A.S. Troelstra, D. van Dalen, Constructivism in Mathematics Corrections

Corrections to Volume 1

compiled by Anne S. Troelstra and Joan Rand Moschovakis
19 July 2018

32 Add to E1.3.1: "*Hint.* A good notation helps in bookkeeping. For example, use (q, r) for the pair of proofs q, r; π_1, π_2 for the projections (unpairing functions); for a proof p of $A \vee B$ from a proof of A or a proof q of B we write $\langle 0, p\rangle$ and $\langle 1, q\rangle$ respectively. The lambda notation can be used to describe functions: $\lambda x.t$ is the function described by the term t as function of x."

32, line 1 of E1.3.3 read "…König's lemma in 2.4 for decidable trees"

32, E1.3.3 Change the hint to: "*Hint.* Use a decidable predicate A such that $\forall n(A(n) \vee \neg A(n)$, the n such that $A(n)$, if existing, is unique, but $?(\exists n A(n) \to \exists k(A(2k) \vee \exists k A(2k+1))$."

38 in the first and second group of displayed prooftrees, interchange $\wedge \mathrm{E_r}$ and $\wedge \mathrm{E_l}$.

41^2 change "y not free in A." into "y not free in A; y must be free for x in A."

41^4 change "y not free in A." into "y not free in A; y must be free for x in A."

53_{12} change "(by (1))" to "(by (2))".

58^7 read "left to right" for "right to left".

60_{15} for "context" read "contexts".

64_1 read "I-isolating (I-spreading)".

65^2 read "**IQC**" for "**MQC**".

68_5 read "$\forall x(A \to B)$" in place of "$\exists x(A \to B)$".

68_1 "$x \notin \mathrm{FV}(B)$" in place of "$x \in \mathrm{FV}(B)$".

69_7 "$\forall$I" in place of "$\exists$I".

72_8 replace "or $y \notin \mathrm{FV}(B)$" by "(or $y \notin \mathrm{FV}(B)$ and y free for x in B)".

87_3 read “C-saturated” for “$\mathcal{L}$(C)-saturated”.

88^{16} “$\Gamma^{k+1} = \Gamma^k$” in place of “$\Gamma_{k+1} = \Gamma_k$”.

89_9 “C^*_{n+2}-saturated” instead of “C^*_{n+1}-saturated”.

90 in the diagram on the right, replace “$\langle 0, 1, 2\rangle$” by “$\langle 0, 1, 3\rangle$”.

91^2 read “of [finite] tree models”.

91_2 read “$S(k) = S(k') \vee S(k') = S(t_{k,i})$”.

91, line 3 of 6.11, insert “$k_0 \in K^*$”, after “$K^* \subset K$”.

$92^{8,9}$ read “$k \Vdash B_1 \to B_2$ iff $\forall k' \geq k(k' \Vdash B_1 \Rightarrow k' \Vdash B_2)$, which implies $\forall k' \geq^* k(k' \Vdash^* B_1 \Rightarrow k' \Vdash^* B_2)$, hence $k \Vdash$”.

103_6 delete the first (.

108^{11} “$x \notin \mathrm{FV}(P)$” instead of “$x \notin \mathrm{FV}(A)$”.

109_8 “$k_0 \nVdash' P$ for P prime” should be “for P prime, $k_0 \Vdash' P$ iff for all i $\mathcal{K}_i \Vdash P$”.

$109_{6,5}$ delete “or an existential formula”.

113^9 read “Π^0_2”.

122_{14} “$y \dot{-} 0 = y$” instead of “$y \dot{-} 0 = 0$”.

130 just above (4) read “so assume”.

133^9 read “upper bound”.

133_4 insert space between “$(\vec{z})$” and “(the”.

138_8 drop “D” before “$\vdash$”.

$141^{2,4}$ change “$\vdash$” to “$|$”.

142_{11} read “**5.12.**” for “**5.1.2.**”.

143 Correct the proof of 5.16, by replacing the lines 2–6 of the proof by:

PROOF. The proof of (i) follows, under the assumption $C \in \mathcal{RH}$, by showing that, if D is a s.p.p of C, then $C \vdash D \Rightarrow C|D$, using induction on D. It suffices to show:

(a) if $C \vdash A \wedge B$ and $C \vdash A \Rightarrow C|A$, $C \vdash B \Rightarrow C|B$, then $C|(A \wedge B)$;

(b) if $C \vdash A \to B$ and $C \vdash B \Rightarrow C|B$, then $C|(A \to B)$;

(c) if $C \vdash \forall x A$ and for all n, $C \vdash A[x/\bar{n}] \Rightarrow C|A[x/\bar{n}]$, then $C|\forall x A$;

(d) if P is prime and $C \vdash P$ then $C|P$.

157-158 Subsections 3.7.9 – 3.7.13 have to be replaced; see the corrected version at the end of the list of corrections for volume 1.

160^7 read $\Lambda^1 x.\varphi$ for $\Lambda^1 x.t$ and read $\Lambda^0\alpha.t$ for $\Lambda^0\alpha.\varphi$.

179^2 read "Let $\psi, \chi, \theta, \ldots$".

179_7 read "5.13" for "5.12".

179 replace in E3.5.3 "$\vdash$" everywhere by "$\Gamma\vdash$".

202^{10} for "$(x \simeq_1 x$" read "$\forall y(x \simeq_1 x'$", and for "$(\exists y\, A(x,y)$" read "$\exists\, y(A(x,y)$".

202^{17} for "$\approx_1$" read "$\simeq_1$".

210_{15} for "$\neg\exists y(\delta' y = 0)$" read "$\neg\exists x(\delta' x = 0)$" and for "$\neg\exists y(\delta'' y = 0)$" read "$\neg\exists x(\delta'' x = 0)$".

$202_{11,12,14}$ "$\mathrm{MP_{PR}}$" instead of "$\mathrm{MR_{PR}}$" (four times).

242, in second line of exercise **4.2.2** , replace "DC-D" by "DC-$\mathbb{N}\times$D".

245_3 read "$\beta \in \bar{\alpha}x$" for "$\beta \in \bar{\alpha}y$".

247^1 for "$\forall m \preceq n$" read "$\forall m \prec n$".

262^9 for "$\alpha k + 1$" read "αk" and read "$\leq$" for "$<$".

264_{11} read "$x \# 0$".

274, line 4 in 5.10 read "order-isomorphic".

274_{14} read "and ϕ leaves all rationals".

276^8 for "$\mathbb{R}$" read "$\mathbb{Q}$".

287_3 "$\exists$-PEM" should be "$\forall$-PEM".

293 Remark concerning the proof of 6.1.3. The property of covering is interpreted as: there is an operation which from a sufiiciently good approximation of a point d in the interval tells us either $d \in A$ or $d \in B$. (Both may be true, but the operation makes a choice.)

297^8 read "$m|x - y|$" for "m".

300^{11} read "$a_{n-1}x^{n-i-1}$" for "$a_{n-i}x^{n-i-1}$".

300^{13} read "2^{-1}" for "2^{-i}".

303, line 4 of 3.2, for "$U(2^{-n-1}$" read "$U(2^{-n}$".

306^1 replace "2^{-m}" by "2^{-m+1}" (twice).

306^{10} replace "2^{-m}" by "2^{-m+1}".

306^{12} replace "2^{-m}" by "2^{-m+1}".

308_5 read "$k+m+1$ points".

309_{10} for "$\frac{1}{2}(r_n - s_n)$" read "$\frac{1}{2}(s_n - r_n)$".

310 in (i) of 4.5 read "$J_m \subset I_n^-$" for "$J_n \subset I_n^-$".

310_{13} delete "from $\langle I_n \rangle_n$".

311 in the picture the left end of I_n^- should coincide with the left end of J_{p+3}.

312 in the first proof of 6.4.8, the appeal to 6.4.5 is not needed; a singular cover by means of intervals with rational endpoints suffices.

312^1 for "quasi-order" read "quasi-cover".

312^2 "I_2" should be "I_1".

313_1 for "$\geq |I_n|$" read "$\geq |J_m|$".

313_2 read "$J_m \subset I_n$ for some m".

317^{13} read "$\neg\exists n'\forall mn$" for "$\neg\exists n'mn$".

317_3 displayed formula (1) should end with "$< 2^{-n-1}$)".

319, 3 lines below (6), for "$-q_y|$" read "$-q_{\{z\}}(y)|$", and "$q_{\{z\}\phi(y,\psi y)}|$" for "$q_{\phi(y,\psi y)}|$".

320^1 read "From (9)" for "From (7)".

338 correct Rasiowa (1954) , replacing "1, 229–231" by "2, 121–124"

pages I-IX from all page numbers of the preliminaries, in lower case roman numerals, one has to subtract 4 ("xv" becomes "xi" etc.) (the preliminaries ought to have been inserted after the table of contents and numbered accordingly).

Corrected version of subsections 3.7.9–15

3.7.9. Definition. In **EL** we introduce abbreviations

$$\begin{array}{rcl} \alpha(\beta)=x & := & \exists y(\alpha(\bar\beta y)=x+1\wedge\forall y'{<}y(\alpha(\bar\beta y')=0)\\ \alpha|\beta=\gamma & := & \forall x(\lambda n.\alpha(\hat x * n)(\beta)=\gamma x)\wedge\alpha 0=0,\ \text{or equivalently}\\ & & \forall x\exists y(\alpha(\hat x*\bar\beta y)=\gamma x+1\wedge\forall y'{<}y(\alpha(\hat x*\bar\beta y')=0))\wedge\alpha 0=0. \end{array}$$

We may introduce $|$, $\cdot(\cdot)$ as primitive operators in a conservative extension $\mathbf{EL}^*$ based on E^+-logic, also called LPT, the logic of partial terms. □

Definition. $\mathbf{EL}^*$ is a conservative extension of **EL** based on the logic of partial terms, to which $\lambda\alpha\beta.\alpha|\beta$ and $\lambda\alpha\beta.\alpha(\beta)$ have been added as primitive operations. Numerical lambda-abstraction satisfies:

$$s\downarrow\wedge(\lambda x.t)\downarrow\to(\lambda x.t)s=t[x/s],\quad(\lambda x.t)\downarrow\leftrightarrow\forall x(t\downarrow).$$

For function application we require strictness:

$$\phi t\downarrow\leftrightarrow\phi\downarrow\wedge t\downarrow.$$

(The implication from right to left must hold since $\phi\downarrow$ is supposed to imply totality of the function denoted by ϕ.) For Rec we have

$$\mathrm{Rec}(t,\phi)\downarrow\leftrightarrow t\downarrow\wedge\phi\downarrow.$$

We also require strictness for the operations $\cdot|\cdot$ and $\cdot(\cdot)$, that is to say

$$\phi|\psi\downarrow\to\psi\downarrow\wedge\ \phi\downarrow,\quad\phi(\psi)\downarrow\to\psi\downarrow\wedge\ \phi\downarrow$$

□

3.7.10. Definition. (*The class of neighbourhood functions*)

$$\alpha\in K^*\ :=\ \alpha 0=0\wedge\forall nm(\alpha n>0\to\alpha n=\alpha(n*m))\wedge\forall\beta\exists x(\alpha(\bar\beta x)>0).$$

□

Crucial is the following

3.7.11. Proposition. To each function term ϕ of $\mathbf{EL}^*$, and each numerical term t of $\mathbf{EL}^*$ and free function variable α, we can construct function terms $\Phi^\alpha_\phi\in K^*$, $\Phi^\alpha_t\in K^*$ of **EL** such that if γ is free for α in ϕ or t respectively (and does not occur in ϕ or t unless γ is α):

(i) $\Phi^\alpha_\phi|\gamma\simeq\phi[\alpha/\gamma]$ and in particular $\Phi^\alpha_\phi|\alpha\simeq\phi$;

(ii) $(\Phi^\alpha_t|\gamma)\downarrow$ iff $t[\alpha/\gamma]\downarrow$;

(iii) $t[\alpha/\gamma]\downarrow\to(\Phi^\alpha_t|\gamma)0=t[\alpha/\gamma]$ and in particular $t\downarrow\to(\Phi^\alpha_t|\alpha)0=t$;

(iv) $\mathrm{FV}(\Phi^\alpha_t)\subset\mathrm{FV}(t)\setminus\{\alpha\}$, $\mathrm{FV}(\Phi^\alpha_\phi)\subset\mathrm{FV}(\phi)\setminus\{\alpha\}$, $\Phi^\alpha_t,\Phi^\alpha_\phi$ primitive recursive in their free variables.

Proof. (i)–(iv) are proved by simultaneous induction on the construction of numerical and function terms. Instead of taking exactly the numerical and function terms as specified for $\mathbf{EL}^*$, we give the proof, for reasons of convenience, for a slightly different set interdefinable with the set generated by the primitives of $\mathbf{EL}^*$. In particular, instead of taking $\boldsymbol{r}$ as a primitive, we take its special case It ("iterator") satisfying

$$\mathrm{It}(t,\psi)0 = t, \quad \mathrm{It}(t,\psi)(Sz) = \psi(\mathrm{It}(t,\psi)z).$$

From It we can easily define Rec satisfying

$$\mathrm{Rec}(t,\phi)0 = t,\ \mathrm{Rec}(t,\phi)(Sz) = \phi(\mathrm{Rec}(t,\phi)z, z),$$

by taking

$$\mathrm{Rec}(t,\phi) := \lambda z.j_0(\mathrm{It}(j(t,0), \lambda u.j(\phi u, S(j_1 u)))z),$$

and from Rec we can readily define $\boldsymbol{r}$.

The reason that we need a function term with partial continuous application $\mid$ to represent a numerical term (instead of application $\cdot(\cdot)$) is that a numerical term t may contain function-terms as subterms, which all have to be defined by the strictness condition of the logic of partial terms; this is a Π^0_2-condition and cannot be expressed by definedness of a numerical term.

We consider a few typical cases. In all cases we put $\Phi^\alpha_\phi 0 = 0$, $\Phi^\alpha_t 0 = 0$. If $u \neq 0$ then $u = \hat{z} * \bar{\gamma}n$ for $z = (u)_0$ and every γ such that $(u)_{i+1} = \gamma(i)$ for all $i < n$, where $|u| = n + 1$. For easier comprehension, for $u \neq 0$ we state the definitions of $\Phi^\alpha_\phi u$, $\Phi^\alpha_t u$ in terms of $\hat{z} * \bar{\gamma}n$ (for fresh variables z, n and γ) instead of u.

Case 1. $t \equiv x$. Take $\Phi^\alpha_t(\hat{z} * \bar{\gamma}n) = x + 1$. Similarly for $t \equiv 0$.

Case 2. $\phi \equiv \alpha$. Take

$$\Phi^\alpha_\alpha(\hat{z} * \bar{\gamma}n) = \begin{cases} \gamma z + 1 \text{ if } z < n, \\ 0 \text{ otherwise.} \end{cases}$$

Case 3. $\phi \equiv \beta$, $\beta \not\equiv \alpha$. Put

$$\Phi^\alpha_\beta(\hat{z} * \bar{\gamma}n) = \beta z + 1 \text{ for all } n.$$

Case 4. $\phi \equiv S$. Put

$$\Phi^\alpha_S(\hat{z} * \bar{\gamma}n) = Sz + 1 \text{ for all } n.$$

Case 5. $t \equiv \phi t'$. Put

$$\Phi^\alpha_t(\hat{z} * \bar{\gamma}n) = \begin{cases} \Phi^\alpha_\phi(\langle \Phi^\alpha_{t'}(\hat{0} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) \text{ if } \Phi^\alpha_{t'}(\hat{0} * \bar{\gamma}n) > 0 \\ \qquad \wedge\, \Phi^\alpha_\phi(\langle \Phi^\alpha_{t'}(\hat{0} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) > 0 \\ \qquad \wedge\, \Phi^\alpha_{t'}(\hat{z} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 6. $\phi \equiv \lambda x.t$. Put

$$\Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) = \begin{cases} \Phi^\alpha_{t[x/z]}(\hat{0} * \bar{\gamma}n) \text{ if } \Phi^\alpha_{t[x/z]}(\hat{0} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_{t[x/j_0 z]}(\langle j_1 z\rangle * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 7. $\phi \equiv It(t, \psi)$. We put

$$\Phi^\alpha_\phi(\hat{0} * \bar{\gamma}n) = \begin{cases} \Phi^\alpha_t(\hat{0} * \bar{\gamma}n) \text{ if } \Phi^\alpha_t(\hat{0} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\psi(\hat{0} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

$$\Phi^\alpha_\phi(\langle Sz\rangle * \bar{\gamma}n) = \begin{cases} \phi^\alpha_\psi(\langle \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) \text{ if} \\ \qquad \phi^\alpha_\psi(\langle \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) \dot{-} 1\rangle * \bar{\gamma}n) > 0 \wedge \\ \qquad \Phi^\alpha_\phi(\hat{z} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\psi(\langle Sz\rangle * \bar{\gamma}n) > 0 \wedge \\ \qquad \Phi^\alpha_t(\langle Sz\rangle * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 8. $\phi \equiv j, j_1, j_2$. Easy and left to the reader.
Case 9. $t \equiv \phi(\psi)$. Take

$$\Phi^\alpha_{\phi(\psi)}(\hat{x} * \bar{\gamma}n) = \begin{cases} z + 1 \text{ if } \exists u{<}n \forall y{<}\mathrm{lth}(u)(\Phi^\alpha_\psi(\hat{y} * \bar{\gamma}n) = (u)_y + 1 \wedge \\ \qquad \Phi^\alpha_\phi(u) = z + 1) \wedge \Phi^\alpha_\phi(\hat{x} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\psi(\hat{x} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Case 10. $\phi \equiv \psi|\xi$. Take

$$\Phi^\alpha_{\psi|\xi}(\hat{x} * \bar{\gamma}n) = \begin{cases} z + 1 \text{ if } \exists u{<}n \forall y{<}\mathrm{lth}(u)(\Phi^\alpha_\xi(\hat{y} * \bar{\gamma}n) = (u)_y + 1) \wedge \\ \qquad \Phi^\alpha_\psi(\hat{x} * u) = z + 1) \wedge \Phi^\alpha_\psi(\hat{x} * \bar{\gamma}n) > 0 \wedge \Phi^\alpha_\xi(\hat{x} * \bar{\gamma}n) > 0, \\ 0 \text{ otherwise .} \end{cases}$$

Note that in Cases 5, 7 (for Sz), 9, 10 of this definition the last two conjuncts, and in Case 6 the last conjunct, are needed to guarantee strictness. □

3.7.12. Definition. Let

$$\alpha|(\beta_0, \ldots, \beta_{n-1}) := \alpha|\nu_n(\beta_0, \ldots, \beta_{n-1}),$$

$$\alpha(\beta_0, \ldots, \beta_{n-1}) := \alpha(\nu_n(\beta_0, \ldots, \beta_{n-1})).$$

□

We are now ready to build an analogue of ordinary recursion theory with partial continuous function application instead of partial recursive application. The preceding lemma has as consequence a version of the smn-theorem:

3.7.13. Theorem. (smn-theorem)

(i) There is a primitive recursive binary functional $\wedge_n$ such that

$$(\alpha \wedge_n \beta_0)|(\beta_1, \ldots, \beta_n) \simeq \alpha|(\beta_0, \ldots, \beta_n).$$

(ii) There is a primitive recursive binary functional $\wedge'_n$ such that

$$(\alpha \wedge'_n \beta_0)(\beta_1, \ldots, \beta_n) \simeq \alpha(\beta_0, \ldots, \beta_n).$$

Proof. Straightforward by the preceding lemma. □

3.7.14. (No change from **3.7.14.**)

3.7.15. Notation ($\Lambda^0 x$, $\Lambda^1 x$, $\Lambda^0 \alpha$, $\Lambda^1 \alpha$). On the basis of Proposition 3.7.11, for each numerical term t and each function term ϕ of **EL*** we can now define function terms of **EL**:

$$\Lambda^0 \alpha.t = \Phi^{\alpha}_{t},$$

$$\Lambda^1 \alpha.\phi = \Phi^{\alpha}_{\phi},$$

$$\Lambda^1 x.\phi = \Phi^{\alpha}_{\phi'},$$

where $\phi'[\alpha] := \phi[x/\alpha 0]$, and with the properties $(\Lambda^0 \alpha.t)|\alpha)(0) \simeq t$, $(\Lambda^1 \alpha.\phi)|\alpha \simeq \phi$ and $(\Lambda^1 x.\phi)|\lambda y.x \simeq \phi$.

Using Theorem 3.7.14, for each numerical term t of **EL*** there is a term t' (which we will denote by $\Lambda^0 x.t$) of **EL**, primitive recursive in the parameters of t minus x, such that $\{t'\}(x) \simeq t$ for all x.

Corrections to Volume 2

compiled by Anne S. Troelstra and Joan Rand Moschovakis
19 July 2018

357^{10} read "$\forall y \neg\neg A$" for "$\forall y\, A$".

357^{11} read "and so for $[x]_M$ a point of M, using Markov's principle" for "and so".

360 Delete formula (5) and replace the next line (line 11) by"(exercise).".

360^{16} read "(4)" for "(5)".

360_9 delete "For a ... see E7.3.2".

381 Delete exercise 7.3.2.

452 Due to an oversight, the axioms for $\mathbf{HA}_0^\omega$ as stated are too weak. Instead of the equality axioms as stated, one should use the formulation of in section 1.6.15 of A.S. Troelstra (editor), *Metamathematical Investigation of intuitionistic Arithmetic and Analysis*, Berlin 1973. The system $\mathbf{HA}_0^\omega$ is there called simply $\mathbf{HA}^\omega$. The equality axioms there required replacement in an arbitary context, for example $t[\mathbf{k}t_1t_2] = t[t_1]$ and $t[\mathbf{s}t_1t_2t_3] = t[t_1t_3(t_2t_3)]$. Another solution was proposed by Benno van den Berg in (*A note on arithmetic in finite types*, arXiv 1408.3557v2 [math.LO] 20 Sep 2016) namely adding to the axioms of $\mathbf{HA}_0^\omega$ one new congruence law $x =_0 y \to fx =_0 fy$, together with the axioms $\mathbf{b}xyz = x(yz)$ and $\mathbf{q}xyz = x(zy)$ for two new combinators $\mathbf{b}$ and $\mathbf{q}$, and defining equality at higher types in terms of equality at type 0 according to the principle of observational equivalence: $x =_\sigma y := \forall f^{\sigma\to 0}(fx =_0 fy)$. The congruence laws for equality at all finite types are provable in this version of $\mathbf{HA}_0^\omega$, correcting a circularity in the proof on pages 452-453. Then $\mathbf{HA}^\omega$ proves $\mathbf{b} = \mathbf{s}(\mathbf{ks})\mathbf{k}$ and $\mathbf{q} = \mathbf{b}(\mathbf{s}(\mathbf{bbs})(\mathbf{kk}))\mathbf{b}$.

476, 477 See at the end of this list.

478_{12} read "$h := \lambda\bar{x}.\mathbf{r}(f\bar{x})(\lambda uv.g\bar{x}(Pv)u)$".

542^7 read "$\Gamma \setminus \{A\} \vdash'$" for "$\Gamma \setminus \{A\} \vdash$".

543^{14} Read "S-successor" for "S-successor set".

543^{15} replace "$\theta \equiv$" by "The set $\theta \equiv$".

543_{17} delete "set".

543_1 read "$(P \to Q) \to P$" for "$P \to (Q \to P)$".

544 The first line of formulas should end with "$\ldots P \vdash P\}$".

547 line 7 of 5.6, read "$Et_0 \vee Et_1$" for "$Et_0 \wedge Et_1$".

565 last line of 9.4 read "maps" for "mappings".

566_1 Interchange "$\cap$" and "$\cup$".

583 line 5 of 2.11, read "$\mathrm{FV}(A_{i+j})$" for "$\mathrm{FV}(A_n)$".

656^{13} read "$\bar{\alpha}$" for "$\vec{\alpha}$".

660_{10} read "$(\#(\alpha_1, \ldots, \alpha_p) \rightarrow$" for "$((\alpha_1, \ldots, \alpha_p) \rightarrow$".

660_6 read "$A(\vec{\alpha}, x))$" for "$A(\vec{\alpha}, x)$".

660_3 read "$A(\vec{\alpha}, \vec{\beta})$" for "$A(\alpha, \beta)$".

661_6 read "$\dot{\forall}$" for "$\dot{\exists}$".

663^{10} read "$\vee$" for "$\wedge$".

663_3 read "$\vec{n} \circledast \vec{m}$" for "$\vec{n} * \vec{m}$".

680, line 1 of 1.5, insert after "*Beth model*": "The definition of Beth model obviously extends to the case where $(K, \leq_K)$ is a collection of spreads".

680, line 3 of 1.5, insert before "as follows": " ,where $(K', \preceq')$ is a set of spreads, ".

680, line 4 of 1.5, read "finite, inhabited, nondecreasing".

680_2 read "$(\bar{\alpha}x \Vdash A$ or" for "$(\bar{\alpha}x \Vdash$ or".

681 remark concerning the proof of the theorem in 1.5. If the Kripke model has no root, the corresponding Beth model becomes rather a collection of spreads instead of a single spread, which is more general than permitted by our definition of Beth model. But this does not otherwise affect the proof.

681 add at the end of 1.5:
REMARK. The construction permits many slight variations. If we restrict attention to Kripke models with a root, and restrict K' to finite inhabited nondecreasing sequences starting with a root, the construction works equally well; this variant has been illustrated in fig. 13.1".

681^{17} read "k_n" for "k".

683_4 $\mathrm{LS_K}$, in a degenerate case, may consist of a single sequence, i.e., lawlike and lawless coincide. But this does not affect the argument.

687 replace line 4 of 2.4. by "$k \Vdash P := \exists z \forall k' \succeq_z k(\vdash \Gamma_{k'} \to A)$.".

687 replace line 4 of 2.5. by "**Case 1.** For A prime apply lemma 2.3."

689^6 $\vdash_{x+lth(k)}$ instead of $\vdash_x$ (twice).

689_9 for "lemma 2.3" read "the covering property (lemma 1.2(i))".

853^4 read "recovered" for "removed".

872 correct Rasiowa (1954) , replacing "1, 229–231" by "2, 121–124"

XXX under "Howard, W.A. 1980" read "565" for "564".

Corrections to pages 476, 477.

It is not generally true that if $x \notin \mathrm{FV}(t')$, $y \not\equiv x$, then

$$\lambda x.(t[y/t']) \simeq (\lambda x.t)[y/t'],$$

(consider e.g. $t \equiv y$, $t' \equiv \boldsymbol{kk}$), but if $x \notin \mathrm{FV}(t')$, $y \notin \mathrm{FV}(t'')$, $y \not\equiv x$, then

$$\boldsymbol{E}t'' \to ((\lambda x.t)[y/t'])t'' \simeq t[x/t''][y/t'] \simeq t[y/t'][x/t''].$$

The failure of the first equation is due to the fact that $\lambda x.t$ has been defined by induction on the complexity of t. This necessitates some repairs. For example, the argument in $476^{1,2}$ should read:

"$\chi\chi \simeq (\lambda zy.x(zz)y)\chi \simeq (\lambda y.x(zz)y)[z/\chi]$, and since an expression $\lambda x. \ldots$ always exists, uniformly in the parameters, i.e. remains "existing" if we substitute existing objects for the free variables, we see that $\boldsymbol{E}(\mathrm{fix}(x))$; also ...".

Corresponding corrections (i.e. postponement of substitution in a defined lambda-term) has to be made in $476_{6,4}$ and 477_{10}. Lines 476_{6-3} are to be replaced by:

$$\begin{aligned} \boldsymbol{r}tt'0 &\simeq \phi\rho 0 \simeq \rho(\phi\rho)0 \simeq \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)((\lambda z.y(\phi\rho(Pz))z)[y/t'])000 \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)(t'(\phi\rho(P0))00 \\ &\simeq \boldsymbol{k}t0 \simeq t. \end{aligned}$$

If $n \in \mathbb{N}$, $n \neq 0$, then

$$\begin{aligned} \boldsymbol{r}tt'n &\simeq \phi\rho n \simeq \rho(\phi\rho)n \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)((\lambda z.y(\phi\rho(Pz))z)[y/t'])n0n \\ &\simeq \boldsymbol{d}(\boldsymbol{k}t)(t'(\phi\rho(Pn))n)0n \\ &\simeq t'(\phi\rho(Pn))n \simeq t'(\boldsymbol{r}tt'(Pn))n \quad \Box \end{aligned}$$

Lines 477_{10-7} are to be replaced by

$$\mu f \simeq \phi M f \simeq M(\phi M) f \simeq M \mu f \simeq \boldsymbol{d}(\boldsymbol{k}0)((\lambda g.S(xg))[x/\mu])(f0)0f^{+},$$

and this latter expression is equal to

$$\boldsymbol{k}0f^{+} = 0 \text{ if } f0 = 0, \text{ and}$$

$$(\lambda g.S(xg)[x/\mu])f^{+} \simeq S(\mu f^{+}) \text{ if } f0 > 0.$$