result for all simple functions g. Finally, for arbitrary g, if g_ε is simple and has $\|g - g_\varepsilon\|_\infty < \varepsilon$,
$$\mathbb{E}[fg|\mathcal{N}] = \mathbb{E}[f|\mathcal{N}]g + \mathbb{E}[f(g-g_\varepsilon)|\mathcal{N}] + \mathbb{E}[f|\mathcal{N}](g_\varepsilon - g)$$
which implies
$$\begin{aligned}\|\mathbb{E}[fg|\mathcal{N}] - \mathbb{E}[f|\mathcal{N}]g\|_p &\leq \|\mathbb{E}[f(g-g_\varepsilon)|\mathcal{N}]\|_p + \|\mathbb{E}[f|\mathcal{N}](g_\varepsilon - g)\|_p \\ &\leq \|f(g-g_\varepsilon)\|_p + \|f\|_p\|g_\varepsilon - g\|_\infty \\ &\leq 2\|f\|_p\varepsilon\end{aligned}$$
Sending $\varepsilon \to 0$, we get
$$\mathbb{E}[fg|\mathcal{N}] = \mathbb{E}[f|\mathcal{N}]g$$
for any $g \in L^\infty(\nu, \mathcal{N})$, as claimed. □

Theorem 4: $\mathbb{E}[\,\cdot\,|\mathcal{N}]$ restricts to a mapping $L^2(\mu;\Sigma) \to L^2(\nu,\mathcal{N})$. Regarded as such, it is the orthogonal projection onto that subspace.

Proof. The first part of this statement is enclosed in Theorem 2. We need to demonstrate that, for any $f \in L^2(\mu, \Sigma)$,
$$f - \mathbb{E}[f|\mathcal{N}] \perp L^2(\nu, \mathcal{N})$$
If $g \in L^\infty(\nu, \mathcal{N})$,
$$\int f\bar{g}d\mu = \int \mathbb{E}[f\bar{g}|\mathcal{N}]d\nu = \int \mathbb{E}[f|\mathcal{N}]\bar{g}d\nu$$
so
$$\int (f - \mathbb{E}[f|\mathcal{N}])\,\bar{g}d\mu = 0$$

Now, if $g \in L^2(\nu, \mathcal{N})$, we may find $g_\varepsilon \in L^\infty(\nu, \mathcal{N})$ such that $\|g - g_\varepsilon\|_2 < \varepsilon$. Then

$$\begin{aligned} |\int (f - \mathbb{E}[f|\,\mathcal{N}])\,\bar{g}d\mu| &= |\int (f - \mathbb{E}[f|\,\mathcal{N}])\,\overline{g - g_\varepsilon}d\mu| \quad \text{by the above} \\ &\le \|f - \mathbb{E}[f|\,\mathcal{N}]\|_2\|g - g_\varepsilon\|_2 \quad \text{by Hölder} \\ &\le 2\|f\|_2\varepsilon \quad \text{by Theorem 2} \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get that $f - \mathbb{E}[f|\,\mathcal{N}] \perp L^2(\nu, \mathcal{N})$, as claimed. □

One special case of this comes from the setting $\mathcal{N} = \sigma(Y)$ for a suitable Σ-measurable $\mathbb{R}$-valued function Y. In this case, one can show that:

Proposition 5 [The "Doob-Dynkin lemma"]: g is $\sigma(Y)$-measurable if and only if there exists $h : \mathbb{R} \to \mathbb{R}$ Borel such that $g = h(Y)$.

Proof. Clearly any function of the form $h(Y)$ with h Borel is $\sigma(Y)$-measurable, so we consider the converse. Suppose g is $\sigma(Y)$-measurable. In particular, $\sigma(g) \subseteq \sigma(Y)$. Fix $n \in \mathbb{N}$ and consider the mesh $\{m2^{-n}\}_{m\in\mathbb{Z}}$. Then, for each (m, n), $g^{-1}[m2^{-n}, (m + 1)2^{-n}) \in \sigma(g)$, so belongs to $\sigma(Y)$, i.e. there is a Borel set $B_{m,n} \subseteq \mathbb{R}$ such that

$$Y^{-1}(B_{m,n}) = g^{-1}[m2^{-n}, (m + 1)2^{-n})$$

Define $h_n(x) = \sum_{m\in\mathbb{Z}} m2^{-n}1_{B_{m,n}}$. Then $\|h_n \circ Y - g\|_\infty \le 2^{-n}$. On the other hand, $B_{m,n} = B_{2m,n+1} \cup B_{2m+1,n+1}$ so on $B_{m,n}$ we have

$$(h_{n+1} - h_n)(x) = \begin{cases} 0 & x \in B_{2m,n+1} \\ 2^{-n-1} & x \in B_{2m+1,n+1} \end{cases}$$

so the sequence h_n is monotone increasing. It is also clearly bounded above, so converges pointwise to some Borel h. Finally, notice that for each n

$$h_n(Y) \le g \le h_n(Y) + 2^{-n}$$

so by taking limits we obtain $h(Y) = g$, as claimed.

□

Remark 1: one frequently writes $\mathbb{E}[f|\,Y]$ as shorthand for $\mathbb{E}[f|\,\sigma(Y)]$.
Remark 2: as an immediate consequence, there exists a Borel function e_f such that $\mathbb{E}[f|\,Y] = e_f(Y)$.

Corollary 1: for any Borel function h,

$$\int |f - \mathbb{E}[f|\,\sigma(Y)]|^2 \le \int |f - h(Y)|^2$$

and, moreover,

$$\int |f - \mathbb{E}[f|\,\sigma(Y)]|^2 = \inf_{h \text{ Borel}} \int |f - h(Y)|^2$$

Proof. The first statement is an immediate consequence of Theorem 4 and Proposition 5. The second statement follows from noticing that $\mathbb{E}[f|\,\sigma(Y)]$ is $\sigma(Y)$-measurable, so we can again use Proposition 5. □

Remark: one can find situations where there exists non-Lebesgue measurable $h : \mathbb{R} \to \mathbb{R}$ such that $\int |f - h(Y)|^2$ is defined, and is strictly smaller than $\int |f - \mathbb{E}[f|\,\sigma(Y)]|^2$. We will not explore this here.

We conclude our discussion by considering the subspace $L^\infty(\nu, \sigma(Y)) \subseteq L^\infty(\mu, \Sigma)$ for Y a bounded Σ-measurable function. It is a vector subspace, but it is clearly not the subspace spanned by Y; $L^\infty(\nu, \sigma(Y))$ contains all linear combinations of indicators of preimages of Y, for example. If p is any polynomial, then $p(Y) \in L^\infty(\nu, \sigma(Y))$. In fact, that's almost everything:

Theorem 6: $L^\infty(\nu, \sigma(Y))$ is the "weak-*" closure of the set $\{p(Y)\}_{p \in \mathrm{Poly}}$ in $L^\infty(\mu, \Sigma)$; here the "weak-* topology" is the topology generated by the prebase

$$U^\varepsilon_{f,g} = \{h \in L^\infty(\mu, \Sigma) : |\int f(g-h)d\mu| < \varepsilon\}$$

This is wildly outside the scope of things provable in an appendix, so I don't prove this here.

7 : Week 7

For the purposes of this document, we will assume the following covering theorem.

Besicovitch covering theorem. For every dimension d, there is a constant $c_d \in \mathbb{N}$ satisfying the following. Suppose V is a set of open balls in $\mathbb{R}^d$ and A is the set of centers of the balls in V. Then there exist subsets $V_1, \ldots, V_{c_d} \subseteq V$ such that, for each i, the elements of V_i are pairwise disjoint, and

$$A \subseteq \bigcup_{i=1}^{c_d} \bigcup_{B \in V_i} B$$

Spring 2018 Problem 3: Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2 + \varepsilon^2} dxdy < \infty$$

Show that $f = 0$ almost everywhere.

Proof. The proof is via Lebesgue differentiation. The moral version is this. Suppose $[a, b]$ is an interval (of positive length) such that $f \geq c$ on $[a, b]$, for a suitable constant $c > 0$. Then the integral diverges:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2 + \varepsilon^2} dxdy \geq \int_{[a,b]^2} \frac{c^2}{|x-y|^2 + \varepsilon^2} dxdy \to \int_{[a,b]^2} \frac{c^2}{|x-y|^2} dxdy = +\infty$$

We will assume that f is nontrivial, and use Lebesgue differentiation to find a Lebesgue point $x \in \mathbb{R}$ such that $f(x) = c > 0$ (replacing f with $-f$, if necessary), hence find small intervals on which f is mostly large (e.g. $\geq c/2$). With some arbitrage, this will be strong enough to recover divergence.

We proceed to the argument. For simplicity, we take f to be real-valued; note from the complex inequalities $|z| \geq |\mathrm{Re}(z)|, |\mathrm{Im}(z)|$ that this case suffices. Let $x \in \mathbb{R}$ be a Lebesgue point for f. It suffices to show that $f(x) = 0$. For the sake of contradiction, we assume $f(x) \neq 0$; by symmetry, we may assume $f(x) = c > 0$. For arbitrary $\frac{1}{4} > \delta > 0$, we may find $\varepsilon = \varepsilon(\delta) > 0$ be such that

$$\frac{1}{2\varepsilon} \int_{|y-x|<\varepsilon} |f(y) - c| dy < \frac{\delta c}{2}$$

In particular, if $U = \{y \in \mathbb{R} : f(y) \geq \frac{c}{2}\}$, we see by Markov's inequality (using the implication $f(y) < \frac{c}{2} \implies |f(y) - c| \geq \frac{c}{2}$)

$$\lambda\left(U \cap (x - \varepsilon, x + \varepsilon)\right) \geq 2\varepsilon - \frac{2}{c} \int_{(x-\varepsilon, x+\varepsilon) \setminus U} |f(y) - c| dy > 2\varepsilon - 2\delta\varepsilon = (1 - \delta)2\varepsilon$$

If we directly apply these estimates to the iterated integral, we obtain a lower bound of the form $\gtrsim c^2(1-\delta)^2$, with some absolute constants. This does not suffice to show divergence; similarly, if we try to "zoom" in to the diagonal $z = y$, i.e. constrain the integral to an interval of the form $(x - \eta\varepsilon, x + \eta\varepsilon)$ with $\eta \ll 1$, then we quickly lose density estimates on U in the set. It transpires that no single scale suffices to show blowup. Instead, we will consider many ($\sim \log \delta^{-1}$) scales, each of which will give $\sim c^2$ as a lower bound, and sum.

For each $n \in \mathbb{N}$, write $L_n = \{y \in \mathbb{R} : |y - x| \in [\varepsilon 2^{n-1}\delta, \varepsilon 2^n \delta)\}$; when $n \leq \log_2(\delta^{-1})$, we have $L_n \subseteq (x - \varepsilon, x + \varepsilon)$. By the union bound, we compute

$$\lambda(U \cap L_n) \geq 2\varepsilon(1 - \delta - (1 - \delta 2^{n-1})) = 2\varepsilon\delta(2^{n-1} - 1)$$

Observe then that the L_n are pairwise disjoint, contained in $(x - \varepsilon, x + \varepsilon)$, and $U \cap L_n$ has nontrivial density in L_n.

Then we have, for each $\rho > 0$,

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(z)f(y)|}{|z-y|^2+\rho^2} dz dy &\geq \iint_{z,y \in U \cap (x-\varepsilon, x+\varepsilon)} \frac{|f(z)f(y)|}{|z-y|^2+\rho^2} dz dy \\
&\geq \sum_{1 \leq n \leq \log_2(\delta^{-1})} \iint_{z,y \in U \cap L_n} \frac{|f(z)f(y)|}{|z-y|^2+\rho^2} dz dy \\
&\geq \sum_{1 \leq n \leq \log_2(\delta^{-1})} \frac{c^2}{4(4\varepsilon^2\delta^2 2^{2n} + \rho^2)} \lambda(U \cap L_n)^2 \\
&\geq \sum_{1 \leq n \leq \log_2(\delta^{-1})} \frac{c^2 \varepsilon^2 \delta^2 (2^{n-1}-1)^2}{4\varepsilon^2\delta^2 2^{2n} + \rho^2} \\
&\geq \frac{c^2}{16} \sum_{2 \leq n \leq \log_2(\delta^{-1})} \frac{1}{4 + \varepsilon^{-2}\delta^{-2}2^{-2n}\rho^2}
\end{aligned}$$

Note that δ, ε were unrelated to ρ. Sending ρ to 0, we conclude

$$\liminf_{\rho \to 0} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(z)f(y)|}{|z-y|^2+\rho^2} dz dy \geq \frac{c^2}{16} \sum_{2 \leq n \leq \log_2(\delta^{-1})} \frac{1}{4} = \frac{c^2}{64 \log 2} \log(\delta^{-1})$$

We have demonstrated this bound for arbitrarily small $\delta > 0$, so we conclude that the $\liminf$ is $+\infty$, as was to be shown. □

Fall 2016 Problem 2: Let μ be a finite positive Borel measure on $\mathbb{R}$ that is singular to Lebesgue measure. Show that

$$\lim_{r \to 0^+} \frac{\mu([x-r, x+r])}{2r} = +\infty$$

for μ-a.e. $x \in \mathbb{R}$.

Proof. Let A be a Borel subset of $\mathbb{R}$ such that $m(A) = 0$, $\mu(\mathbb{R} \setminus A) = 0$. For each $\varepsilon > 0$, let $A^\varepsilon \supseteq A$ be open such that $m(A^\varepsilon) < \varepsilon$. For each $\alpha > 0$, define A_α to be the subset

$$A_\alpha := \left\{x \in A : \exists \{r_n\}_{n=1}^\infty, r_n \downarrow 0, \mu((x - r_n, x + r_n)) \leq 2\alpha r_n\right\}$$

We leave it to the reader to verify that A_α is Borel as well. For each $x \in A_\alpha$, we may in particular find some sequence $\{r_n\}_{n=1}^\infty$ as above for which $[x - r_n, x + r_n] \subseteq A^\varepsilon$. Let V be the (fine!) covering defined by these intervals. By the Besicovitch covering theorem, there is some universal constant N[9] such that V posses N subfamilies $V_1, \dots, V_N$ for which

$$A_\alpha \subseteq \bigcup_{i=1}^N \bigcup_{I \in V_i} I$$

and

$$\forall I \neq J \in V_i, I \cap J = \emptyset$$

Then

$$\mu(A_\alpha) \leq \sum_{i=1}^N \mu(\bigcup_{I \in V_i} I) \leq \alpha \sum_{i=1}^N m(\bigcup_{I \in V_i} I) < \alpha N \varepsilon$$

Since A_α was independent of ε, we may take $\varepsilon \to 0$ to get $\mu(A_\alpha) = 0$ for all $\alpha > 0$.

Lastly, the set of points x for which

$$\frac{\mu([x - r, x + r])}{2r} \not\to +\infty$$

is contained in the union of the A_α, which is equal to $\bigcup_{n=1}^\infty A_n$. Since each A_n is μ-null, the set of problem points is μ-null, and we're done.

$\square$

Fall 2023 Problem 5: Let $\omega : \mathbb{R} \to [0, \infty)$ be a locally integrable function to which we associate a Borel measure via

$$\omega(E) = \int_E \omega(x) dx.$$

Let M denote the (centered) Hardy-Littlewood maximal function:

$$(Mf)(x) = \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| dy.$$

Assume that the function $\frac{1}{\omega}$ is locally integrable and that there exists $C > 0$ so that

$$\omega(\{x \in \mathbb{R} : |(Mf)(x)| > \lambda\}) \leq \frac{C}{\lambda^2} \int_{\mathbb{R}} |f(x)|^2 dx$$

uniformly in $\lambda > 0$ and functions $f : \mathbb{R} \to \mathbb{R}$ for which the right-hand side above is finite. Prove that

$$\sup_{x \in \mathbb{R}, r > 0} \left(\frac{1}{2r} \int_{x-r}^{x+r} \omega(y) dy \right) \left(\frac{1}{2r} \int_{x-r}^{x+r} \frac{1}{\omega(y)} dy \right) < \infty.$$

Hint: Apply the hypothesis to a well-chosen function f and constant λ.

[9] I believe $N = 2$ here.

Proof. Fix $x \in \mathbb{R}$ and $r > 0$. Write $f(y) = 1_{[x-r,x+r]}(y)\frac{1}{\omega(y)^{1/2}}$ and $\lambda = \frac{1}{8r}\int_{x-r}^{x+r}\frac{1}{\omega(y)}dy$, which we may assume is positive. Then, by the hypothesis,

$$\omega(\{y \in \mathbb{R} : |(Mf)(y)| > \lambda\}) \leq \frac{C\int_{x-r}^{x+r}\frac{1}{\omega(y)}dy}{\left(\frac{1}{8r}\int_{x-r}^{x+r}\frac{1}{\omega(y)}dy\right)^2} = \frac{32Cr}{\frac{1}{2r}\int_{x-r}^{x+r}\frac{1}{\omega(y)}dy}$$

If $|y - x| \leq r$,

$$Mf(y) \geq \frac{1}{4r}\int_{x-r}^{x+r}\frac{1}{\omega(t)}dt = 2\lambda$$

so we reach the conclusion that $\{y \in \mathbb{R} : |(Mf)(y)| > \lambda\}$ contains all of $[x - r, x + r]$. Thus

$$\omega(\{y \in \mathbb{R} : |(Mf)(y)| > \lambda\}) \geq \int_{x-r}^{x+r}\omega(y)dy$$

and we conclude by rearranging the first inequality that

$$\left(\frac{1}{2r}\int_{x-r}^{x+r}\omega(y)dy\right)\left(\frac{1}{2r}\int_{x-r}^{x+r}\frac{1}{\omega(y)}dy\right) \leq 16C$$

$\square$

Bonus problem

Spring 2022 Problem 5: Let μ be a Borel measure on $\mathbb{R}^2$, and assume it has the following property: for every fixed $r > 0$, the quantity $\mu(B(x,r))$ is finite and independent of x, where $B(x,r)$ is the open ball of radius r around x.

(a) Prove that there is a finite constant c such that $\mu(B(x,r)) \le cr^2$ whenever $0 < r \le 1$.

[We did this one in a previous week, so we'll skip it for today.]

(b) Prove that μ is a constant multiple of Lebesgue measure.

Proof. (b): When this problem came up previously we established that μ is absolutely continuous with respect to Lebesgue measure, so we'll take that for granted now.

Thus we may write $d\mu = f d\lambda$ for some nonnegative locally integrable Borel function f. By the assumption, the average of f on $B(x,r)$ is independent of x. If f is nonconstant, then there is some $\varepsilon > 0$ and positive measure sets A, B such that $\sup_{x\in A} f(x)+\varepsilon < \inf_{x\in B} f(x)$. By Lebesgue differentiation, a.e. point of A (resp. B) is a Lebesgue point for A (resp. for B). Consequently, we may find some $x \in A$, $y \in B$ and $r > 0$ such that $\mu(B(x,r)) < \mu(B(y,r))$, contradicting our assumption. Thus f is constant a.e., so μ is a constant multiple of Lebesgue measure. $\square$

Hints and remarks about the preceding problems.

Spring 2018 Problem 3.

Hint, part 1: Lebesgue differentiation.

Hint, part 2: if $f(y) \sim c > 0$ for most $y \sim x$, the double integral should be large as $\varepsilon \to 0$.

Hint, part 3: assume $f(x) = c > 0$ and x is a Lebesgue point. Write U for the "good" set where $f(y) \geq \frac{c}{2}$, and conclude that it has arbitrarily high density near x. Decompose the integral into many scales $|y - z| \sim 2^{-n}$, over which the integral is always large. You will need to estimate the concentration of $U \times U$ on small sets; for this, use the union bound on the complement.

Remark. The upshot of this problem is that, for nontrivial f, the double integral diverges as $\varepsilon \to 0$. In contrast to other singular integral qual problems, where the goal is to show convergence, we cannot use approximation. Indeed, it is easy to see that one expects any error term to also diverge, and we cannot subtract $\infty - \infty$.

In general, Lebesgue differentiation is a good thing to try when the goal is to show that a singular integral operator is badly behaved.

Fall 2016 Problem 2.

Hint, part 1: for each constant $\alpha > 0$, show that the set of x for which the quotient is bounded by α infinitely often, is μ-null.

Hint, part 2: It will be helpful to consider a set $A \subseteq \mathbb{R}$ such that $\lambda(A) = 0, \mu(\mathbb{R} \setminus A) = 0$, and consider an open neighborhood U which has small λ-mass. Use Besicovitch to bound the "bad" sets from the previous hint by something like α times $\lambda(U)$.

Remark. Compare with Fall 2009 Problem 4, where the goal is to show that for λ-a.e. $x \in \mathbb{R}$, the quotient limits to 0.

Fall 2023 Problem 5.

Hint, part 1: try to arrange for $\omega(\{x \in \mathbb{R} : |(Mf)(x)| > \lambda\}) \geq \int_{x-r}^{x+r} \omega(y)dy$ and $\lambda^{-2} \int_{\mathbb{R}} |f(x)|^2 dx \lesssim \frac{2r}{\frac{1}{2r}\int_{x-r}^{x+r} \frac{1}{\omega(y)} dy}$.

Hint, part 2: Suppose f is supported on $[x - r, x + r]$. Then, whenever $|y - x| \leq r$, one has $Mf(y) \geq \frac{1}{4r} \int_{x-r}^{x+r} |f(t)| dt$.

Hint, part 3: Take $f(y) = 1_{[x-r,x+r]}(y) \frac{1}{\omega(y)^{1/2}}$.

Remark. The celebrated Muckenhoupt theorem says that the following are equivalent, for each $\omega : \mathbb{R}^n \to [0, \infty)$ and each $1 < p < \infty$:

(a) The centered Hardy-Littlewood maximal function M is bounded on $L^p(\omega(x)dx)$, i.e. there is a constant $C > 0$ such that for which

$$\int |M(f)(x)|^p \omega(x)dx \leq C \int |f(x)|^p \omega(x)dx, \quad \forall f \in L^p(\omega(x)dx)$$

(b) ω is locally integrable, and ω satisfies the A_p condition: there is some $C > 0$ such that, for any ball B in $\mathbb{R}^n$,

$$\left(\frac{1}{\lambda_n(B)} \int_B \omega(x)dx\right) \left(\frac{1}{\lambda_n(B)} \int_B \omega(B)^{-\frac{1}{p-1}} dx\right)^{p-1} \leq C < \infty$$

In this problem, we have verified that, when $p = 2$, the A_p condition is necessary for M to be bounded in $L^p(\omega(x)dx)$.

A component of the A_p condition is that ω satisfies "reverse Hölder inequalities," which should morally be thought of as ω being nonzero and slowly varying.

Spring 2022 Problem 5.

Hint: we previously showed that μ is absolutely continuous with respect to λ, so we may write $d\mu = f d\lambda$ for a suitable f. It remains to show that f is nonconstant; use Lebesgue differentiation.

Appendix: the algebraic (Hamel) dimension of Banach spaces

Recall that a Banach space is a complete normed vector space (for us, over $\mathbb{R}$). Such a space V is in particular a vector space, so we can ask about its dimension (the axiom of choice implies, and is equivalent to, the statement that every vector space has a basis). This notion of dimension is called the *Hamel, linear*, or *algebraic dimension*. In most functional-analytic contexts, this is *not* the notion of dimension that people usually refer to, as what is often more important is to consider sets whose *closed* linear span is the full space; we won't discuss the latter here, as in the Banach context this becomes extremely delicate. We'll write $\dim_{\text{alg}}$ for the algebraic dimension. In this note, we will give intuition for the following result:

Theorem 7.1. If V is an infinite-dimensional Banach space, then $\dim_{\text{alg}}(V) \geq \mathfrak{c}$, where $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0}$.

To compare, if V is an infinite-dimensional *separable* Banach space, then a simple argument implies $|V| = \mathfrak{c}$, so certainly the algebraic dimension of V is at most $\mathfrak{c}$. We will discuss this directly, namely:

Theorem 7.2. Let B be an infinite-dimensional separable Banach space. Then the algebraic dimension of B is $\mathfrak{c}$.

Observe that Theorem 1 is straightforward to prove from Theorem 2. Indeed, given V, from the infinite dimensions we may find a countably infinite linearly independent set $\{v_n\}_{n=1}^{\infty}$, and apply Theorem 2 to $\overline{\text{span}\{v_n\}_{n=1}^{\infty}}$.

We first establish a lemma:

Lemma 7.3. There exists a family $\mathscr{A} \subseteq \mathscr{P}(\mathbb{N})$ such that $|\mathscr{A}| = \mathfrak{c}$ and, if $A \neq B \in \mathscr{A}$, then $A \cap B$ is finite.

Proof of lemma. We follow Gillman and Jerison's proof in *Rings of Continuous Functions*.

Let $\varphi : \mathbb{N} \to \mathbb{Q}$ be a bijection. For each irrational number r, fix an increasing sequence of rational numbers $s_1 < s_2 < \ldots$ such that $\lim_n s_n = r$, and define $A_r = \{\varphi^{-1}(s_n) : n \in \mathbb{N}\}$. Let $\mathscr{A} = \{A_r : r \in \mathbb{R} \setminus \mathbb{Q}\}$.

We verify that $\mathscr{A}$ has the right properties. For each $r \in \mathbb{R} \setminus \mathbb{Q}$, $r = \sup A_r$, so the sets A_r are all distinct, and hence $|\mathscr{A}| = \mathfrak{c}$. On the other hand, if $r_1 \neq r_2 \in \mathbb{R} \setminus \mathbb{Q}$, say $r_1 < r_2$, then there is some n such that, if $A_{r_2} = \{s_1 < s_2 < \ldots\}$, then $s_n \geq r_1$, so $A_{r_1} \cap A_{r_2}$ is contained in the set $\{s_1, \ldots, s_{n-1}\}$ (hence is finite).

□

We now proceed to Lacey's proof of Theorem 2.

Proof of Theorem 2. By induction, we find a sequence of elements $x_n \in B$ and $f_n \in B'$ such that

$$f_n(x_n) \neq 0, \quad f_n(x_m) = 0 \; \forall m \neq n$$

Indeed, pick first $x_1 \neq 0$ and f_1 such that $f_1(x_1) \neq 0$. Once $\{x_1, f_1, \ldots, x_n, f_n\}$ have been selected with $\{x_1, \ldots, x_n\}$ linearly independent, take $x_{n+1} \in \ker(f_1) \cap \cdots \cap \ker(f_n) \setminus \{0\}$ to be arbitrary; then certainly $\{x_1, \ldots, x_{n+1}\}$ is linearly independent.. Define g_{n+1} to be the functional on $\{x_1, \ldots, x_{n+1}\}$ defined by $g_{n+1}(x_{n+1}) = 1$ and $g_n(x_j) = 0$ for all $1 \leq j \leq n$, and extend by Hahn-Banach to $f_{n+1} \in B'$. Thus we have the full countably infinite family.

For $0 < t < 1$ irrational, define $x_t = \sum_{n \in A_t} x_n 2^{-n}$ with A_t as in the preceding lemma. We claim that the family $\{x_t\}_{t \in (0,1) \setminus \mathbb{Q}}$ is linearly independent. Indeed, for any $t_1, \ldots, t_k$ distinct, the intersection

$A_{t_1} \cap \cdots \cap A_{t_k}$ is finite, hence (since A_{t_1} is infinite) there is some $n \in A_{t_1} \setminus (A_{t_2} \cup \cdots \cup A_{t_k})$, so given any linear relation

$$\alpha_1 x_{t_1} + \ldots + \alpha_k x_{t_k} = 0$$

after applying f_n we obtain

$$\alpha_1 2^{-n} f_n(x_n) = 0$$

so $\alpha_1 = 0$. Running the same argument for the other t_j, we see that all linear relations are trivial, hence the x_t are linearly independent, as claimed.

Lastly, B itself has cardinality $\mathfrak{c}$, so we conclude that B has algebraic dimension $\mathfrak{c}$.

$\square$

We briefly remark as well that, if $\lambda \geq \mathfrak{c}$, then a Banach space has cardinality λ if and only if it has algebraic dimension λ. Not every λ can be the cardinality/dimension of a Banach space; there is at least one constraint in the "cofinality" of λ. On the other hand, if $\lambda = \kappa^{\aleph_0}$ for some κ, then $\ell^2(\lambda)$ (suitably interpreted) has cardinality $\lambda^{\aleph_0} = \kappa^{\aleph_0^2} = \lambda$, so there at least exist "large" Banach spaces in some sense.

We have now established that the separable Banach spaces have continuum algebraic dimension, though this involved a somewhat difficult argument. We would like to offer evidence in support of Theorem 1, which does not rely on clever combinatorial constructions. In particular, we'll give a short argument that the algebraic dimension of an infinite dimensional Banach space must be uncountable.

Lemma 7.4. Suppose $W \subseteq V$ is a finite-dimensional algebraic subspace of a Banach space V. Then W is closed in V.

Proof. You did this on your homework 5. $\square$

Corollary 7.5. Any infinite-dimensional Banach space has uncountable algebraic dimension.

Remark. We showed this in discussion, in the special case of Hilbert spaces.

Proof. Each linearly-independent set $\{v_1, v_2, \ldots\}$ defines an increasing sequence $V_n = \text{span}(v_1, \ldots, v_n)$ of finite-dimensional vector subspaces. By the previous lemma, each V_n is closed. Since V is infinite-dimensional and V_n is finite-dimensional, each V_n has empty interior in V. By the Baire category theorem (since V is a complete metric space), $\bigcup_{n\geq 1} V_n$ has empty interior in V. On the other hand, $\text{span}(\{v_j\}_{j=1}^{\infty}) = \bigcup_{n\geq 1} V_n$. In particular, V is not the span of the v_j, so any basis must be uncountable. $\square$

So goes the usual formulation of the uncountability reult. Note carefully that, unless we assume the continuum hypothesis, this doesn't quite prove Theorem 1.

8 : Week 8

Spring 2022 Problem 4. Let $f : [0, \infty) \to [0, \infty)$. Assume that $f(0) = 0$ and that f is **convex**, meaning that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall x, y \geq 0,\ 0 < t < 1.$$

Prove that

$$f(x) = \int_0^x g(y)dy$$

for some increasing function $g : [0, \infty) \to [0, \infty)$. [Hint: the question does not tell you that f is differentiable, or even continuous.]

Proof. We start by picking $T > 0$ and showing that f is absolutely continuous on $[0, T]$. Observe the following facts about f:

1. f is monotone increasing on $[0, \infty)$.
2. If $0 \leq a < b < c < \infty$, then
$$\frac{f(c) - f(a)}{c - a} \geq \frac{f(b) - f(a)}{b - a}.$$
3. If $0 \leq a < b < c < \infty$, then
$$\frac{f(c) - f(b)}{c - b} \geq \frac{f(b) - f(a)}{b - a}.$$

We justify each fact in turn. For (1): if $0 \leq x < y$, then

$$f(x) \leq \frac{x}{y} f(0) + \frac{y - x}{y} f(y) \leq f(y)$$

so indeed f is monotone increasing.

For (2): the claim is equivalent to the inequality

$$f(b) \leq f(a) + \frac{b - a}{c - a} [f(c) - f(a)]$$

But note that the right-hand side may be written as

$$\frac{b - a}{c - a} f(c) + \frac{c - b}{c - a} f(a)$$

so the inequality follows directly from the definition of convexity.

For (3): the claim is equivalent to the inequality

$$f(b) \leq \frac{(c - b)(b - a)}{c - a} \left[\frac{f(c)}{c - b} + \frac{f(a)}{b - a} \right]$$

But the right-hand side is just

$$\frac{b - a}{c - a} f(c) + \frac{c - b}{c - a} f(a)$$

which we considered previously.

We apply these facts to the problem. Let $L = f(T+1) - f(T)$. Let $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_k < b_k \le T$ be a sequence of intervals in $[0, T]$. Then, by repeated application of the above facts,

$$\frac{f(b_j) - f(a_j)}{b_j - a_j} \le L \quad \forall j = 1, \ldots, k$$

Thus, if we take $\varepsilon > 0$ arbitrary and write $\delta = \frac{\varepsilon}{L}$, if $a_1, b_1, \ldots, a_k, b_k$ is a sequence of intervals with $\sum_{j=1}^k b_j - a_j < \delta$, we have

$$\sum_{j=1}^k |f(b_j) - f(a_j)| \le \sum_{j=1}^k L|b_j - a_j| < \varepsilon$$

and we conclude that f is absolutely continuous on $[0, T]$.

In particular, we may find a Radon-Nikodym derivative $g \in L^1_{\text{loc}}(0, \infty)$ such that

$$f(x) = f([0, x]) = \int_0^x g(y)dy.$$

Since f is nonnegative and increasing, g takes values in $[0, \infty]$. If $x < y$ are Lebesgue points for g, we have

$$g(y) - g(x) = \lim_{n\to\infty} \frac{f(y + \frac{1}{n}) - f(y - \frac{1}{n})}{2/n} - \lim_{n\to\infty} \frac{f(x + \frac{1}{n}) - f(x - \frac{1}{n})}{2/n}$$

For each particular $n > \frac{2}{y-x}$, we may appeal to to the facts about f to conclude that $g(y) - g(x) \ge 0$. Thus g is increasing on the set of Lebesgue points, so we may replace g with an almost everywhere equivalent function that is increasing (say, $\tilde{g}(x) = \sup_{y \le x \text{ Lebesgue}} g(y)$). Since f is finite everywhere and g is increasing, we must have g is finite. Thus we have shown all claimed facts about g. $\square$

Spring 2018 Problem 4:

(a) Fix $1 < p < \infty$. Show that

$$f \mapsto [Mf](x, y) = \sup_{r>0, \rho>0} \frac{1}{4r\rho} \int_{-r}^{r} \int_{-\rho}^{\rho} f(x + h, y + \ell)dhd\ell$$

is bounded on $L^p(\mathbb{R}^2)$.

(b) Show that

$$[A_r f](x, y) = \frac{1}{4r^3} \int_{-r}^{r} \int_{-r^2}^{r^2} f(x + h, y + \ell)dhd\ell$$

converges to f a.e. in the plane as $r \to 0$.

Proof. (a): Since M is subadditive, it suffices to exhibit a bound for the dense subset $C_c^\infty(\mathbb{R}^2)$. Let $f \in C_c^\infty(\mathbb{R}^2)$ be arbitrary. Let M_1, M_2 be the maximal functions defined by

$$[M_j f](x, y) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} f((x, y) + he_j)dh \quad (j = 1, 2)$$

where e_1, e_2 are the standard unit vectors in $\mathbb{R}^2$. Note that

$$[Mf](x,y) \leq [M_2[M_1 f]](x,y)$$

so that

$$\|Mf\|_{L^p} \leq \|M_2[M_1 f]\|_{L^p(\mathbb{R}^2)}$$

We wish to appeal to boundedness of the usual Hardy-Littlewood maximal function. Expanding the right-hand side,

$$\|M_2[M_1 f]\|_{L^p(\mathbb{R}^2)}^p = \int_{\mathbb{R}} \int_{\mathbb{R}} |M_2[M_1 f]|^p(x,y) dy dx$$

Let $g = M_1 f$. Then g is nonnegative and measurable, and by boundedness of the Hardy-Littlewood maximal function, we have the estimate

$$\int_{\mathbb{R}} |M_2 g|^p(x,y) dy \lesssim_p \int_{\mathbb{R}} |g|^p(x,y) dy$$

for each $x \in \mathbb{R}$. It follows that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |M_2[M_1 f]|^p(x,y) dy dx \lesssim_p \int_{\mathbb{R}} \int_{\mathbb{R}} |M_1 f|^p(x,y) dy dx = \int_{\mathbb{R}} \int_{\mathbb{R}} |M_1 f|^p(x,y) dx dy$$

where we may appeal to boundedness again to see

$$\|M_2[M_1 f]\|_{L^p(\mathbb{R}^2)}^p \lesssim_p \int_{\mathbb{R}^2} |f|^p(x,y) dx dy$$

so that M is bounded, as was to be shown.

(b): Observe that $A_r f \to f$ uniformly when $f \in C_c^\infty(\mathbb{R}^2)$. Consider now $f \in L^p(\mathbb{R}^2)$ arbitrary, $\varepsilon > 0$, and $g \in C_c^\infty(\mathbb{R}^2)$ so that $\|f - g\|_{L^p} \leq \varepsilon$. Write T for the (subadditive) operator

$$Tu(x) = \limsup_{r \to 0^+} A_r[u - u(x)](x)$$

We clearly have $Tg = 0$ and $Tf \leq T[f-g]$. Additionally, we have the pointwise bound

$$A_r[f - g - f(x) + g(x)] \leq M[f-g](x) + |f(x) - g(x)|$$

so that

$$T[f-g] \leq M[f-g] + |f-g|$$

and hence

$$\|T[f-g]\|_{L^p(\mathbb{R}^2)} \lesssim_p \varepsilon$$

It follows that, for each $\alpha > 0$,

$$\lambda_2(\{x : Tf(x) > \alpha\}) \leq \lambda_2(\{x : T[f-g](x) > \alpha\}) \leq \frac{1}{\alpha^p} \int |T[f-g]|^p \lesssim \frac{\varepsilon^p}{\alpha^p}$$

This holds for every choice of α, ε. Sending $\varepsilon \to 0$, we conclude

$$\lambda_2(\{x : Tf(x) > \alpha\}) = 0$$

Since this holds for each $\alpha > 0$, we conclude that $Tf = 0$ a.e., as was to be shown. $\square$

Fall 2013 Problem 12: Let $f : [0,1] \to \mathbb{R}$ be a continuous function that is absolutely continuous on each interval $[\varepsilon, 1]$ with $0 < \varepsilon \leq 1$.

(a) Show that f is not necessarily absolutely continuous on $[0,1]$.

(b) Show that if f is of bounded variation on $[0,1]$, then f is absolutely continuous on $[0,1]$.

Proof. (a): Let $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Clearly $f : [0,1] \to \mathbb{R}$ is continuous. If $0 < \varepsilon \leq 1$, then from the derivative estimate

$$|f'(x)| = |\sin(1/x) - x^{-1}\cos(1/x)| \leq 1 + \varepsilon^{-1}$$

we see that, for any collection of disjoint intervals $(a_1, b_1), \ldots, (a_k, b_k)$ in $[\varepsilon, 1]$ with total length at most $\frac{\rho}{1+\varepsilon^{-1}}$, we have

$$\sum_{j=1}^{k} |f(b_j) - f(a_j)| \leq \sum_{j=1}^{k} \int_{a_j}^{b_j} |f'(x)|dx \leq \frac{\rho}{1+\varepsilon^{-1}}(1+\varepsilon^{-1}) \leq \rho$$

from which absolute continuity on $[\varepsilon, 1]$ follows.

It remains to establish that f is not absolutely continuous on $[0,1]$. To this end, observe that for $x_k = \frac{\pi}{k}$ ($n > 3$), we have for any $n > 3$

$$\sum_{k=n}^{N} |f(x_{n+1}) - f(x_n)| = 2(N - n - 1)$$

In particular, choosing $\varepsilon := 1$, we see that for any $\delta > 0$ we may choose $n > \pi\delta^{-1} + 1$ and $N = 2n + 1$ to obtain

$$\sum_{k=n}^{N} |f(x_{n+1}) - f(x_n)| > \pi\delta^{-1} + 1 > \varepsilon$$

whereas

$$\sum_{k=n}^{N} |x_{n+1} - x_n| < \frac{\pi}{n} < \delta$$

violating the condition for absolute continuity, if we choose $a_j = x_j$ and $b_j = x_{j+1}$.

(b): We first note the following lemma.

Lemma. Suppose f is continuous and of finite variation on $[0,1]$. Then, for each $\varepsilon > 0$, there is $\delta > 0$ such that the total variation of f on $[0,\delta]$ is at most ε.

Proof of lemma. Suppose not. Let $\varepsilon > 0$ be such that the total variations of f on any $[0,\delta]$ is greater than $\varepsilon > 0$. Let δ be such that $0 \leq y \leq \delta$ implies $|f(y) - f(0)| < \varepsilon/4$. Let $M < +\infty$ be the total variation of f on $[0,1]$, and let $0 = x_0 < x_1 < \ldots < x_n = 1$ be such that

$$\sum_{j=1}^{n} |f(x_k) - f(x_{k-1})| > M - \varepsilon/4$$

We may freely assume that $x_1 < \delta'$, since adjoining additional points only increases variation. By assumption, we may find $0 = y_0 < y_1 < \ldots < y_N = \delta$ such that

$$\sum_{j=1}^{N} |f(y_j) - f(y_{j-1})| > \varepsilon$$

Then the new sequence $\{z_j\}_{j=0}^{N+n}$ defined by

$$z_j = \begin{cases} y_j & 0 \leq j \leq N \\ x_{j-N} & N+1 \leq j \leq N+n \end{cases}$$

satisfies

$$\begin{aligned}\sum_{j=1}^{N+n} |f(z_j) - f(z_{j-1})| &= \sum_{j=1}^{N} |f(y_j) - f(y_{j-1})| + \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| - |f(x_1) - f(x_0)| \\ &> (M - \varepsilon/4) + \varepsilon - \varepsilon/4 > M\end{aligned}$$

which violates that $M < +\infty$ was the total variation.

$\square$

We now use the lemma. Let $\varepsilon > 0$ be arbitrary. Let $\delta_1 > 0$ be such that the variation of f on $[0, \delta_1]$ is less than $\varepsilon/2$; this is possible, since f is continuous and bounded variation. By assumption, f is absolutely continuous on $[\delta_1, 1]$, so there is a $\delta_2 > 0$ such that, for any tuple of disjoint intervals $(a_1, b_1), \ldots, (a_k, b_k)$, we have

$$\sum_{j=1}^{k} |b_j - a_j| < \delta_2 \implies \sum_{j=1}^{k} |f(b_j) - f(a_j)| < \varepsilon/2$$

We claim that $\delta := \min(\delta_1, \delta_2)$ suffices. Let $(a_1, b_1), \ldots, (a_k, b_k)$ be an increasing sequence of pairwise disjoint intervals with $\sum_{j=1}^{k} |b_j - a_j| < \delta$. Suppose j_* is the first index such that $b_{j_*} \geq \delta_1$. Then the intervals $(a_{j_*}, b_{j_*}) \cap [\delta_1, 1], \ldots, (a_k, b_k) \cap [\delta_1, 1]$ are pairwise disjoint of total length $< \delta_2$, so we have

$$|f(b_{j_*}) - f(\delta_1)| + \sum_{j=j_*+1}^{k} |f(b_j) - f(a_j)| < \varepsilon/2$$

Similarly, since f has total variation less than $\varepsilon/2$ on $[0, \delta_1]$, we have

$$\sum_{1 \leq j < j_*} |f(b_j) - f(a_j)| + |f(\delta_1) - f(a_{j_*})| 1_{a_{j_*} \leq \delta_1} < \varepsilon/2$$

so in total we have

$$\sum_{j=1}^{k} |f(b_j) - f(a_j)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

as claimed.

$\square$

Bonus problem

Fall 2018 Problem 4. Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$ and for each $\alpha \in \mathbb{T}$ define the rotation map $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $R_\alpha(z) = \alpha z$. A Borel probability measure μ on $\mathbb{T}$ is called α-**invariant** if $\mu(R_\alpha(E)) = \mu(E)$ for all Borel sets $E \subseteq \mathbb{T}$.

(a) Let m be Lebesgue measure on $\mathbb{T}$ (defined, for instance, by identifying $\mathbb{T}$ with $[0, 1)$ through the exponential function). Show that for every $\alpha \in \mathbb{T}$, m is α-invariant.

(b) Prove that if α is not a root of unity, then the set of powers $\{\alpha^n : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$.

(c) Prove that if α is not a root of unity, then m is the only α-invariant Borel probability measure on $\mathbb{T}$.

Proof. (a): We omit this, other than noticing that it is an obvious consequence of translation invariance of Lebesgue measure on $\mathbb{R}$ and an easy mod 1 rearrangment.

(b): Since $\mathbb{T}$ is compact, there is some $\beta \in \mathbb{T}$ and $k \mapsto n_k$ subsequence such that $\alpha^{n_k} \to \beta$ as $k \to \infty$. If $\varepsilon > 0$ is arbitrary and K is large enough so that $k \geq K$ implies $|\alpha^{n_k} - \beta| < \varepsilon/2$, then we also have

$$k > K \implies |\alpha^{n_k - n_K} - 1| = |\alpha^{n_k} - \alpha^{n_K}| < \varepsilon$$

Since α is not a root of unity and $n_k - n_K \neq 0$, we have that $\alpha^{n_k - n_K}$ is a nontrivial element of $\mathbb{T}$. Thus, 0 is an accumulation point of $\{\alpha^n\}_{n \in \mathbb{Z}}$.

Lastly, if $\beta \in \mathbb{T}$ is arbitrary and $\varepsilon > 0$, we may find $n \in \mathbb{Z}$ such that $|\alpha^n - 1| < \varepsilon$. Then the sequence $\alpha^n, \alpha^{2n}, \ldots, \alpha^{kn}$, with $k > \varepsilon^{-1}$, has the property that any $\gamma \in \mathbb{T}$ is within $\varepsilon > 0$ of some α^{jn}. But in particular β is within $\varepsilon > 0$ of a power of α. Thus the sequence of powers is dense in $\mathbb{T}$.

(c): Let μ be an α-invariant Borel probability measure. Note that the orbit $n \mapsto \alpha^n$ is infinite, so μ has no pure points.

It is convenient to identify $\mathbb{T}$ with $[0, 1)$ by a complex logarithm. Write θ for the element of $[0, 1)$ such that $e^{2\pi i\theta} = \alpha$. We claim that, for every interval I in $[0, 1)$, we have the inequality

$$\mu(I) \leq 3m(I)$$

We first demonstrate this for intervals of the form $[0, \frac{1}{2n})$ with a constant of 2. By part (b), we may find $k_1, \ldots, k_{n-1} \in \mathbb{Z}$ such that

$$\frac{j - 1/2}{n} < k_j\theta < \frac{j}{n}, \quad j = 1, \ldots, n-1$$

It follows that

$$\frac{1}{2n} < k_1\theta, \quad k_{n-1}\theta + \frac{1}{2n} < 1, \quad k_{j-1}\theta + \frac{1}{2n} < k_j\theta \quad j = 2, \ldots, n-1$$

so that the intervals $I_0, \ldots, I_{n-1}$ defined by

$$I_0 = [0, \frac{1}{2n}), \quad I_j = [k_j\theta, k_j\theta + \frac{1}{2n}) \quad j = 1, \ldots, n-1$$

are pairwise disjoint and contained in $[0,1)$. Since they all have the same length, their μ-value is the same, so

$$n\mu([0,\frac{1}{2n}) = \mu\left(\bigcup_{j=0}^{n-1} I_j\right) \leq 1$$

i.e. $\mu([0,\frac{1}{2n})) \leq 2 \cdot \frac{1}{2n}$, as claimed.

The result follows for arbitrary intervals; indeed, if $I \subseteq [0,1)$ has length ρ, then for each $N > \rho^{-1}$ we may find $I_1, \ldots, I_K$ pairwise disjoint intervals covering I of length $\frac{1}{2N}$ and $K \leq 2N\rho + 2$, so that

$$\mu(I) \leq 2\frac{K}{2N} \leq 2(\rho + \frac{1}{2N}) \leq 3\rho$$

as claimed.

Finally, observe that this immediately implies that $\mu \ll \lambda$, so $d\mu = f d\lambda$ for a suitable nonnegative L^1 function f of total mass 1. It remains to establish that $f \equiv 1$ Lebesgue a.e. But notice that, for any Lebesgue point x,

$$f(x) = \lim_{r \to 0^+} \frac{\mu((x-r, x+r))}{2r}$$

Let x, y be any pair of Lebesgue points and $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ be such that, for any $r \leq \delta$,

$$\left|\frac{\mu((x-r,x+r))}{2r} - f(x)\right|, \left|\frac{\mu((y-r,y+r))}{2r} - f(y)\right| < \varepsilon$$

Let $n, k \in \mathbb{Z}$ be such that $|n\theta + x - y - k| < \delta\varepsilon/6$. We have

$$(x + n\theta - \delta, x + n\theta + \delta) \subseteq (y + k - \delta - \delta\varepsilon/6, y + k + \delta + \delta\varepsilon/6)$$

so that, appealing to the upper bound on μ from earlier,

$$\left|\frac{\mu((x-\delta,x+\delta))}{2\delta} - \frac{\mu((y-\delta,y+\delta))}{2\delta}\right| \leq \frac{4\delta\varepsilon}{4\delta} = \varepsilon$$

Thus $|f(x) - f(y)| < 2\varepsilon$. Since $\varepsilon > 0$, we conclude that all Lebesgue points have the same f-value. Since the Lebesgue points are a full-measure set, we conclude that $f \equiv 1$ almost everywhere, so $\mu = \lambda$ as claimed.

$\square$

Hints and remarks about the preceding problems

Spring 2022 Problem 4.

Hint, part 1: as a warm-up, show that convex functions $\mathbb{R} \to \mathbb{R}$ *are necessarily continuous.*

Hint, part 2: show that f *is absolutely continuous on each* $[0, T]$*. Then, analyze the Radon-Nikodym derivative.*

Hint, part 3: to show that f *is absolutely continuous on each* $[0, T]$*, control the slopes of secant lines. Find inequalities to relate the sizes of secant lines between points* $(a, f(a)), (b, f(b)), (c, f(c))$ *with* $a < b < c$*. To find the inequalities that should be true, draw pictures of typical convex functions.*

Remark. This implies in particular that convex functions have well-behaved distributional derivatives (e.g. FTC holds). It turns out that they also have well-behaved second-order derivatives as well, which arises from studying the Lebesgue-Stieltjes measure arising from monotone increasing functions g.

Spring 2018 Problem 4.

Hint, part (a): control M *by two applications of the one-dimensional HL maximal function, one for each dimension.*

Hint, part (b): follow the proof of the Lebesgue differentiation theorem.

Remark. On the homework, you're tasked with showing that you can perform Lebesgue differentiation with "balls" replaced with sets B_r satisfying $B_r \subseteq B(x, r)$ and $\lambda_n(B_r) \gtrsim \lambda_n(B(x, r))$. In particular, that problem does not apply here.

Thus we have justified that you can do Lebesgue differentiation for sets that are comparable to metric balls, and with axis-parallel rectangles of arbitrary eccentricity. It turns out that things break down when you permit (a) arbitrary eccentricity and (b) arbitrary rotations, simultaneously! This is related to the so-called *Kakeya problem*.

Fall 2013 Problem 12.

Hint for (a): try something with unbounded variation near 0*.*

Hint for (b), part 1: break f *into a "small variation" part near* 0*, and an "absolutely continuous" part away from* 0*.*

Hint for (b), part 2: try showing the following lemma: "If f *is continuous and of finite variation on* $[0, 1]$*, then for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that the total variation of* f *on* $[0, \delta]$ *is at most* $\varepsilon > 0$*." Observe carefully that the assumption of continuity is necessary!*

Fall 2018 Problem 4.

Hint for (a): appeal to translation-invariance of usual Lebesgue measure, together with a simple cut-and-rearrange procedure on $[0, 1)$ *mod* 1*.*

Hint for (b): use compactness of $\mathbb{T}$ *to find an accumulation point. Rotate to* 1 *to find a dense mesh.*

Hint for (c): remember, we don't have arbitrary translation invariance. Instead, show that μ *is necessarily absolutely continuous with respect to* λ*, and then use Lebesgue differentiation. As a warm-up, convince yourself that* μ *has no pure point component.*

Remark. Although we don't have arbitrary translation invariance, this problem is still a bit easier than when we showed that Lebesgue measure was uniquely specified by special values and translation invariance. This is because of Lebesgue differentiation, which allows us to skip a lot of technical manipulations.

By the "Krein-Milman theorem," you can in fact show that Lebesgue measure is automatically "ergodic" for irrational rotations R_α. The argument goes as follows: invariant measures are convex combinations of ergodic measures. But, the space of invariant measures is just $\{\lambda\}$, and ergodic measures are necessarily invariant, so λ itself has to be ergodic.

9 : Week 9

Fall 2021 Problem 1: Let $f : [0, 2\pi] \to \mathbb{C}$ belong to L^1 and assume that

$$\int_0^{2\pi} f(x) \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4} \right) dx = 0$$

whenever $\varphi : \mathbb{R} \to \mathbb{C}$ is smooth and (2π)-periodic. Prove that

$$f(x) = a + be^{ix} + ce^{-ix} \quad \text{a.e.}$$

for some complex scalars a, b, c.

Proof. We first consider the warm-up. Suppose $f \in L^1([0, 2\pi])$ is such that $\int f\varphi' dx = 0$ whenever φ is smooth and (2π)-periodic. Extend f periodically to all of $\mathbb{R}$. Let η be a mollifier; then, for each $\varepsilon > 0$,

$$\begin{aligned} \int_0^{2\pi} (f * \eta_\varepsilon)\varphi' dx &= \int_{\mathbb{R}} \int_0^{2\pi} f(x-t)\eta_\varepsilon(t)\varphi'(x) dx dt \\ &= \int_{\mathbb{R}} \eta_\varepsilon(t) \int_{-t}^{2\pi - t} f(x)\varphi'(x+t) dx dt \end{aligned}$$

If $\tilde{\varphi}(x) = \varphi(x+t)$, then $\tilde{\varphi}'(x) = \varphi'(x+t)$ and $\tilde{\varphi}$ is smooth and (2π)-periodic. We conclude that

$$\int_0^{2\pi} (f * \eta_\varepsilon)\varphi' dx = 0$$

for all choices of φ. Integrating by parts,

$$0 = \int_0^{2\pi} (f * \eta_\varepsilon)\varphi' dx = -\int_0^{2\pi} (f * \eta_\varepsilon)' \varphi dx$$

Since φ can be taken to be an approximate identity near any point, we conclude that $(f * \eta_\varepsilon)'$ is a.e. zero. Since $f * \eta_\varepsilon$ is smooth, we conclude that $f * \eta_\varepsilon \equiv c_\varepsilon$ for a suitable constant c_ε.

Finally, if x is any Lebesgue point for f,

$$f(x) = \lim_{\varepsilon \to 0^+} f * \eta_\varepsilon(x) = \lim_{\varepsilon \to 0^+} c_\varepsilon$$

In particular, the c_ε are convergent as $\varepsilon \to 0$; it follows immediately that f is a.e. constant.

We now consider the current case. Towards the end of the argument, we will need a lemma:

Lemma. Suppose φ is a mollifier in $\mathbb{R}^n$. Then, for any $f \in L^1(\mathbb{R}^n)$, we have

$$\|f * \varphi_t - f\|_1 \to 0 \quad \text{as } t \to 0^+$$

Proof of lemma. Let $\varepsilon > 0$ be arbitrary. Let $g \in C_c^\infty(\mathbb{R}^n)$ be such that $\|f - g\|_1 < \varepsilon/3$. By elementary considerations, $g * \varphi_t \to g$ uniformly as $t \to 0$, so we may find $\delta > 0$ such that $0 < t < \delta$ implies $\|g * \varphi_t - g\|_1 < \varepsilon/3$. Consequently,

$$\|f * \varphi_t - f\|_1 \le \|f * \varphi_t - g * \varphi_t\|_1 + \|g * \varphi_t - g\|_1 + \|g - f\|_1 < \varepsilon$$

where we have used the operator norm bound $\|h * \varphi_t\|_1 \le \|h\|_1$. □

We now proceed to the argument. Extend f periodically to all of $\mathbb{R}$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be smooth, nonnegative, $\equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, supported in $[-1, 1]$, such that $\int \eta = 1$. For $\varepsilon > 0$, write $\eta_\varepsilon(t) = \varepsilon^{-1}(\varepsilon^{-1}t)$, so that $\int \eta_\varepsilon = 1$ and $(f * \eta_\varepsilon)(x) \to f(x)$ as $\varepsilon \to 0^+$, whenever x is a Lebesgue point of f.

Then $f * \eta_\varepsilon$ is smooth; further,

$$(f * \eta_\varepsilon)(x + 2\pi) = \int f(x + 2\pi - t)\eta_\varepsilon(t)dt = \int f(x - t)\eta_\varepsilon(t)dt = (f * \eta_\varepsilon)(x)$$

so that $f * \eta_\varepsilon$ is (2π)-periodic. Lastly, if $\varphi : \mathbb{R} \to \mathbb{C}$ is smooth and (2π)-periodic, then by Fubini-Tonelli

$$\begin{aligned}
\int_0^{2\pi} (f * \eta_\varepsilon)(x) \left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4}\right] dx &= \int_0^{2\pi} \int_{\mathbb{R}} f(x - t)\eta_\varepsilon(t) \left[\varphi^{(2)}(x) + \varphi^{(4)}(x)\right] dtdx \\
&= \int_{\mathbb{R}} \eta_\varepsilon(t) \int_0^{2\pi} f(x - t) \left[\varphi^{(2)}(x) + \varphi^{(4)}(x)\right] dxdt \\
&= \int_{\mathbb{R}} \eta_\varepsilon(t) \int_0^{2\pi} f(x - t) \left[\tilde{\varphi}^{(2)}(x - t) + \tilde{\varphi}^{(4)}(x - t)\right] dxdt
\end{aligned}$$

where we write $\tilde{\varphi}$ for the function $\tilde{\varphi}(x) = \varphi(x + t)$; note that $\tilde{\varphi}$ is still smooth and (2π)-periodic. By the assumption on f, together with periodicity,

$$\int_0^{2\pi} f(x - t) \left[\tilde{\varphi}^{(2)}(x - t) + \tilde{\varphi}^{(4)}(x - t)\right] dx = \int_0^{2\pi} f(x) \left[\tilde{\varphi}^{(2)}(x) + \tilde{\varphi}^{(4)}(x)\right] dx = 0$$

so that

$$\int_0^{2\pi} (f * \eta_\varepsilon)(x) \left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4}\right] dx = 0$$

as well, for all $\varepsilon > 0$ and all smooth (2π)-periodic φ .

Abbreviate $g_\varepsilon = f * \eta_\varepsilon$. Then g_ε is smooth and (2π)-periodic. Integrating by parts twice, we obtain

$$\int_0^{2\pi} g_\varepsilon^{(2)}(x) \left[\varphi(x) + \varphi^{(2)}(x)\right] dx = 0$$

for all φ smooth (2π)-periodic. Thus, for such φ,

$$\int_0^{2\pi} g_\varepsilon^{(2)}(x)\varphi(x)dx = -\int_0^{2\pi} g_\varepsilon^{(2)}(x)\varphi^{(2)}(x)dx$$

and, integrating the second expression by parts twice,

$$\int_0^{2\pi} g_\varepsilon^{(2)}(x)\varphi(x)dx = -\int_0^{2\pi} g_\varepsilon^{(4)}(x)\varphi(x)dx,$$

i.e. the function $x \mapsto g_\varepsilon^{(2)}(x) + g_\varepsilon^{(4)}(x)$ is orthogonal to all smooth (2π)-periodic functions φ. Thus $g_\varepsilon^{(2)}(x) + g_\varepsilon^{(4)}(x) = 0$ for all $x \in \mathbb{R}$.

Write $h_\varepsilon = g_\varepsilon^{(2)}$, so that $h_\varepsilon = -h_\varepsilon^{(2)}$. By standard ODE theory, it follows that

$$h_\varepsilon(x) = a_\varepsilon e^{ix} + b_\varepsilon e^{-ix}$$

for suitable complex constants $a_\varepsilon, b_\varepsilon \in \mathbb{C}$. By FTC, it follows that

$$g_\varepsilon(x) = -a_\varepsilon e^{ix} - b_\varepsilon e^{-ix} + c_\varepsilon x + d_\varepsilon$$

for suitable complex constants c, d; since g_ε is periodic, $c_\varepsilon = 0$.

We have concluded that, for every $\varepsilon > 0$, there are complex constants $a_\varepsilon, b_\varepsilon, d_\varepsilon$ so that

$$(f * \eta_\varepsilon)(x) = a_\varepsilon e^{ix} + b_\varepsilon e^{-ix} + c_\varepsilon$$

For any $x \in [0, 2\pi]$ Lebesgue point for f, we have

$$\lim_{\varepsilon \to 0^+} a_\varepsilon e^{ix} + b_\varepsilon e^{-ix} + c_\varepsilon = f(x)$$

We claim that the coefficients $a_\varepsilon, b_\varepsilon, c_\varepsilon$ are Cauchy as $\varepsilon \to 0^+$. Since

$$\frac{1}{2\pi}\int_0^{2\pi} (f * \eta_\varepsilon)(x)dx = c_\varepsilon$$

and $(f * \eta_\varepsilon) \to f$ in $L^1([-\pi, 3\pi])$ (say), we see that

$$\frac{1}{2\pi}\int_0^{2\pi} (f * \eta_\varepsilon)(x)dx \to \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

Thus c_ε is Cauchy. Similarly,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ix} f(x)dx \leftarrow \frac{1}{2\pi}\int_0^{2\pi} e^{-ix}(f * \eta_\varepsilon)(x)dx = a_\varepsilon$$

so a_ε is Cauchy; the same holds for b_ε. Thus the coefficients $a_\varepsilon, b_\varepsilon, c_\varepsilon$ are Cauchy, so converge to some a, b, c, and hence

$$(f * \eta_\varepsilon)(x) \to a e^{ix} + b e^{-ix} + c$$

pointwise everywhere as $\varepsilon \to 0$. Since the left-hand side converges a.e. to f, we conclude that f agrees with the right-hand side a.e. $\square$

Fall 2011 Problem 3: Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Fix $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$.

(a) Show that

$$[f * g](x) := \int_{\mathbb{R}^3} f(x - y)g(y)dy$$

defines a continuous function on $\mathbb{R}^3$.

(b) Moreover, show that $[f * g](x) \to 0$ as $|x| \to \infty$.

Proof. (a): We first claim that, for each $g \in L^q(\mathbb{R}^3)$, the function $v \mapsto \tau_v g$, where $\tau_v g(x) = g(x + v)$, is continuous. Indeed, if $g \in C_c(\mathbb{R}^3)$ and $v_n \to v$, then $\tau_{v_n} g \to \tau_v g$ uniformly, hence in $L^q(\mathbb{R}^3)$. If $g \in L^q(\mathbb{R}^3)$, $v \in \mathbb{R}^3$, and $\varepsilon > 0$ arbitrary, then we may find $g' \in C_c(\mathbb{R}^3)$ with $\|g - g'\|_{L^q} < \varepsilon/4$. Let $\delta > 0$ be such that $\|v' - v\| < \delta$ implies $\|\tau_{v'} g' - \tau_v g'\|_{L^q} < \varepsilon/2$. Then

$$\|\tau_{v'} g - \tau_v g\|_{L^q(\mathbb{R}^3)} \leq \|\tau_{v'} g - \tau_{v'} g'\|_{L^q(\mathbb{R}^3)} + \|\tau_{v'} g' - \tau_v g'\|_{L^q(\mathbb{R}^3)} + \|\tau_v g' - \tau_v g\|_{L^q(\mathbb{R}^3)} < \varepsilon$$

since $\tau_v, \tau_{v'}$ are linear and preserve L^q. Thus $v \mapsto \tau_v g$ is continuous for each choice of $g \in L^q(\mathbb{R}^3)$.

By L^p-L^q duality, for each choice of f, g, the function

$$x \mapsto \int_{\mathbb{R}^3} f(-y)\tau_x g(y)dy = \int_{\mathbb{R}^3} f(-y)g(y+x)dy$$

is a composition of two continuous functions, hence is continuous. Changing variables, we conclude that

$$x \mapsto \int_{\mathbb{R}^3} f(y)g(x-y)dy = [f * g](x)$$

is continuous, as was to be shown.

(b): We may assume $\|f\|_{L^p} = 1 = \|g\|_{L^q}$. Let $\varepsilon > 0$ be arbitrary. Since the sequence of functions $\{1_{B(0,n)}(x)|f(x)|^p\}_{n=1}^{\infty}$ is uniformly dominated by the integrable function $|f|^p$, we see by DCT that

$$\lim_{n\to\infty} \int_{B(0,n)} |f(x)|^p dx = \int_{\mathbb{R}^3} |f(x)|^p$$

In particular, we may find an $N_1 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^3 \setminus B(0,N_1)} |f(x)|^p dx < (\varepsilon/2)^p$$

Similarly, we may find $N_2 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^3 \setminus B(0,N_2)} |g(x)|^q dx < (\varepsilon/2)^q$$

Suppose $\|x\| > N_1 + N_2$. Then

$$|[f * g](x)| \leq \int_{B(0,N_2)} |f(x-y)g(y)|dy + \int_{\mathbb{R}^3 \setminus B(0,N_2)} |f(x-y)g(y)|dy$$

By Hölder, we have

$$\int_{B(0,N_2)} |f(x-y)g(y)|dy \leq \|g\|_{L^q} \left(\int_{B(x,N_2)} |f(y)|^p dy \right)^{1/p} < \varepsilon/2$$

since $B(x, N_2)$ is disjoint from $B(0, N_1)$. Similarly,

$$\int_{\mathbb{R}^3 \setminus B(0,N_2)} |f(x-y)g(y)|dy \leq \|f\|_{L^p(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3 \setminus B(0,N_2)} |g(y)|^q \right)^{1/q} < \varepsilon/2$$

by the assumption on N_2. Thus $|[f * g](x)| < \varepsilon$ for large $\|x\|$, as was to be demonstrated. $\square$

Winter 2007 Problem 3: Let $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$ denote the convolution of f and g. Fix $g \in L^1(\mathbb{R})$. Do the following:

(a) Show that $A_g(f) := f * g$ is a bounded operator $L^1(\mathbb{R}) \to L^1(\mathbb{R})$.

(b) Suppose in addition that $g \geq 0$. Find the corresponding norm $\|A_g\|_{L^1 \to L^1}$.

Proof. (a): By Tonelli,

$$\|f * g\|_{L^1(\mathbb{R})} \leq \int \int |f(x-y)||g(y)| dy dx = \int |g(y)| \int |f(x-y)| dx dy = \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$$

so that A_g is bounded, and $\|A_g\|_{L^1 \to L^1} \leq \|g\|_{L^1}$.

(b): Let $f \geq 0$ be any measurable function with $\|f\|_{L^1(\mathbb{R})} = 1$. Then

$$\|A_g(f)\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(y) dy dx = \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$$

so that $\|A_g\|_{L^1 \to L^1} = \|g\|_{L^1}$, using the fact mentioned in (a) above. $\square$

We omit (c) here.

Hints and remarks about the preceding problems

Fall 2021 Problem 1.

Hint, part 1: as a warm-up, show that $\int f\varphi' dx = 0$ *for all* φ *smooth and periodic implies that* f *is a.e. constant.*

Hint, part 2: mollify f.

Hint, part 3: consider mollifications $f * \eta_t$, *as considered on the homework. The mollified* f *satisfies the same identity by Fubini-Tonelli. So we may assume* f *is smooth; then, integrate by parts and solve an ODE.*

Remark. It's also possible to handle this directly by Fourier analysis techniques.

Fall 2011 Problem 3.

Hint for (a): Hölder + uniform integrability.

Hint for (b): Hölder + dominated convergence.

Winter 2007 Problem 3.

Hint for (a): Fubini-Tonelli.

Hint for (b): for one inequality, use Fubini-Tonelli. For the other, sample $g \approx \delta_0$.

Remark. Part (c) asks you to demonstrate that the only $f \in L^1(\mathbb{R})$ with $f * f = f$ is $f = 0$. This is best handled with Fourier analysis techniques. Another approach is to consider the $g \geq 0$ case, and consider the function $\eta(t) = -t \log t$, together with Jensen.

10 : Week 10

The following is a collection of problems that I might have used if the quarter ran several weeks longer. Some of them are directly relevant to what we have done so far (e.g. the Banach-Alaoglu and weak-$*$ topology), and some would have been relevant later (uniform boundedness, open mapping).

I suggest the document be used as such:

- For this week, I will consider the following problems: Fall 2021 Problem 4, Spring 2018 Problem 6, and Spring 2017 Problem 4.
- I strongly suggest that you think about Fall 2019 Problem 9 (in the uniform boundedness principle section), and the non-qual open mapping theorem problem, once you have learned those two results.
- The remaining problems I will leave as options if you want to further build on your familiarity with the techniques.

10.1 Weak and weak-$*$ topologies, and Banach-Alaoglu

Non-qual problem: Let X be a compact metric space and $T : X \to X$ a homeomorphism. Show that there exists an *invariant* Radon probability measure for T, i.e. a Radon probability measure μ such that $T_*\mu := \mu \circ T^{-1}$ is equal to μ.

Fall 2021 Problem 4: Let $r_1 > r_2 > \cdots > 0$. For each positive integer n, let $\mathcal{C}_n$ be a pairwise disjoint collection of 2^n closed disks of radius r_n in $[0,1]^2$, and assume that every member of $\mathcal{C}_n$ contains exactly two members of $\mathcal{C}_{n+1}$. Let $K_n = \bigcup_{D\in\mathcal{C}_n} D$, and let $K = \bigcap_{n=1}^{\infty} K_n$.

(a) Prove that there is a Borel probability measure μ such that $\mu(K) = 1$ and $\mu(D) = 2^{-n}$ for every $D \in \mathcal{C}_n$.

(b) Prove that K is the support of μ; that is, the smallest closed set whose measure equals 1.

Proof. (a): For each n and $D \in \mathcal{C}_n$, let c_D be the center of D, and we write

$$\mu_n = 2^{-n} \sum_{D\in\mathcal{C}_n} \delta_{c_D}$$

where as usual δ_{c_D} denotes the point mass at c_D. We claim that $\{\mu_n\}_{n=1}^{\infty}$ is weak-$*$ convergent. To do this, let $f \in C([0,1]^2)$ be arbitrary; we will show that the pairings $\langle f, \mu_n\rangle$ converge in $\mathbb{R}$.

Let $\varepsilon > 0$ be arbitrary and $\delta > 0$ be such that $\|x - y\| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for any $x, y \in [0,1]^2$; here $\|\cdot\|$ is the usual Euclidean distance. Let $N \in \mathbb{N}$ be such that $r_N < \delta/2$.

Then, for any $n, m \geq N$, where we assume $m > n$ without loss of generality,

$$\langle f, \mu_n \rangle - \langle f, \mu_m \rangle = 2^{-n} \sum_{D \in \mathcal{C}_n} f(c_D) - 2^{-m} \sum_{D \in \mathcal{C}_m} f(c_D)$$
$$= 2^{-n} \sum_{D \in \mathcal{C}_n} \left[f(c_D) - 2^{n-m} \sum_{\substack{D' \in \mathcal{C}_m \\ D' \subseteq D}} f(c_{D'}) \right]$$

because each $D' \in \mathcal{C}_m$ is a subset of a unique $D \in \mathcal{C}_n$; furthermore, for each $D \in \mathcal{C}_n$, there are 2^{m-n}-many $D' \in \mathcal{C}_m$ such that $D' \subseteq D$. Thus,

$$|\langle f, \mu_n \rangle - \langle f, \mu_m \rangle| \leq 2^{-n} \sum_{D \in \mathcal{C}_n} \left| f(c_D) - 2^{n-m} \sum_{\substack{D' \in \mathcal{C}_m \\ D' \subseteq D}} f(c_{D'}) \right|$$
$$\leq 2^{-n} \sum_{D \in \mathcal{C}_n} 2^{n-m} \sum_{\substack{D' \in \mathcal{C}_m \\ D' \subseteq D}} |f(c_D) - f(c_{D'})|$$
$$< 2^{-n} \sum_{D \in \mathcal{C}_n} 2^{n-m} \sum_{\substack{D' \in \mathcal{C}_m \\ D' \subseteq D}} \varepsilon = \varepsilon$$

Thus we have shown that $\langle f, \mu_n \rangle$ is Cauchy, hence convergent.

Finally, since $\{\mu_n\}_{n=1}^{\infty}$ is a sequence in the (norm-closed) unit ball $(M([0,1]^2))_1$, and the latter is a compact metrizable space in the weak-$*$ topology, it follows that there is some $\mu \in M([0,1]^2)$ for which $\mu_{n_k} \rightharpoonup \mu$ in the weak-$*$ topology for a suitable subsequence $k \mapsto n_k$. Thus $\langle f, \mu_{n_k} \rangle \to \langle f, \mu \rangle$ as $k \to \infty$ for each $f \in C([0,1]^2)$, so we further have $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$ as $n \to \infty$. Thus in fact $\mu_n \rightharpoonup \mu$ in the weak-$*$ topology.

We claim that this μ does the job. Testing against $f \equiv 1$, we see that $\mu([0,1]^2) = 1$. For each $n \in \mathbb{N}$ and $D \in \mathcal{C}_n$, we may find a sequence $\{f_k\}_{k=1}^{\infty}$ in $[0,1]^2$ such that $f_k \downarrow 1_D$, $f_k \equiv 1$ on D, $0 \leq f_k \leq 1$, $f_k \equiv 0$ on each $D' \in \mathcal{C}_n$ other than D. Then

$$\langle f_k, \mu \rangle = \lim_j \langle f_k, \mu_j \rangle = 2^{-n}$$

since the value is 2^{-n} for all $j \geq n$. By dominated convergence,

$$\mu(D) = \lim_k \int f_k d\mu = 2^{-n}$$

as claimed.

In particular, $\mu(K_n) = 1$ for all n. Since the K_n are nested, continuity from above implies

$$\mu(K) = \lim_n \mu(K_n) = 1$$

as was to be shown.

(b): Since μ is positive and $\mu([0,1]^2) = \mu(K)$, we see that $\operatorname{supp}(\mu) \subseteq K$. Pick any $x \in K$ and $\varepsilon > 0$. In particular, $x \in K_n$ for every n, so for each n we may find $D_{n,x} \in \mathcal{C}_n$ such that $x \in D_{n,x}$. Let $n \in \mathbb{N}$ be such that $r_n < \varepsilon/2$. Then $B(x, \varepsilon) \supseteq D_{n,x}$, so

$$\mu(B(x, \varepsilon)) \geq \mu(D_{n,x}) = 2^{-n} > 0.$$

Since $\varepsilon > 0$ was arbitrary, we see that $x \in \operatorname{supp}(\mu)$. Since $x \in K$ was arbitrary, we conclude that $\operatorname{supp}(\mu) = K$, as was to be shown. $\square$

Spring 2018 Problem 6 [slightly modified]: Let $\mathcal{P}([0,1])$ denote the space of Borel probability measures on $[0,1]$ and $\mathcal{P}([0,1]^2)$ denote the space of Borel probability measures on $[0,1]^2$. Fix $\mu, \nu \in \mathcal{P}([0,1])$ and define

$$\mathcal{M} = \Big\{ \gamma \in \mathcal{P}([0,1]^2) : \iint_{[0,1]^2} f(x)g(y)d\gamma(x,y) = \int_{[0,1]} f(x)d\mu(x) \int_{[0,1]} g(y)d\nu(y) \\ \text{for all } f, g \in C([0,1]) \Big\}$$

Show that $F : \mathcal{M} \to \mathbb{R}$ defined by

$$F(\gamma) = \iint_{[0,1]^2} \sin^2(\pi(\theta - \phi))d\gamma(\theta, \phi)$$

achieves its infimum on $\mathcal{M}$.

Proof. We claim that, if $\mathcal{M}$ is equipped with the weak-$*$ topology, then $\mathcal{M}$ is compact and F is continuous.

First, we demonstrate that $\mathcal{M}$ is precompact, i.e. for any sequence $\{\gamma_n\}_{n=1}^\infty$ in $\mathcal{M}$ there is some $\gamma \in \mathcal{P}([0,1]^2)$ such that $\gamma_n \rightharpoonup \gamma$. Note that $[0,1]^2$ is compact and every $\gamma \in \mathcal{P}([0,1]^2)$ has total mass 1, so by Banach-Alaoglu, $\mathcal{P}([0,1]^2)$ is compact. Thus the closure of $\mathcal{M}$ in $\mathcal{P}([0,1]^2)$ is compact, hence $\mathcal{M}$ is precompact, as claimed.

Next, we demonstrate that in fact $\mathcal{M}$ is closed. Suppose $\gamma_n \rightharpoonup \gamma$. Then, for each $\varepsilon > 0$ and $f, g \in C([0,1])$, there is $n \in \mathbb{N}$ such that

$$\begin{aligned} \iint_{[0,1]^2} f(x)g(y)d\gamma(x,y) &= \iint_{[0,1]^2} f(x)g(y)d\gamma_n(x,y) + O(\varepsilon) \\ &= \int_{[0,1]} f(x)d\mu(x) \int_{[0,1]} g(y)d\nu(y) + O(\varepsilon) \end{aligned}$$

Thus, for each $f, g \in C([0,1])$ and each $\varepsilon > 0$,

$$\left| \iint_{[0,1]^2} f(x)g(y)d\gamma(x,y) - \int_{[0,1]} f(x)d\mu(x) \int_{[0,1]} g(y)d\nu(y) \right| < \varepsilon$$

so in particular

$$\iint_{[0,1]^2} f(x)g(y)d\gamma(x,y) = \int_{[0,1]} f(x)d\mu(x) \int_{[0,1]} g(y)d\nu(y)$$

and hence $\gamma \in \mathcal{M}$, i.e. $\mathcal{M}$ is closed.

Lastly, we demonstrate that F is continuous. If $\gamma_n \rightharpoonup \gamma$, then in particular for

$$h(x,y) = \sin^2(\pi(x-y)) \in C([0,1])$$

we get

$$\iint_{[0,1]^2} h(x,y)d\gamma_n(x,y) \to \iint_{[0,1]^2} h(x,y)d\gamma(x,y)$$

which is the statement

$$F(\gamma_n) \to F(\gamma)$$

i.e. F is continuous (since $\mathcal{M}$ is metrizable, sequences are all that is needed!). Thus, F is a continuous function on a compact space, so achieves its minimum. $\square$

Fall 2020 Problem 2: Show that there is a constant $c \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_0^1 f(x)\cos(\sin(n\pi x))dx = c\int_0^1 f(x)dx$$

for every $f \in L^1([0,1])$.

Spring 2022 Problem 3: Let X be a real Banach space and let X' be its dual. If $Y \subseteq X$, then let

$$Y^\perp := \{\ell \in X' : \ell(y) = 0 \forall y \in Y\}$$

On the other hand, if $Z \subseteq X'$, then let

$${}^\perp Z := \{x \in X : \ell(x) = 0 \forall \ell \in Z\}$$

(a) Prove that ${}^\perp(Y^\perp)$ is the closed linear span of Y in X for any $Y \subseteq X$.

(b) Provide an example of a real Banach space X and a subset $Z \subseteq X'$ for which $({}^\perp Z)^\perp$ is not the closed linear span of Z in X'. [Hint: try something involving the spaces $L^1(m), C([0,1])$, and $L^\infty(m)$, where m is Lebesgue measure on $[0,1]$.]

For the next problem, you may find it convenient to quote the following result of Baire:

Theorem (Baire). If X is a compact metric space and $\phi : X \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous, then there is a sequence of functions $f_n : X \to \mathbb{R}$ such that $f_n \le f_{n+1}$ and $\phi = \lim_n f_n$ pointwise.

Fall 2016 Problem 3: If X is a compact metric space, we denote by $\mathcal{P}(X)$ the set of positive Borel measures μ on X with $\mu(X) = 1$. By a theorem of Baire, one can prove the following:

(a) Let $\phi : X \to [0,\infty]$ be a lower semi-continuous function on a compact metric space X. If μ and μ_n for $n \in \mathbb{N}$ are in $\mathcal{P}(X)$ and $\mu_n \rightharpoonup \mu$ with respect to the weak topology on $\mathcal{P}(X)$, then

$$\int \phi d\mu \le \liminf_{n\to\infty} \int \phi d\mu_n \tag{10.1}$$

(b) Let $K \subseteq \mathbb{R}^d$ be a compact set. For $\mu \in \mathcal{P}(K)$, we define

$$E(\mu) := \int_K \int_K \frac{1}{\|x-y\|} d\mu(x)d\mu(y)$$

Here $\|z\|$ denotes the Euclidean norm of $z \in \mathbb{R}^d$.

Show that the function $E : \mathcal{P}(K) \to [0,\infty]$ attains its minimum on $\mathcal{P}(K)$ (possibly ∞).

Reminder: a function f is said to be lower semi-continuous *if* $\liminf_{y\to x} f(y) \geq f(x)$ *for all* x.

Spring 2017 Problem 4: For $n \geq 1$, let $a_n : [0,1) \to \{0,1\}$ denote the n^{th} digit in the binary expansion of x, so that

$$x = \sum_{n\geq 1} a_n(x)2^{-n} \quad \text{for all } x \in [0,1)$$

where we remove digit expansion ambiguity by requiring that $\liminf_n a_n(x) = 0$ for all $x \in [0,1)$. Let $M([0,1))$ denote the space of finite signed Borel measures on $[0,1)$ and define linear functionals L_n on $M([0,1))$ via

$$L_n(\mu) = \int_0^1 a_n(x)d\mu(x)$$

Show that no subsequence of the L_n converge in the weak-$*$ topology on $M([0,1))^*$.

Solution taken from here. It suffices, for any subsequence n_k, to identify $\mu \in M([0,1))$ such that the sequence $L_{n_k}(\mu)$ does not converge as $k \to \infty$. Notice that, if $\mu = \delta_b$ for some $b \in [0,1)$, then $L_{n_k}(\mu)$ is the entry in position n_k in the binary expansion of b. Thus, if we take

$$b = \sum_{k=1}^{\infty} (k \bmod 2)2^{-n_k}$$

then $L_{n_k}(\delta_b) = (k \bmod 2)$, which does not converge as $k \to \infty$. $\square$

Fall 2017 Problem 4: Consider the Banach space $V = C([0,1])$ of all real-valued continuous functions on $[0,1]$ equipped with the supremum norm. Let $B = \{f \in V : \|f\| \leq 1\}$ be the closed unit ball in V.

Show that there exists a bounded linear functional $\Lambda : V \to \mathbb{R}$ such that $\Lambda(B)$ is an open subset of $\mathbb{R}$.

Proof. Let Λ be defined by

$$\Lambda(f) = \int_0^1 f(x)dx + f(1) - f(0)$$

It is easy to see that Λ is a bounded linear functional. We claim that $\Lambda(B) = (-3,3)$. Since $\Lambda(B)$ is clearly connected and symmetric, for the $\supseteq$ containment it suffices to find f_n such that $\Lambda(f_n) > 3 - \frac{1}{n}$. To this end, define

$$f_n(x) = \begin{cases} 2nx - 1 & x \leq \frac{1}{n} \\ 1 & \frac{1}{n} < x \leq 1 \end{cases}$$

Clearly $f_n \in B$ for each n. Also,

$$\Lambda(f) = \frac{1}{2n} + (1 - \frac{1}{n}) + 1 - (-1) = 3 - \frac{1}{2n} > 3 - \frac{1}{n}$$

so $(-3,3) \subseteq \Lambda(B)$.

We consider the reverse inclusion. Note that

$$|\Lambda(B)| \leq \int_0^1 |f(x)|dx + |f(1) - f(0)| \leq 3,$$

and equality can only hold if $|f(x)| = 1$ a.e. and $|f(0)| = |f(1)| = 1$ while $f(0), f(1)$ have opposite signs. We claim this cannot happen; indeed, we may assume $f(1) = 1 = -f(0)$, and then we will find $x \in (0,1)$ and $\varepsilon \in (0,1)$ such that $|f(y)| \leq \varepsilon$ for $|y - x| \leq \varepsilon$, so that $|f(x)|$ is not a.e. 1. Thus $\Lambda(B) \subseteq [-3,3]$ and $3 \notin \Lambda(B)$, so by symmetry $-3 \notin \Lambda(B)$ as well. Thus $\Lambda(B) = (-3,3)$, as was to be shown. $\square$

10.2 Uniform boundedness principle

Fall 2019 Problem 5: Let $\mathcal{H}$ be a Hilbert space with the scalar product of x, y denoted by $\langle x, y \rangle$, and let $A, B : \mathcal{H} \to \mathcal{H}$ be (everywhere-defined) linear operators with

$$\forall x, y \in \mathcal{H} : \quad \langle Bx, y \rangle = \langle x, Ay \rangle$$

Then A and B are both bounded (and thus continuous).

Problem from mathoverflow: Consider the Hilbert space $\ell^2(\mathbb{N})$, and consider a matrix $A = [a_{ij}]_{i,j}$, consisting of nonnegative entries, such that, for all $y \in \ell^2(\mathbb{N})$, the entries of the vector Ay all converge, and the vector Ay also belongs to $\ell^2(\mathbb{N})$. Show that A is a bounded linear map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$.

Important remark: we are not here claiming that every linear map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is bounded!

The statement is also true when the entries are assumed only to be real numbers.

10.3 Open mapping theorem

Non-qual problem.[10] The following display is a *false* statement:

If V is a vector space and $\|\cdot\|_1, \|\cdot\|_2$ are two norms on V such that $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ are complete, then $\|\cdot\|_1, \|\cdot\|_2$ are equivalent.

What follows is a "proof" of this statement. Identify the mistake in the argument!

Denote $\|\cdot\|_3 = \|\cdot\|_1 + \|\cdot\|_2$. Then $\|\cdot\|_3$ is another norm on V: indeed,

$$\|x+y\|_3 = \|x+y\|_1 + \|x+y\|_2 \leq \|x\|_1 + \|y\|_1 + \|x\|_2 + \|y\|_2 = \|x\|_3 + \|y\|_3$$

and the other axioms are obvious.

[10] Taken from an old mathoverflow post that I cannot find.

We claim that $(V, \|\cdot\|_3)$ is complete as well. Indeed, if $\{x_n\}_{n=1}^\infty$ is Cauchy in $\|\cdot\|_3$, then it is clearly Cauchy in $\|\cdot\|_1, \|\cdot\|_2$, so by completeness we may find $y \in V$ such that $x_n \to y$ in $\|\cdot\|_1, \|\cdot\|_2$. But then

$$\|x_n - y\|_3 = \|x_n - y\|_1 + \|x_n - y\|_2 \to 0$$

so $x_n \to y$ in $\|\cdot\|_3$. Thus $(V, \|\cdot\|_3)$ is complete.

Let $T : (V, \|\cdot\|_3) \to (V, \|\cdot\|_1)$ be the identity map on V. Then T is clearly linear. Furthermore, T is bounded:

$$\|Tv\|_1 = \|v\|_1 \leq \|v\|_1 + \|v\|_2 = \|v\|_3$$

Additionally, T is bijective. In particular, T is surjective, so by the open mapping theorem we see that T is open. Thus T is an open continuous bijection, so T is a homeomorphism. Since T is a linear homeomorphism, we in particular have that $\|\cdot\|_1$ and $\|\cdot\|_3$ are equivalent.

By the same argument, $\|\cdot\|_2$ and $\|\cdot\|_3$ are equivalent. But, as we have seen, equivalence of norms is transitive, so $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, as was to be shown.

Remark. As an example to demonstrate that the statement itself cannot be true: one can demonstrate without too much difficulty that $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$ are isomorphic as vector spaces, but it turns out that they are not isomorphic as normed vector spaces (here, "isomorphism" means a linear bijection that is open and bounded); this can be shown with a little more difficulty by considering adjoints. Consequently, we can regard $\ell^1(\mathbb{N}), \ell^2(\mathbb{N})$ as the same vector space with two incomparable complete norms.

Hints and remarks about the preceding problems

Fall 2021 Problem 4.

Hint: approximate μ by models μ_n at level n. Use compactness to find limit points.

Remark. Recall that our construction of the Cantor measure last quarter was very involved. With Banach-Alaoglu, we can instead find Cantor measure by a simple approximation procedure.

Spring 2018 Problem 6.

Hint: compactness and continuity.

Remark. Many optimization problems can be characterized by attempting to find suitable topologies such that the functional (in our case, F) is continuous (or semi-continuous), and the domain is compact. Depending on the subject, this might be the weak-$*$ topology on measures, L^p topology, C^k topology, etc.

Spring 2017 Problem 4.

Hint: it suffices to consider arbitrary subsequence $k \mapsto n_k$ and find μ such that $L_n(\mu)$ does not converge. Try something of the form $\mu = \delta_b$ for suitably-defined $b \in [0, 1)$.

Remark. Why doesn't this contradict Banach-Alaoglu?