# Markov's Principle, Markov's Rule and the Notion of Constructive Proof

Intuitionism, Computation and Proof:
Selected themes from the research of G. Kreisel
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## What is Markov's Principle?

The original version asserts that if a recursive algorithm cannot fail to converge, then it converges. Over intuitionistic arithmetic **HA** this can be expressed equivalently by the formula

- ★ $\mathrm{MP}_0$: $\forall e \forall x[\neg\forall y \neg T(e,x,y) \rightarrow \exists y T(e,x,y)]$, or the schema
- ★ $\mathrm{MP}_{\mathrm{QF}}$: $\neg\forall x \neg A(x) \rightarrow \exists x A(x)$, where $A(x)$ has only bounded quantifiers but may have additional variables free.

In the language of predicate logic Markov's Principle becomes

- ★ $\mathrm{MP}_{\mathrm{D}}$: $\forall x(A(x) \vee \neg A(x))$ & $\neg\forall x \neg A(x) \rightarrow \exists x A(x)$.

**Proposition.** **HA** $+ \mathrm{MP}_{\mathrm{D}} \vdash \mathrm{MP}_{\mathrm{QF}}$, since if $A(x)$ has only bounded quantifiers then **HA** $\vdash \forall x(A(x) \vee \neg A(x))$.

**Theorem.** [Kreisel, modified number-realizability] **HA** $\nvdash \mathrm{MP}_{\mathrm{QF}}$.

**Corollary.** Intuitionistic predicate logic **IQC** $\nvdash \mathrm{MP}_D$.

**Theorem.** [cf. Smorynski 1973] **HA** $+ \mathrm{MP}_{\mathrm{QF}} \nvdash \mathrm{MP}_{\mathrm{D}}$.

## What is Markov's Rule?

Versions of Markov's Rule corresponding to $\mathrm{MP}_{\mathrm{D}}$ and to $\mathrm{MP}_{\mathrm{QF}}$:

- $\mathrm{MR}_{\mathrm{D}}$: If $\vdash \forall x(A(x) \vee \neg A(x))$ & $\neg\forall x \neg A(x)$ then $\vdash \exists x A(x)$.
- $\mathrm{MR}_{\mathrm{QF}}$: If $\vdash \neg\forall x \neg A(x)$ where $A(x)$ has only bounded quantifiers, then $\vdash \exists x A(x)$.

**Definition.** A rule of the form "If $\vdash A$ then $\vdash B$ " is *admissible* for a formal system **T** in a language $\mathcal{L}$, if whenever $\sigma$ substitutes formulas $F_i$ of $\mathcal{L}$ uniformly for the predicate letters $\mathrm{P}_i$ in $A$ and $B$:

$$\text{If } \mathbf{T} \vdash \sigma(A) \text{ then } \mathbf{T} \vdash \sigma(B).$$

Every nontrivial admissible rule of classical predicate logic **CQC** is *derivable*, in the strong sense that **CQC** proves the corresponding implication. In contrast, **IQC** has nonderivable admissible rules.

**Theorem.** [(a) Smorynski; (b) Friedman, Dragalin independently]

(a) $\mathrm{MR}_{\mathrm{QF}}$ is admissible for **HA**.

(b) $\mathrm{MR}_{\mathrm{D}}$ is admissible for **IQC** and for **HA**.

## What is the common notion of constructive proof?

This is harder. In [*Infinitistic methods from a finitist point of view*, in Proc. Symp. Found. Math. (Warsaw 1959), Pergamon, Oxford (1961) 185-192] Heyting quotes Kreisel [*Mathematical significance of consistency proofs*, JSL 23 (1958) 133-181]:

> *. . . the notion of constructive proof is vague.*

Heyting objects that

> *. . . the notion of vagueness is vague in itself; what we need, is a precise notion of precision. As far as I know, the only notion of this sort is based on a formal system.*

But even for a formal system with a recursive proof predicate,

> *. . . the difficulty reappears if we ask what it means that a given formula A is provable,*

for instance if $\neg\neg\exists p(p \vdash A)$ rather than $\exists p(p \vdash A)$, which leads directly to Markov's Principle, which Heyting did not accept.

In his case study [*Church's Thesis and the ideal of informal rigour*, NDJFL 28 no.4 (1987)] Kreisel writes:

> *IR, short for 'informal rigour,' is a venerable ideal in the broad tradition of analysing precisely common notions or, as one sometimes says, notions implicit in common reasoning. CT, short for 'Church's thesis,' concerns the common notion of effective computability, and is thus a candidate for IR. . . . there are two, possibly alternating stages in work on IR: first, the possibilities of* pursuing *IR, and secondly, of* examining *the pursuit, that is, its contribution to the broad area of knowledge to which the notions . . . belong. . . . A familiar directive for this [second] kind of investigation is:* dégager les hypothèses utiles. *Since a lot of work has been done around CT it is a candidate for use in examinations of IR too.*

MP concerns the common notions of constructive existence, effective computability and constructive proof so is a candidate.

**Church's Thesis** is often identified with recursive choice:

CT: $\forall x \exists y A(x,y) \rightarrow \exists e \forall x \exists y (T(e,x,y) \,\&\, A(x,U(y)))$

where $T(e,x,y)$ is a quantifier-free formula, $U$ is primitive recursive and **HA** $\vdash \forall y \forall z (T(e,x,y) \,\&\, T(e,x,z) \rightarrow y = z)$.

A more precise rendition is recursive comprehension CT! with the stronger hypothesis $\forall x \exists ! y A(x,y)$, where ! expresses uniqueness.

**Proposition.** **HA** + $\mathrm{MP}_{\mathrm{QF}}$ + CT! $\vdash \mathrm{MP}_{\mathrm{D}}$, because

**HA** $\vdash \forall x (A(x) \vee \neg A(x)) \rightarrow \forall x \exists ! y (y \leq 1 \,\&\, (y = 0 \leftrightarrow A(x)))$.

**Church's Rule** for an arithmetical theory is often stated

CR: If $\vdash \forall x \exists y A(x,y)$ then $\vdash \exists e \forall x \exists y (T(e,x,y) \,\&\, A(x,U(y)))$.

**Theorem.** [Kleene-Nelson, plain and formalized q-realizability]

(a) CT is consistent with **HA** and CR is admissible for **HA**.

(b) If **HA** $\vdash \forall x \exists y A(x,y)$ where $\forall x \exists y A(x,y)$ is closed, then **HA** $\vdash \forall x \exists y (T(\mathbf{n},x,y) \,\&\, A(x,U(y)))$ for some numeral **n**.

- But is there a uniform way to extract effectively, from any proof in **HA** of a closed formula $\forall x \exists y A(x, y)$, a Gödel number $n$ for which $\mathbf{HA} \vdash \forall x \exists y (T(\mathbf{n}, x, y) \& A(x, U(y)))$?
- **PA** and **HA** prove the same $\Pi^0_2$ statements, by the Gödel-Gentzen negative translation with $\mathrm{MR}_{\mathrm{QF}}$. Is "proving the same $\Pi^0_2$ statements as the corresponding classical theory" a necessary or desirable feature of any constructive theory?
- Realizability establishes that CT is consistent relative to **HA**. Formalized q-realizability proves that CR is admissible for **HA**. Both results extend to semiclassical subsystems of **PA** all of whose axioms are realizable, but e.g. the realizability of $\mathrm{MP}_{\mathrm{QF}}$ is established using Markov's Principle. Is there a simpler way?

Kreisel ends his case study with the

> **Conjecture:** *There are simple conditions, easily verified for current intuitionistic systems, that imply easily the consistency of CT and closure under Church's Rule.*

Kreisel made several contributions to the study of Markov's Principle and the notion of constructive proof.

One, his invention of modified number-realizability to show that **HA** $\nvdash$ $\mathsf{MP}_{\mathrm{QF}}$, has already been mentioned.

Another was a proof, outlined by Gödel and filled in by Kreisel, that the Beth and Kripke completeness theorems for intuitionistic predicate logic entail instances of Markov's Principle. Others here (e.g. D. McCarty) know much more about this than I do. By introducing "exploding nodes" which force all formulas, Veldman proved an alternate Kripke-style completeness theorem avoiding the use of MP (but involving seriously infinitistic reasoning).

**Questions:** What is the effect of adding an appropriate version of Markov's Principle to **IQC**, **HA**, or (subsystems of) Kleene and Vesley's intuitionistic analysis **FIM**?

Could MP legitimately enhance the notion of constructive proof?

The usual argument for arithmetical Markov's Principle involves the common notions of constructive existence and effective computability:

1. Surely the constructive natural numbers $\mathbb{N}$ have order type $\omega$.
2. If an arithmetical property $P(n)$ is effectively decidable on the constructive natural numbers, there is a computable function $\varphi : \omega \to \{0, 1\}$ such that for each $n \in \omega$: $\varphi(n) = 1 \leftrightarrow P(n)$.
3. If this $\varphi$ satisfies $\neg\forall n \in \omega(\varphi(n) = 0)$ then $\forall n \in \omega(\varphi(n) = 0)$ is inconsistent (since intuitionistic negation expresses inconsistency), so cannot be true.
4. So generating the natural numbers $n$ one by one, and computing $\varphi(n)$ for each, must eventually produce an $n \in \omega$ with $\varphi(n) = 1$ for which $P(n)$ holds (by $\omega$-consistency).
5. So Markov's Principle holds for constructive arithmetic.

Most constructive mathematicians do not accept this argument.

**Definition.** A formula $E$ in a language $\mathcal{L}$ is *persistently consistent* with a theory $\mathbf{T}$ in $\mathcal{L}$, if for every extension $\mathbf{S}$ of $\mathbf{T}$ in $\mathcal{L}$:

if $\mathbf{S}$ is consistent then $\mathbf{S} + E$ is consistent.

A schema is *persistently consistent* with a theory $\mathbf{T}$ in $\mathcal{L}$ if whenever $\mathbf{S}$ is a consistent extension of $\mathbf{T}$ in $\mathcal{L}$, the theory obtained by adding to $\mathbf{S}$ all $\mathcal{L}$-instances $E$ of the schema is consistent.

**Note:** $E$ is persistently consistent with $\mathbf{T}$ if and only if $\mathbf{T} \vdash \neg\neg E$.

**Example.** $\mathrm{MP_D}$ is persistently consistent with **IQC**, since $\mathbf{IQC} \vdash \neg\neg(\forall x(A(x) \vee \neg A(x))\ \&\ \neg\forall x \neg A(x) \rightarrow \exists x A(x))$. Briefly,

- $\mathbf{IQC} \vdash \neg\neg\ \mathrm{MP_D}$. But $\mathbf{IQC} \nvdash \mathrm{MP_D}$ as we have seen, and
- $\mathbf{IQC} \nvdash \neg\neg\forall(\mathrm{MP_D})$ by the following argument:

If $E(y)$ is $\forall x(P(x,y) \vee \neg P(x,y))\ \&\ \neg\forall x \neg P(x,y) \rightarrow \exists x P(x,y)$, then $\mathbf{IQC} \nvdash \neg\neg\forall y E(y)$ by a linear Kripke countermodel with root 0, nodes $n \in \omega$, $\mathrm{D}(n) = \{\mathbf{0}, \ldots, \mathbf{n}\}$, and $n+1 \Vdash P(\mathbf{n+1}, \mathbf{n})$.

This leads us to the "double negation shift" schema

**??** DNS: $\forall x \neg\neg A(x) \rightarrow \neg\neg\forall x A(x)$.

Evidently **IQC** + DNS $\vdash \neg\neg\forall(\mathrm{MP_D})$. Moreover,

- **IQC** + DNS $\vdash (\neg\neg E \leftrightarrow E^g)$ (Kuroda's Theorem).

But DNS is *not* persistently consistent with **IQC**, since

- **IQC** $\vdash$ (DNS $\leftrightarrow \neg\neg$DNS) and **IQC** $\nvdash$ DNS.

Analytical versions of DNS and $\forall(\mathrm{MP_D})$ can fail constructively.
If **FIM** is Kleene's formal system of intuitionistic analysis, then

- **FIM** $\vdash \neg\forall\alpha(\forall x \alpha(x) = 0 \vee \neg\forall x \alpha(x) = 0)$.

If VS is Vesley's Schema, so **FIM** + VS is consistent and proves Brouwer's "creating subject" counterexamples, then

- **FIM** + VS $\vdash \neg\forall\alpha(\neg\neg\exists x \alpha(x) = 0 \rightarrow \exists x \alpha(x) = 0)$.

So neither DNS nor $\mathrm{MP_D}$ can reasonably be added to **IQC**, although $\mathrm{MP_D}$ seems plausible for effectively enumerable domains.

**Some reasons to prefer HA + $\mathrm{MP_{QF}}$ to HA:**

- $\mathrm{MP_{QF}}$ says "we have only the standard natural numbers." It holds in every Kripke model of **HA** with constant domain.
- In the Effective Topos the standard model of **HA** is the only model (van den Berg-van Oosten [2014]).
- Any attempt to identify constructive arithmetical truth with provability in a particular consistent r.e. extension of **HA** is doomed to failure by the incompleteness theorem. A broader notion of constructive proof respects admissible rules.
- $\mathrm{MR_{QF}}$ is admissible for **HA** and for all the usual formal systems for constructive and intuitionistic analysis, by the (uniform, syntactic) Friedman-Dragalin translation.
- $\mathrm{MR_{QF}}$ is admissible for **HA**$^*$ = **HA** extended with transfinite induction over all recursive well-orderings (Leivant [1990]).
- So only computational bounds (cf. Kohlenbach) are lost by extending **HA** by $\mathrm{MP_{QF}}$, and conceptual simplicity is gained.

## From arithmetic to analysis

- *Constructive arithmetic* can be axiomatized by **HA** (or **HA** + $\mathrm{MP_{QF}}$), or more efficiently by $\mathbf{IA}_0$ (or $\mathbf{IA}_0$ + $\mathrm{MP_{QF}}$), where $\mathbf{IA}_0$ is intuitionistic arithmetic with $0, ', +, \cdot$ as primitives.
- Varieties of *constructive analysis* can be axiomatized using intuitionistic logic with two sorts of variables and quantifiers. Type-0 variables $x, y, z, x_1, \ldots$ range over the constructive natural numbers. Type-1 variables $\alpha, \beta, \gamma, \alpha_1, \ldots$ range over infinite sequences of natural numbers.
- A neutral subsystem $\mathbf{IA}_1$ of Kleene's intuitionistic analysis **FIM** extends $\mathbf{IA}_0$ to a two-sorted language.
- Additional mathematical axioms (countable choice, fan, bar, continuity) distinguish one variety of constructive analysis from another, and may be thought of as determining the intended range of the sequence variables (the *universe*).

Kleene adds to $\mathbf{IA}_1$ a countable choice axiom schema

- ★ $\mathrm{AC}_{01}$: $\forall x \exists \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, \lambda y.\beta(x, y))$

but he seldom needs more than *countable choice for numbers*:

- ★ $\mathrm{AC}_{00}$: $\forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$

or even just *function comprehension* ("unique choice"):

- ★ $\mathrm{AC}_{00}!$: $\forall x \exists! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$.

Troelstra's **EL** ([1973,1988]) has only *quantifier-free choice*:

- ★ qf-$\mathrm{AC}_{00}$: $\mathrm{AC}_{00}$ for bounded-quantifier $A(x, y)$.

Veldman's **BIM** has a *minimal axiom of countable choice*:

- ★ min-$\mathrm{AC}_{00}$: $\forall x \exists y \alpha(x, y) = 0 \rightarrow \exists \beta \forall x \alpha(x, \beta(x)) = 0$,

and calls a subset of $\mathbb{N}$ *decidable* if it has a characteristic function.

**Theorem.** (Vafeiadou [2012]) Intuitionistic recursive analysis **IRA** can be axiomatized equivalently by $\mathbf{IA}_1$ + qf-$\mathrm{AC}_{00}$ or **EL** or **BIM**. Each pair of these systems has a common definitional extension.

In the language of **FIM**, Markov's Principle can be expressed by

★ $\mathrm{MP}_1$: $\forall\alpha[\neg\neg\exists x\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0]$.

**Interesting constructive or semi-constructive universes:**

- The primitive recursive universe $\mathbb{U}_0$.
- Markov's universe $\mathbb{U}_M$: all total recursive functions.
- The arithmetical universe $\mathbb{U}_{Ar}$: all arithmetical functions.
- The hyperarithmetical (or $\Delta^1_1$) universe $\mathbb{U}_{\Delta^1_1}$: all $\Delta^1_1$-definable number-theoretic functions.
- The projective universe $\mathbb{U}_{An}$: all analytically definable number-theoretic functions.
- Brouwer's "reduced" universe $\mathbb{U}_D$: all "lawlike" sequences.
- Brouwer's universe $\mathbb{U}_{Br} = \mathbb{N}^{\mathbb{N}}$: all choice sequences.

Clearly $\mathbb{U}_0 \subsetneq \mathbb{U}_M \subsetneq \mathbb{U}_{Ar} \subsetneq \mathbb{U}_{\Delta^1_1} \subsetneq \mathbb{U}_{An} \subsetneq \mathbb{U}_{Br}$ and $\mathbb{U}_D \subsetneq \mathbb{U}_{Br}$.

## Markov's Principle has descriptive power.

Constructive mathematicians use intuitionistic logic to reason about constructive mathematical objects: natural numbers, number-theoretic functions, sets, .... The mathematical axioms are determined by the particular constructive universe under consideration, but may describe that universe in different ways.

- **IRA** $\simeq$ $\mathbf{IA}_1$ + qf-$AC_{00}$ $\simeq$ **EL** $\simeq$ **BIM**.
- **IRA** + $MP_1$ + $\forall\alpha\exists e\forall x\{e\}(x) \simeq \alpha(x)$ describes $(\mathbb{N}, \mathbb{U}_M)$ but is inconsistent with Kleene's intuitionistic analysis **FIM**.
- **IRA** + $\forall\alpha\neg\neg\exists e\forall x\{e\}(x) \simeq \alpha(x)$ has the same classical models and is consistent with **FIM**.
- **FIM** can only prove the existence of recursive sequences. But there are classically correct subsystems **T** of **FIM** with the property that **T** + $MP_1$ proves that all arithmetical sequences are *unavoidable* (cannot fail to exist).

**Goal:** Find the *strongest constructively acceptable axioms* describing the *intended range* $\mathbb{U}$ *of the sequence variables*.

**Example**: $\mathbb{U}_M \subsetneq \mathbb{U}_{Ar} \subsetneq \mathbb{U}_{An}$. Constructive axioms for $(\mathbb{N}, \mathbb{U}_{Ar})$ should include an *arithmetical comprehension* axiom stronger than qf-$\mathrm{AC}_{00}$, but weaker than $\mathrm{AC}_{00}!$. A formula is *arithmetical* if it has only number quantifiers, but may have free sequence variables.

★ $\mathrm{AC}_{00}^{-}!$: $\forall x \exists ! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$ ($A$ arithmetical).

$\mathbf{IA}_1 + \mathrm{AC}_{00}^{-}!$ has $(\mathbb{N}, \mathbb{U}_{Ar})$ as a classical model. But Brouwer accepted the *Fan Theorem*, whose functional version is

★ $\mathrm{FT}_1$: $\forall \alpha [\forall x \alpha(x) \leq 1 \rightarrow \exists y \rho(\overline{\alpha}(y) = 0)]$

$$\rightarrow \exists n \forall \alpha [\forall x \alpha(x) \leq 1 \rightarrow \exists y \leq n \rho(\overline{\alpha}(y) = 0)].$$

$\mathbf{IFT} \equiv \mathbf{IA}_1 + \mathrm{AC}_{00}^{-}! + \mathrm{FT}_1$ and $\mathbf{IFT} + \mathrm{MP}_1$ also have $(\mathbb{N}, \mathbb{U}_{Ar})$ as a classical model and are *stronger, better, more descriptive* constructive axiomatizations of $(\mathbb{N}, \mathbb{U}_{Ar})$.

**A Short Story**. A few years ago R. Solovay wanted to prove that a classical system **S** with arithmetical comprehension $\mathrm{AC}_{00}^{-}!$ and bar induction could be negatively interpreted in Kleene's neutral subsystem $\mathbf{B} = \mathbf{IA}_1 + \mathrm{AC}_{01} + \mathrm{BI}_1$ of **FIM**. For the negative interpretation of $\mathrm{AC}_{00}^{-}!$ he appeared to need

★ $\mathrm{DNS}_0^-$: $\forall x \neg\neg A(x) \rightarrow \neg\neg \forall x A(x)$ ($A(x)$ arithmetical).

Clearly Markov's Principle $\mathrm{MP}_1$: $\neg\neg\exists x\, \alpha(x) = 0 \rightarrow \exists x\, \alpha(x) = 0$, together with Kleene's bar induction schema *in the form* $^x$26.3b:

★ $\mathrm{BI}_1$: $\forall \alpha \exists x \rho(\overline{\alpha}(x)) = 0 \;\&\; \forall u[Seq(u) \;\&\; \rho(u) = 0 \rightarrow A(u)]$

$$\& \;\forall u[Seq(u) \;\&\; \forall n A(u * \langle n \rangle) \rightarrow A(u)] \rightarrow A(\langle\, \rangle)$$

proves the negative interpretation of $\mathrm{BI}_1$. (This was *not* obvious for the other forms $^x$26.3a,c,d of BI in Kleene-Vesley [1965].)

Finally Solovay finessed the issue of arithmetical comprehension and was able to negatively interpret **S** in $\mathbf{B} + \mathrm{MP}_1$, but not in **B**. He was interested in consistency strength, so a question remained.

**Theorem.** (Solovay 2002) $\mathbf{IA}_1 + \mathrm{AC}_{00}^{-}! + \mathrm{BI}_1 + \mathrm{MP}_1$ proves:

1. $\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow \exists y\alpha(x, y) = 0]$.
2. $\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow A(x, \alpha)]$ for $A(x, \alpha)$ arithmetical.
3. Arithmetical Kuroda's Principle $\mathrm{DNS}_0^{-}$.

**Corollary.** (JRM [2003]) $\mathbf{IA}_1 + \mathrm{AC}_{00}^{-}! + \mathrm{BI}_1 + \mathrm{MP}_1$ proves

1. The constructive arithmetical hierarchy is proper.
2. An intuitionistic version of $\Delta^1_1$ comprehension $\neg\neg(\dot{\Delta}^1_1\text{-CF})$:
$$\forall x[\neg\neg\exists\alpha\forall z\beta(x, \overline{\alpha}(z)) = 0 \leftrightarrow \forall\beta\exists z\gamma(x, \overline{\beta}(z)) = 0]$$
$$\rightarrow \neg\neg\exists\delta\forall x[\delta(x) = 0 \leftrightarrow \forall\beta\exists z\gamma(x, \overline{\beta}(z)) = 0].$$

If $\mathbf{T}$ is a theory extending **IRA**, and $A(x, y)$ is a formula with only $x, y$ free, then if $\mathbf{T} \vdash \neg\neg\exists!\alpha\forall xA(x, \alpha(x))$ we say that the sequence *classically defined by* $A(x, y)$ is *unavoidable over* $\mathbf{T}$. (JRM [2010])

**Remark:** (YNM) $\alpha$ is unavoidable over $\mathbf{T}$ iff $\alpha$ is in the domain of the sequence variables at some node of every Kripke $\omega$-model of $\mathbf{T}$.

**Example**: Every sequence classically defined by an arithmetical formula $A(x, y)$ with only $x, y$ free is unavoidable over $\mathbf{B} + \mathrm{MP}_1$.

**Question:** Is $\mathrm{MP}_1$ $\Pi^0_2$-conservative over **B** and **FIM**? We know

- **FIM** + $\mathrm{MP}_1$ only proves that recursive sequences exist.
- **FIM** is consistent with "there are no non-recursive sequences" so only recursive sequences are unavoidable over **FIM**.
- But by Solovay's theorem, all arithmetical sequences are unavoidable over (a subsystem of) **B** + $\mathrm{MP}_1$.

Coquand and Hofmann [1999] extended the Friedman-Dragalin translation dynamically to theories with Markov's Principle, confirming an analogue for MP of Kreisel's conjecture about CT. Their method works smoothly for intuitionistic analysis, using binary sequences $\alpha$ for the translation of formulas $E$ by $E^\alpha$, since binary sequences behave well under termwise multiplication.

**Theorem.** If **T** is $\mathbf{IA}_1$, **B**, **FIM** or any subsystem of **FIM** obtained by adding to $\mathbf{IA}_1$ any of the schemas qf-$\mathrm{AC}_{00}$, $\mathrm{AC}_{00}$, $\mathrm{AC}_{01}$, $\mathrm{BI}_1$ and/or $\mathrm{CC}_{10}$, then **T** + $\mathrm{MP}_1$ and **T** prove the same $\Pi^0_2$ formulas.

**Constructive reverse mathematics** owes a lot to the work of W. Veldman, whose version of **IRA** is **BIM**. Recently he studied equivalents of the principles OI($\mathcal{C}$) and OI([0,1]) of Open Induction, on Cantor Space and the unit interval. To prove OI($\mathcal{C}$) Coquand used monotone bar induction on a spread with at most binary branching – an "almost fan" in Veldman's terminology. Veldman isolated the principle of $\Sigma^0_1$ *monotone bar induction*:

★ $\Sigma^0_1$-BI$^m$: $\forall\alpha\exists x\exists y\rho(\overline{\alpha}(x), y) = 0$ &
$\forall w(Seq(w) \to (\exists y\rho(w, y) = 0 \leftrightarrow \forall n\exists y\rho(w * \langle n\rangle, y) = 0))$
$\to \exists y\rho(\langle\rangle, y) = 0.$

**Theorem.** (Coquand) **IRA** + $\Sigma^0_1$-BI$^m$ $\vdash$ OI($\mathcal{C}$).

**Theorem**. (Veldman [2016] ms) Over **BIM** + MP$_1$:

$$\neg\neg(\Sigma^0_1\text{-CF}) \;\Leftrightarrow\; \Sigma^0_1\text{-BI}^m \;\Leftrightarrow\; \text{OI}(\mathcal{C}).$$

where $\neg\neg(\Sigma^0_1$-CF) is the key lemma for Solovay's theorem:

★ $\neg\neg(\Sigma^0_1$-CF): $\forall\alpha\neg\neg\exists\zeta\forall x[\zeta(x) = 0 \leftrightarrow \exists y\alpha(x, y) = 0]$

But is $(\mathbb{N}, \mathbb{U}_{\Delta^1_1})$ a classical model of **IRA** + $\mathrm{MP}_1$ + $\Sigma^0_1$-$\mathrm{BI}^m$? Probably not, since the $\Delta^1_1$ sequences do not satisfy the bar theorem (Kleene). A good axiomatization of $(\mathbb{N}, \mathbb{U}_{\Delta^1_1})$ might be

$$\mathbf{T}_{\Delta^1_1} \equiv \mathbf{IFT} + \neg\neg(\Sigma^0_1\text{-CF}) + \dot{\Delta}^1_1\text{-CF}^{\neg\neg} + (*)$$

where $(*)$ says "only classically $\Sigma^1_1$ sequences are unavoidable":

$(*)$ $\quad \forall\alpha\neg\neg\exists \mathrm{e}\forall \mathrm{x}\forall \mathrm{y}[\alpha(\mathrm{x}) = \mathrm{y} \leftrightarrow \neg\neg\exists\beta\forall \mathrm{z}\neg \mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{y}, \overline{\beta}(\mathrm{z}))].$

$\mathbf{T}_{\Delta^1_1}$ is consistent with **FIM** by ${}^{\Delta^1_1}$realizability (JRM [2010]), which does not verify $\mathrm{MP}_1$ but does verify

$(**)$: $\quad \forall x\neg\neg\exists y\alpha(x, y) = 0 \to \neg\neg\forall x\exists y\alpha(x, y) = 0,$

which is equivalent over **IRA** to classical quantifier-free choice

★ qf-$\mathrm{AC}^\circ_{00}$: $\quad \forall x\neg\neg\exists y\alpha(x, y) = 0 \to \neg\neg\exists\beta\forall x\alpha(x, \beta(y)) = 0,$

which is consistent with **FIM** + $\forall\alpha\neg\neg\exists e\forall x(\alpha(x) = \{e\}(x))$ by ${}^{G}$realizability.

**Question:** Does $\mathrm{MP}_1$ have any interesting equivalents over **FIM**?

"Weak König's Lemma" WKL is

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

Adding a strong effective uniqueness hypothesis gives WKL!:

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0 \ \& $$
$$\forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\exists x \alpha(x) \neq \beta(x) \rightarrow \exists x [\rho(\overline{\alpha}(x)) \neq 0 \vee \rho(\overline{\beta}(x)) \neq 0]]$$
$$\rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

**Theorem**. (Ishihara, J. Berger, Schwichtenberg [2005]) Over $\mathbf{M} = \mathbf{IA}_1 + \mathrm{AC}_{01}!$:

$$\mathrm{WKL!} \Leftrightarrow \mathrm{FT}_1.$$

Weakening the uniqueness hypothesis in WKL! gives WKL!!:

$$\forall y \exists \alpha \in 2^{\mathbb{N}} \forall x \leq y \rho(\overline{\alpha}(x)) = 0$$
$$\& \ \forall \alpha \in 2^{\mathbb{N}} \forall \beta \in 2^{\mathbb{N}} [\forall x \rho(\overline{\alpha}(x)) = 0 \ \& \ \forall x \rho(\overline{\beta}(x)) = 0 \rightarrow \alpha = \beta]$$
$$\rightarrow \exists \alpha \in 2^{\mathbb{N}} \forall x \rho(\overline{\alpha}(x)) = 0.$$

**Proposition**. Over **IRA**, WKL $\Rightarrow$ WKL!! $\Rightarrow$ WKL!.

Recall Ishihara's [1993] decomposition of Markov's Principle into the conjunction of $\mathrm{MP}^{\vee}$:

$\neg\neg\exists x(\alpha(x) \neq 0 \vee \beta(x) \neq 0) \rightarrow \neg\neg\exists x\alpha(x) \neq 0 \vee \neg\neg\exists x\beta(x) \neq 0.$

and "weak Markov's Principle" WMP:

$\forall\beta[\neg\forall n\beta(n) = 0 \vee \neg\forall n(\beta(n) = 0 \rightarrow \alpha(n) = 0)] \rightarrow \exists n\alpha(n) \neq 0.$

which is intuitionistically true by weak continuous choice.

**Theorem.** (JRM [2012])

1. Over **M**: WKL!! $\Leftrightarrow$ $\mathrm{MP}^{\vee}$ $+ \neg\neg$ WKL.
2. Over **M**: WKL! $\not\Rightarrow$ WKL!! $\not\Rightarrow$ WKL.
3. Over **M** + $\mathrm{MP}_1$: $\mathrm{FT}_1 \Leftrightarrow \neg\neg$WKL $\Leftrightarrow$ WKL!!.

**Corollary.** Over **FIM**: WKL!! $\Leftrightarrow$ $\mathrm{MP}^{\vee}$ $\Leftrightarrow$ $\mathrm{MP}_1$.

Much more could be done towards an IR investigation of MP.

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