Minimum Classical Extensions of Constructive Theories

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Abstract. Reverse constructive mathematics, based on the pioneering work of Kleene, Vesley, Kreisel, Troelstra, Bishop, Bridges and Ishihara, is currently under development. Bishop constructivists tend to emulate the classical reverse mathematics of Friedman and Simpson. Veldman's reverse intuitionistic analysis and descriptive set theory split notions in the style of Brouwer. Kohlenbach's proof mining uses interpretations and translations to extract computational information from classical proofs. We identify the *classical content* of a constructive mathematical theory with the Gentzen negative interpretation of its classically correct part. In this sense **HA** and **PA** have the same classical content but intuitionistic and classical two-sorted recursive arithmetic with quantifier-free countable choice do not; Σ^0_1 numerical double negation shift expresses the precise difference. Other double negation shift and weak comprehension principles clarify the classical content of stronger constructive theories. Any consistent axiomatic theory **S** based on intuitionistic logic has a *minimum classical extension* $\mathbf{S}^{+g}$, obtained by adding to **S** the negative interpretations of its classically correct consequences. Subsystems of Kleene's intuitionistic analysis and supersystems of Bishop's constructive analysis provide interesting examples, with the help of constructive decomposition theorems.

1 There is virtue in simplicity.

The negative translations proposed by Gödel [7] and Gentzen [6] are straightforward syntactic methods for converting formulas of a full logical language into classically equivalent formulas not involving $\vee$ or $\exists$. The Gentzen negative translation E^g of a formula E replaces $\vee$ and $\exists$ by their classical equivalents in terms of $\neg$, & and $\forall$, but does not change $\rightarrow$. The Gödel translation also replaces $\rightarrow$ by its classical equivalent in terms of $\neg$ and &, but the simpler Gentzen version is more transparent and will be used in what follows. When necessary to guarantee the intuitionistic equivalence of $\neg\neg\mathrm{E}^g$ and E^g, prime formulas are replaced by their double negations; this step is omitted in applications where prime formulas are stable under double negation.

The negative translations of classical logical axioms and rules are correct by intuitionistic logic, so if E follows from Γ by classical logic then E^g follows from Γ^g by intuitionistic logic. With classical logic E and E^g are equivalent. Even with intuitionistic logic, $\neg\neg\mathrm{E}^g$ and E^g are equivalent.

1.1 Classical content in arithmetic and analysis

In what follows, the "language of arithmetic" may be any first-order language with equality, constants 0, $'$, $+, \cdot$ and possibly other constants for primitive recursive functions. Prime formulas are equations $\mathrm{s} = \mathrm{t}$ between terms. Classical arithmetic **PA** and intuitionistic arithmetic **HA** are expressed in this language.

The "language of analysis" is any two-sorted language extending the language of arithmetic, with variables $\mathrm{m}, \mathrm{n}, \ldots, \mathrm{x}, \mathrm{y}, \mathrm{z}$ over natural numbers and variables $\alpha, \beta, \gamma, \ldots$ over infinite sequences of natural numbers (i.e. one-place number-theoretic functions), with constants for additional primitive recursive functions and functionals. Function application is denoted by $\alpha(\mathrm{x})$, and equality at type 1 is defined extensionally. Troelstra's **EL** and Kleene and Vesley's **I** are expressed in the language of analysis; cf. [19],[21], [12].

Some applications involve intuitionistic arithmetic of arbitrary finite types $\mathbf{HA}^{\omega}$, an extension of **HA** in a language with variables over, and terms for, primitive recursive functions of all finite types. Prime formulas are equations between terms of the same type. There are intensional and extensional versions of $\mathbf{HA}^{\omega}$; for details see [19].

Definition 1. *The* **classical content** *of a formula* E *in the language of arithmetic, analysis, or the arithmetic of finite types is its Gentzen negative translation* E^g*. The* **classical content** Γ^g *of a collection* Γ *of formulas consistent with classical logic is the closure under intuitionistic logic of the set* $\{\mathrm{E}^g \colon \mathrm{E} \in \Gamma\}$.

Remark 1. If **S** is a formal system, based on intuitionistic logic but consistent with classical logic, and if **T** comes from **S** by adding one or more classically correct *logical* axiom schemas, then $\mathbf{S}^g = \mathbf{T}^g$ so the classical content of **S** is determined by the negative translations of its *mathematical* axioms. Following Kleene [11] we denote $\mathbf{S} + (\neg\neg \mathrm{A} \rightarrow \mathrm{A})$ by $\mathbf{S}^{\circ}$.

1.2 Minimum classical extension of a constructive theory

Definition 2. *If* **S** *is a formal system based on intuitionistic logic, in a language including* &, $\vee$, $\rightarrow$, $\neg$, *and quantifiers* $\forall$ *and* $\exists$ *of one or more sorts, then the* **classical subtheory** $\mathrm{cls}(\mathbf{S})$ *of* **S** *is the set of all classically correct theorems of* **S***; the* **classical content** *of* **S** *is* $(\mathrm{cls}(\mathbf{S}))^g$*; and the* **minimum classical extension** $\mathbf{S}^{+g}$ *of* **S** *is the closure under intuitionistic logic of* $\mathbf{S} \cup (\mathrm{cls}(\mathbf{S}))^g$.

Remark 2. If **S** is an intuitionistic subsystem of a (consistent) classical theory then $\mathrm{cls}(\mathbf{S}) = \mathbf{S}$, so $\mathbf{S}^{+g}$ is the closure under intuitionistic logic of $\mathbf{S} \cup \mathbf{S}^g$. Intuitionistic arithmetic is its own minimum classical extension because the negative interpretations of all the mathematical axioms of **HA** (and **PA**) are provable in **HA**. The prime formulas of arithmetic are equations between terms of type 0, which do not change under the translation because **HA** proves that they are decidable, hence stable under double negation.

However, the neutral (classically correct) "basic" subsystem **B** of Kleene and Vesley's formal system **I** for intuitionistic analysis does not contain its classical content, nor does Troelstra's recursive analysis **EL**. In each of these cases the classical content is nevertheless completely determined by the negative translations of the mathematical axioms of the system.

In the case that **S** is consistent but $\mathbf{S}^\circ$ is not, cls(**S**) may include more than the consequences of the classically correct axioms of **S**. Intuitionistic analysis **I** differs from **B** by just one axiom schema (which conflicts with $\mathbf{B}^\circ$), but **I** proves monotone bar induction (which is consistent with $\mathbf{B}^\circ$) while **B** does not. The question then is how to determine the classical subtheory of a subsystem **S** of **I** with a classically false continuity axiom or schema, in order to identify $\mathbf{S}^{+g}$.

2 Double negation shift and weak comprehension axioms

Double negation shift principles have long been studied as weaker alternatives to constructively questionable axioms like Markov's Principle (cf. [18], [1], [16], [5]). In [2] Brouwer himself used double negation shift to prove that the intuitionistic real numbers form a closed species, though he later rejected this argument.

The most general double negation shift schema, whose addition to intuitionistic predicate logic would suffice to prove Glivenko's Theorem, is

$$\mathrm{DNS}: \quad \forall \mathrm{x} \neg\neg \mathrm{A}(\mathrm{x}) \rightarrow \neg\neg \forall \mathrm{x} \mathrm{A}(\mathrm{x}).$$

The converse holds by intuitionistic predicate logic. The strength of an instance of double negation shift depends on the logical complexity of the formula A(x) and the domain of the variable x.

2.1 "Double negation shift for numbers" $\mathbf{DNS_0}$

DNS_0 denotes the restriction of DNS to cases where x is a number variable. AC_0 denotes the axiom schema of countable choice, which was accepted by Bishop and Brouwer. Danko Ilik argues in [8] that $\mathbf{HA}^\omega + \mathrm{AC}_0 + \mathrm{DNS}_0$ "is a distinct variety of Constructive Mathematics" because it satisfies existential instantiation, proves the double negation of Bishop's Limited Principle of Omniscience for numbers, refutes the recursive choice principle CT_0, and contains its classical content. He also observes that DNS_0 can replace Markov's Principle in consistency proofs for classical analysis (cf. [15]).

Proposition 1. $\mathbf{HA}^\omega + \mathrm{AC}_0 + \mathrm{DNS}_0$ *is its own minimum classical extension.*

Proof. $(\mathbf{HA}^\omega + \mathrm{AC}_0 + \mathrm{DNS}_0)^g = (\mathbf{HA}^\omega + \mathrm{AC}_0)^g$ by Remark 1 since DNS_0 is a classical logical schema. $(\mathbf{HA}^\omega)^g \subseteq \mathbf{HA}^\omega$ and $(\mathrm{AC}_0)^g \subseteq \mathbf{HA}^\omega + \mathrm{AC}_0 + \mathrm{DNS}_0$.

2.2 $\mathbf{\Sigma^0_1}$ double negation shift for numbers

Definition 3. *In the language of arithmetic or analysis,* Σ^0_1**-double negation shift for numbers** *is the schema*

$$\Sigma^0_1\text{-DNS}_0: \quad \forall \mathrm{x} \neg\neg \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \rightarrow \neg\neg \forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y})$$

where $\mathrm{A(x,y)}$ *is a formula with only bounded number quantifiers and no sequence quantifiers, but perhaps containing additional free variables.*

Proposition 2. $\mathbf{EL} + \Sigma^0_1\text{-DNS}_0$ *is its own minimum classical extension.*

Proof. The only axiom or schema of $\mathbf{EL} + \Sigma^0_1\text{-DNS}_0$ whose negative interpretation is not provable in **EL** is the quantifier-free countable choice schema

$$\text{QF-AC}_{00}: \quad \forall x \exists y A(x,y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$$

where $\mathrm{A(x,y)}$ may have additional free variables of both sorts but only bounded numerical quantifiers. Its negative interpretation is intuitionistically equivalent to $\forall x \neg\neg \exists y A^g(x,y) \rightarrow \neg\neg \exists \alpha \forall x A^g(x, \alpha(x))$, which follows easily from $\Sigma^0_1\text{-DNS}_0$ and QF-AC_{00} by intuitionisitic logic.

Definition 4. Two-sorted intuitionistic arithmetic $\mathbf{IA_1}$ *is the subsystem of Kleene's* **B** *obtained by omitting the axiom schemas of countable choice and bar induction.* **Intuitionistic recursive analysis IRA** *comes from* $\mathbf{IA_1}$ *by adding the recursive comprehension axiom*

$$\forall \rho [\forall x \exists y \rho(\langle x, y \rangle) = 0 \rightarrow \exists \alpha \forall x \rho(\langle x, \alpha(x) \rangle) = 0]^1$$

Remark 3. This special case is equivalent, over $\mathbf{IA}_1$, to QF-AC_{00}. Vafeiadou proved in [23] that **IRA** and **EL** are mathematically equivalent, in the sense of having a common definitional extension. See [17] for a precise description of $\mathbf{IA}_1$ and [22] for her comparison of **EL** with **IRA**.

Proposition 3. *Over* **EL** *and* **IRA**, $\Sigma^0_1\text{-DNS}_0$ *is interderivable with the case*

$$\forall \rho [\forall x \neg\neg \exists y \rho(\langle x, y \rangle) = 0 \rightarrow \neg\neg \forall x \exists y \rho(\langle x, y \rangle) = 0].$$

Proof. Each formula $\mathrm{A(x,y)}$ of the language of analysis with only bounded number quantifiers and no sequence quantifiers expresses a primitive recursive relation of its free variables, such that **EL** and **IRA** both prove

$$\exists \rho \forall x \forall y [A(x,y) \leftrightarrow \rho(\langle x, y \rangle) = 0].$$

Corollary 1. *(to Propositions 2, 3):*

(a) $(\mathbf{EL}_0)^{+g} = \mathbf{EL}_0$, *where* $\mathbf{EL}_0$ *comes from* **EL** *by omitting* QF-AC_{00}.
(b) $\mathbf{EL}^{+g} = \mathbf{EL} + \Sigma^0_1\text{-DNS}_0$.
(c) $(\mathbf{IA}_1)^{+g} = \mathbf{IA}_1$.
(d) $\mathbf{IRA}^{+g} = \mathbf{IRA} + \Sigma^0_1\text{-DNS}_0$

Proof. $\mathbf{EL}_0$ and $\mathbf{IA}_1$ prove the converse of QF-AC_{00}, and so $\Sigma^0_1\text{-DNS}_0$ follows from $(\text{QF-AC}_{00})^g$ in **EL** and **IRA**.

[1] $\langle x, y \rangle = 2^x \cdot 3^y$ is Kleene's code for the ordered pair of x and y; similarly for n-tuples.

2.3 Stronger restricted versions of $\mathbf{DNS_0}$

The full strength of DNS_0 is not needed to negatively interpret AC_0. In constructive or intuitionistic arithmetic and analysis, if $\mathrm{A(x)}$ is a *negative* formula (i.e. has no occurrences of $\exists$ or $\vee$) then $\neg\neg\mathrm{A(x)} \leftrightarrow \mathrm{A(x)}$ is provable, so double negation shift holds trivially for all negative formulas $\mathrm{A(x)}$.

Definition 5. *In the language of arithmetic or analysis,* DNS^-_{00} *denotes the restriction of* DNS *to the case where* x *is a number variable and* $\mathrm{A(x)}$ *is of the form* $\exists \mathrm{yA(x,y)}$ *where* $\mathrm{A(x,y)}$ *is negative. In the language of analysis,* DNS^-_{01} *denotes the restriction of* DNS *to the case where* x *is a number variable and* $\mathrm{A(x)}$ *is of the form* $\exists\alpha \mathrm{A(x,}\alpha)$ *where* $\mathrm{A(x,}\alpha)$ *is negative. In the language of* $\mathbf{HA}^\omega$, $\mathrm{DNS}^-_{0\infty}$ *includes all* $\mathrm{DNS}^-_{0\sigma}$ *for finite types* σ*.*

These restricted double negation shift schemas characterize the minimum classical extensions of theories with AC_{00}, AC_{01}, or the collection $\mathrm{AC}_{0\infty}$ of all $\mathrm{AC}_{0\sigma}$ for finite types σ. In particular, AC_{01} is the strong countable choice axiom schema of Kleene's **B**:

$$\mathrm{AC}_{01}: \quad \forall \mathrm{x} \exists \alpha \mathrm{A(x,}\alpha) \rightarrow \exists \alpha \forall \mathrm{x A(x,} \lambda \mathrm{y}.\alpha(\langle \mathrm{x,y}\rangle))$$

where x is free for α in $\mathrm{A(x,}\alpha)$, and AC_{00} is like $\mathrm{QF\text{-}AC}_{00}$ but with no restriction on $\mathrm{A(x,y)}$ except that the substitution of $\alpha(\mathrm{x})$ for y must be free.

Proposition 4. *Each of* $\mathbf{HA}^\omega + \mathrm{AC}_{0\infty} + \mathrm{DNS}^-_{0\infty}$, $\mathbf{EL}_0 + \mathrm{AC}_{00} + \mathrm{DNS}^-_{00}$, $\mathbf{IA}_1 + \mathrm{AC}_{00} + \mathrm{DNS}^-_{00}$, $\mathbf{EL}_0 + \mathrm{AC}_{01} + \mathrm{DNS}^-_{01}$ *and* $\mathbf{IA}_1 + \mathrm{AC}_{01} + \mathrm{DNS}^-_{01}$ *is its own minimum classical extension.*

Proof. As for Propositions 1 and 2.

Corollary 2. *(to Proposition 4):*

(a) $(\mathbf{HA}^\omega + \mathrm{AC}_{0\infty})^{+g} = \mathbf{HA}^\omega + \mathrm{AC}_{0\infty} + \mathrm{DNS}^-_{0\infty}$.
(b) $(\mathbf{EL}_0 + \mathrm{AC}_{0i})^{+g} = (\mathbf{EL} + \mathrm{AC}_{0i})^{+g} = \mathbf{EL}_0 + \mathrm{AC}_{0i} + \mathrm{DNS}^-_{0i}$ *for i = 0,1.*
(c) $(\mathbf{IA}_1 + \mathrm{AC}_{0i})^{+g} = (\mathbf{IRA} + \mathrm{AC}_{0i})^{+g} = \mathbf{IA}_1 + \mathrm{AC}_{0i} + \mathrm{DNS}^-_{0i}$ *for i = 0,1.*

Proof. As for Corollary 1.

Remark 4. Many syntactic refinements of these results are possible. For example, Proposition 15 in [4] shows that (b) holds for $\Pi^0_1\text{-}\mathrm{AC}_{00}$ (with the hypothesis $\forall \mathrm{x} \exists \mathrm{y} \forall \mathrm{z} \rho(\langle \mathrm{x,y,z}\rangle) = 0$) and $\Sigma^0_2\text{-}\mathrm{DNS}_0$ (with hypothesis $\forall \mathrm{x} \neg\neg \exists \mathrm{y} \forall \mathrm{z} \rho(\langle \mathrm{x,y,z}\rangle) = 0$) in place of AC_{00} and DNS^-_{00}, respectively. Observe that $\Sigma^0_1\text{-}\mathrm{DNS}_0$ and $\Sigma^0_2\text{-}\mathrm{DNS}_0$ are instances of DNS^-_{00}, but e.g. $\Sigma^0_3\text{-}\mathrm{DNS}_0$ is not.

2.4 Weak comprehension principles

Over **EL** or **IRA**, a number-theoretic relation $\mathrm{A(x)}$ (perhaps with number and sequence parameters) has a characteristic function for x only if it satisfies $\forall \mathrm{x}(\mathrm{A(x)} \vee \neg \mathrm{A(x)})$. The weak comprehension schema

$$\neg\neg\ \mathrm{CF}_0: \quad \neg\neg\exists\zeta\forall \mathrm{x}(\zeta(\mathrm{x}) = 0 \leftrightarrow \mathrm{A(x)})$$

asserts only that it is *consistent* to assume that A(x) has a characteristic function for x. Here we consider two restricted versions of $\neg\neg\,\mathrm{CF}_0$. The first one is[2]

$$\neg\neg\,\Pi^0_1\text{-}\mathrm{CF}_0: \quad \forall\alpha\neg\neg\exists\zeta\forall x(\zeta(x) = 0 \leftrightarrow \forall y\alpha(\langle x, y\rangle) = 0).$$

By formula induction, **IRA** + $\neg\neg\,\Pi^0_1\text{-}\mathrm{CF}_0$ proves $\neg\neg\exists\zeta\forall x(\zeta(x) = 0 \leftrightarrow A(x))$ for all negative arithmetical formulas A(x), with universal number quantifiers and free variables of both types allowed. The same holds with **EL** in place of **IRA**.

The second is $\neg\neg\,\mathrm{CF}_0^-$, the restriction of $\neg\neg\,\mathrm{CF}_0$ to negative formulas A(x). Over $\mathbf{IA}_1$ or $\mathbf{EL}_0$, $\neg\neg\,\mathrm{CF}_0^-$ is equivalent to the negative translation of the schema

$$\mathrm{CF_d}: \ \forall x(A(x) \vee \neg A(x)) \rightarrow \exists\alpha\forall x[\alpha(x) \leq 1 \ \& \ (\alpha(x) = 0 \leftrightarrow A(x))].$$

Over $\mathbf{IA}_1$ or $\mathbf{EL}_0$ the conjunction of $\mathrm{CF_d}$ and QF-AC_{00} is equivalent (cf. [23]) to the countable comprehension ("unique choice") schema $\mathrm{AC}_{00}!$ which is like AC_{00} but with hypothesis $\forall x\exists! yA(x, y)$, where in general $\exists! yB(y)$ abbreviates $\exists yB(y) \ \& \ \forall y\forall z(B(y) \ \& \ B(z) \rightarrow y = z)$. Over $\mathbf{IA}_1$ or $\mathbf{EL}_0$, AC_{00} is stronger than $\mathrm{AC}_{00}!$ (which is stronger than QF-AC_{00}), but the negative translations of AC_{00} and $\mathrm{AC}_{00}!$ are equivalent. Putting these facts together gives refinements of Corollary 2(b),(c) (for $i = 0$), and additional characterizations.

Theorem 1. *Let* AC^{Ar}_{00} *be the restriction of* AC_{00} *to* arithmetical *predicates* A(x, y)*, with number quantifiers and free variables of both types allowed. Then*

(a) $(\mathbf{IA}_1 + \mathrm{AC}^{Ar}_{00})^{+g} = \mathbf{IA}_1 + \mathrm{AC}^{Ar}_{00} + \Sigma^0_1\text{-}\mathrm{DNS}_0 + \neg\neg\,\Pi^0_1\text{-}\mathrm{CF}_0$.
(b) $(\mathbf{IA}_1 + \mathrm{AC}_{00})^{+g} = \mathbf{IA}_1 + \mathrm{AC}_{00} + \Sigma^0_1\text{-}\mathrm{DNS}_0 + \neg\neg\,\mathrm{CF}_0^-$.
(c) $(\mathbf{IA}_1 + \mathrm{AC}_{00}!)^{+g} = (\mathbf{IRA} + \mathrm{CF_d})^{+g} = \mathbf{IRA} + \mathrm{CF_d} + \Sigma^0_1\text{-}\mathrm{DNS}_0 + \neg\neg\,\mathrm{CF}_0^-$.

Each of these results remains true with $\mathbf{EL_0}$ *in place of* $\mathbf{IA_1}$*, and* **EL** *in place of* **IRA**.

3 Bar induction, a weak continuity principle, and BD-N

As Iris Loeb [13] observes, constructive reverse mathematics currently lacks a unifying methodology. According to Ishihara [9] its aim is "to classify various theorems in intuitionistic, constructive recursive and classical mathematics by logical principles, function existence axioms and their combinations" over a weak constructive base built on intuitionistic logic. Resulting decomposition theorems can help to extract and compare the classical content of constructive and semi-constructive theories. Two examples, one involving bar induction and the other involving a weak continuity principle, illustrate the method.

Brouwer's bar theorem, although not accepted by Bishop, is of interest to constructive mathematicians. The fan theorem FT, which follows from the bar theorem but is conservative over Heyting arithmetic by [20], has the property that the minimum classical extension of **IRA** + FT proves that intuitionistic predicate logic is complete for its intended interpretation ([3], [14]).

[2] Over **EL** or **IRA**, $\neg\neg\,\Pi^0_1\text{-}\mathrm{CF}_0$ entails the principle $\neg\neg\Pi^0_1$-LEM in [5], and similarly for $\neg\neg\,\Sigma^0_1\text{-}\mathrm{CF}_0$ and $\neg\neg\Sigma^0_1$-LEM.

3.1 Three versions of bar induction

Kleene chose to axiomatize his neutral basic system **B** by $\mathbf{IA}_1$ + AC_{01} + $\mathrm{BI_d}$, where $\mathrm{BI_d}$ is "*decidable* bar induction:"

$$\mathrm{BI_d}: \quad \forall\alpha\exists x R(\overline{\alpha}(x)) \;\&\; \forall w(R(w) \vee \neg R(w)) \;\&\; \forall w(R(w) \to A(w)) \;\&\; \forall w(\forall x A(w * \langle x+1\rangle) \to A(w)) \to A(1).$$

Classical bar induction BI° simply drops the premise $\forall w(R(w) \vee \neg R(w))$, and *monotone* bar induction (which is provable in **I** but not in **B**) is

$$\mathrm{BI_{mon}}: \quad \forall\alpha\exists x R(\overline{\alpha}(x)) \;\&\; \forall w(R(w) \to \forall u R(w * u)) \;\&\; \forall w(R(w) \to A(w)) \;\&\; \forall w(\forall x A(w * \langle x+1\rangle) \to A(w)) \to A(1).$$

Here $\overline{\alpha}(0) = 1$ and $\overline{\alpha}(x+1) = \langle \alpha(0)+1, \ldots, \alpha(x)+1\rangle$. We let w, u vary over Kleene's "sequence numbers" (so w determines the length lh(w) of the sequence w codes); $w * v$ codes the concatenation of the sequences coded by w and v, $\langle x+1\rangle$ codes the sequence whose only term is x, and 1 codes the empty sequence.

Kleene proved ([12] p. 79) that $\mathbf{IA_1} + \mathrm{AC}_{00} + \mathrm{BI_{mon}} \vdash \mathrm{BI_d}$, so $\mathrm{BI_{mon}}$ lies between $\mathrm{BI_d}$ and BI° in strength over $\mathbf{IA_1} + \mathrm{AC}_{00}$.

Proposition 5. *$\mathrm{BI_d}$ has the same classical content as BI° over $\mathbf{IA_1}$ or $\mathbf{EL_0}$.*

Proof. The only difference between $\mathrm{BI_d}$ and BI° is a classically provable premise $\forall w(R(w) \vee \neg R(w))$ whose negative interpretation is provable intuitionistically.

3.2 A double negation shift principle for functions

In the absence of countable choice, the double negation shift principle

$$\mathrm{DNS}_1^-: \quad \forall\alpha\neg\neg\exists x R(\overline{\alpha}(x)) \to \neg\neg\forall\alpha\exists x R(\overline{\alpha}(x)),$$

where R(w) is a negative formula of the language of analysis, is a sufficient addition to prove the double negation translation of $\mathrm{BI_d}$ and $\mathrm{BI_{mon}}$.[3]

Theorem 2. *The minimum classical extensions of* **B** *and its subsystems with* AC_{01} *replaced by* AC_{00} *or by* $\mathrm{QF\text{-}AC}_{00}$ *or omitted altogether are computed as follows. Similar results hold with* $\mathbf{EL_0}$ *in place of* $\mathbf{IA_1}$.

(a) $\mathbf{B}^{+g} \equiv (\mathbf{IA_1} + \mathrm{AC}_{01} + \mathrm{BI_d})^{+g} = \mathbf{B} + (\mathrm{AC}_{01})^g = \mathbf{B} + \mathrm{DNS}_{01}^-$.
(b) $(\mathbf{IA_1} + \mathrm{AC}_{00} + \mathrm{BI_d})^{+g} = \mathbf{IA_1} + \mathrm{AC}_{00} + \mathrm{BI_d} + \mathrm{DNS}_{00}^-$.
(c) $(\mathbf{IRA} + \mathrm{BI_d})^{+g} = \mathbf{IRA} + \mathrm{BI_d} + (\mathrm{BI}^\circ)^g + \Sigma_1^0\text{-}\mathrm{DNS}_0 \subseteq \mathbf{IRA} + \mathrm{BI_d} + \mathrm{DNS}_1^-$.
(d) $(\mathbf{IA_1} + \mathrm{BI_d})^{+g} = \mathbf{IA_1} + \mathrm{BI_d} + (\mathrm{BI}^\circ)^g \subseteq \mathbf{IA_1} + \mathrm{BI_d} + \mathrm{DNS}_1^-$.

Proof. $\mathbf{IA_1} + \mathrm{AC}_{00} + (\neg\neg A \to A) \vdash \mathrm{BI}^\circ$ (*26.1° on p. 53 of [12]) and therefore $(\mathbf{IA_1} + \mathrm{AC}_{00})^g \vdash (\mathrm{BI}^\circ)^g$. Proposition 5, Corollary 2(b),(c), Corollary 1(c),(d) and the (easy) fact that $\mathbf{IA_1} + \mathrm{DNS}_1^- \vdash \Sigma_1^0\text{-}\mathrm{DNS}_0$ complete the argument.

[3] Only the special case $\Sigma_1^0\text{-}\mathrm{DNS}_1$ is needed for the version of bar induction labeled ^{x}26.3b in [12]; cf. [14], [15].

3.3 Applying a typical constructive decomposition theorem

Kleene proved (*27.23 on p. 87 of [12]) that $\mathbf{IRA} + \mathrm{BI}^\circ$ entails the "weak limited principle of omniscience" WLPO, which is inconsistent with $\mathbf{I}$. In [4] Fujiwara proved that BI° is equivalent over $\mathbf{EL}_0$ to $\mathrm{BI}_{\mathrm{mon}} + \mathrm{CD}$, where CD is the constant domain axiom schema $\forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \mathrm{B}) \to (\forall \mathrm{x}\mathrm{A}(\mathrm{x}) \vee \mathrm{B})$ (with x not free in B).

Proposition 6. $\mathrm{BI}_{\mathrm{mon}}$ *has the same classical content as* BI° *over* $\mathbf{IA_1}$ *or* $\mathbf{EL_0}$*, so* $(\mathbf{IA_1} + \mathrm{BI}_{\mathrm{d}})^g = (\mathbf{IA_1} + \mathrm{BI}_{\mathrm{mon}})^g = \mathbf{IA_1} + (\mathrm{BI}^\circ)^g$ *and similarly with* $\mathbf{EL_0}$ *in place of* $\mathbf{IA_1}$*.*

Proof. CD is a classical logical schema whose negative interpretation is provable by intuitionistic logic. Use Fujiwara's decomposition theorem and Proposition 5.

Remark 5. It follows that the neutral subsystem $\mathbf{B}$ of Kleene and Vesley's intuitionistic analysis I has the same classical content as the variant $\mathbf{B'}$ with $\mathrm{BI}_{\mathrm{mon}}$ replacing BI_{d}, and so $(\mathbf{B'})^{+g} \equiv (\mathbf{IA_1} + \mathrm{AC}_{01} + \mathrm{BI}_{\mathrm{mon}})^{+g} = \mathbf{B'} + \mathrm{DNS}_{01}^{-}$.

3.4 Applying an atypical constructive decomposition theorem

In [10], over a constructive base theory $\mathbf{EL'} \equiv \mathbf{EL} + \Pi^0_1\text{-}\mathrm{AC}_{00}$, Ishihara and Schuster decompose a restricted version

$$\begin{aligned}\mathrm{WC\text{-}N'}:\ & \forall\alpha\exists\mathrm{n}\forall\mathrm{k}\,\sigma(\langle\overline{\alpha}(\mathrm{k}),\mathrm{n}\rangle) = 0\\ & \quad \&\ \forall\mathrm{w}\forall\mathrm{m}\forall\mathrm{n}(\sigma(\langle\mathrm{w},\mathrm{m}\rangle) = 0\ \&\ \mathrm{m} \le \mathrm{n} \to \sigma(\langle\mathrm{w},\mathrm{n}\rangle) = 0)\\ & \qquad \to \forall\alpha\exists\mathrm{n}\exists\mathrm{m}\forall\beta \in \overline{\alpha}(\mathrm{m})\forall\mathrm{k}\,\sigma(\langle\overline{\beta}(\mathrm{k}),\mathrm{n}\rangle) = 0\end{aligned}$$

of weak continuity into a classically correct mathematical principle

$$\mathrm{BD\text{-}N}:\ \forall\alpha\exists\mathrm{m}\forall\mathrm{n} \ge \mathrm{m}\,\beta(\alpha(\mathrm{n})) < \mathrm{n} \to \exists\mathrm{m}\forall\mathrm{n}\,\beta(\mathrm{n}) \le \mathrm{m}$$

and a classically false logical principle $\neg\forall\alpha(\exists\mathrm{x}\,\alpha(\mathrm{x}) \ne 0 \vee \forall\mathrm{x}\,\alpha(\mathrm{x}) = 0)$ negating the limited principle of omniscience LPO.

Proposition 7. *The minimum classical extensions of* $\mathbf{EL'}$ *and* $\mathbf{EL'}$ + BD-N *are computed as follows, and similarly for* $\mathbf{IRA} + \Pi^0_1\text{-}\mathrm{AC}_{00}$ $(\equiv \mathbf{IA_1} + \Pi^0_1\text{-}\mathrm{AC}_{00})$ *in place of* $\mathbf{EL'}$*.*

(a) $\mathbf{EL'}^{+g} \equiv (\mathbf{EL}_0 + \Pi^0_1\text{-}\mathrm{AC}_{00})^{+g} = \mathbf{EL'} + \Sigma^0_2\text{-}\mathrm{DNS}_0$.
(b) $(\mathbf{EL'} + \mathrm{BD\text{-}N})^{+g} = \mathbf{EL'} + \mathrm{BD\text{-}N} + \Sigma^0_2\text{-}\mathrm{DNS}_0$.

Proof. (a) holds by Corollary 1(a) and Remark 4. $\mathbf{EL'}$ + BD-N is consistent with classical logic and satisfies (b) because $\mathbf{EL}^{+g}$ proves the contrapositive of $(\mathrm{BD\text{-}N})^g$, which is equivalent to $(\mathrm{BD\text{-}N})^g$ over $\mathbf{EL}$.

Remark 6. $\mathbf{IA}_1 + \Pi^0_1\text{-}\mathrm{AC}_{00} + \mathrm{WC\text{-}N'}$ is a subsystem of Kleene's $\mathbf{I}$ which is consistent (by [12]) but is not consistent with classical logic. The next theorems, discovered by the second author, essentially trivialize the notion of minimum classical extension for intuitionistic systems inconsistent with classical logic.

Theorem 3. $(\mathbf{EL'} + \mathrm{WC\text{-}N'})^{+g} = \mathbf{EL'} + \mathrm{WC\text{-}N'} + (\Gamma^\circ)^g$ *where* Γ° *is the set of all classically true sentences in the language of* $\mathbf{EL'}$*. A corresponding result holds with* $\mathbf{IA_1} + \Pi^0_1\text{-}\mathrm{AC}_{00}$ *in place of* $\mathbf{EL'}$*.*

Proof. $\mathbf{EL}' + \text{WC-N}' \vdash \neg\text{LPO}$ by Ishihara and Schuster's decomposition theorem, therefore $\mathbf{EL}' + \text{WC-N}' \vdash (\neg\text{LPO} \vee \text{E})$ for every formula E. If $\text{E} \in \Gamma^0$ then $(\neg\text{LPO} \vee \text{E}) \in \Gamma^\circ$, so $(\neg\text{LPO} \vee \text{E}) \in \text{cls}(\mathbf{EL}' + \text{WC-N}')$. But $(\neg\text{LPO} \vee \text{E})^g$ is just $\neg(\neg\neg\text{LPO}^g \& \neg\text{E}^g)$, which is equivalent by intuitionistic logic to $\neg\neg\text{E}^g$ and hence to E^g. So $(\Gamma^\circ)^g \subseteq (\text{cls}(\mathbf{EL}' + \text{WC-N}'))^g$, and the reverse inclusion is immediate from the definitions.

Theorem 4. $\mathbf{I}^{+g} = \mathbf{I} + (\Gamma^\circ)^g$ *where now* Γ° *is the set of all classically true sentences in the language of* $\mathbf{I}$.

Proof. $(\text{cls}(\mathbf{I}))^g = (\Gamma^\circ)^g$ by an argument similar to the proof of Theorem 3, but with WLPO ($\equiv \forall\alpha(\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0)$) in place of LPO, using the fact that $\mathbf{I} \vdash \neg\text{WLPO}$ by *27.17 on p. 84 of [12]. Thus $\mathbf{I}^{+g} = \mathbf{I} + (\Gamma^\circ)^g$.

Remark 7. By Lemma 8.4a in [12] every classically true negative sentence of the language of analysis is realizable by a primitive recursive function, so (by Theorem 9.3 of [12]) Kleene's function-realizability guarantees the consistency of $(\mathbf{EL}' + \text{WC-N}')^{+g}$ and of $\mathbf{S}^{+g}$ for every subsystem $\mathbf{S}$ of $\mathbf{I}$.

4 Conclusion

We have suggested a way to define the minimum classical extension $\mathbf{S}^{+g}$ of a mathematical theory $\mathbf{S}$ based on intuitionistic logic, with examples from arithmetic, analysis and the arithmetic of finite types. If $\mathbf{S} + (\neg\neg\text{A} \to \text{A})$ is consistent, then classical and constructive mathematics coexist in $\mathbf{S}^{+g}$ exactly as far as the mathematical axioms of $\mathbf{S}$ permit.

For example, if $\mathbf{S}$ is a classically correct subsystem of Kleene's intuitionistic analysis $\mathbf{I} \equiv \mathbf{B} + \text{CC}_{11}$, then by viewing the choice sequence variables $\alpha, \beta, \ldots$ alternatively as variables over classical one-place number-theoretic functions, restricting the language and logic by omitting the symbols $\vee$ and $\exists$ with their axioms and rules, and replacing each mathematical axiom of $\mathbf{S}$ by its negative translation, one obtains a faithful copy of $\mathbf{S}^\circ \equiv \mathbf{S} + (\neg\neg\text{A} \to \text{A})$ within the extended intuitionistic system $\mathbf{S}^{+g}$. In particular, $\mathbf{B}^{+g}$ includes the negative translation of a system $\mathbf{C} \equiv \mathbf{B}^\circ$ of classical analysis with countable choice.

On the other hand, if $\mathbf{S}$ refutes a classical logical principle, then $\mathbf{S}^{+g}$ includes the negative translations of *all classically true sentences in the language of* $\mathbf{S}$. In particular, $\mathbf{I}^{+g}$ contains a negative version of true classical analysis.

We conclude that only constructive and semi-constructive systems consistent with classical logic have interesting minimum classical extensions, and typical constructive decomposition theorems assist in comparing their classical content.

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