CLASSICAL CONSEQUENCES OF CONSTRUCTIVE SYSTEMS

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Abstract. This is a survey of formal axiomatic systems for the three main varieties of constructive analysis, in a common language and with intuitionistic logic, which are as nearly as possible compatible with classical analysis and with one another. Classically sound consequences of principles of intuitionistic mathematics are emphasized. Compatibility with classical analysis is of two kinds. On the one hand, Bishop's constructive mathematics and a very substantial part of intuitionistic analysis are classically correct, sharing with constructive recursive mathematics a neutral subsystem adequate for recursive function theory and elementary real analysis. On the other, each constructive system considered here is separately consistent with the negative interpretation of each of its classically sound subsystems, establishing internal compatibility with the classical context it is intended to refine. A possibly new criterion for the transposability of unique existential quantifiers, and a recent theorem by Vafeiadou on the internal compatibility of classical and intuitionistic analysis, are included.

1. Introduction

Much attention has been paid to the effective constructive content of classical mathematics. Constructive mathematicians, working with intuitionistic logic and carefully stated versions of classical mathematical axioms, formulate and prove constructively meaningful classical equivalents of theorems of classical arithmetic, analysis and algebra ([2],[9],[19]). Constructive recursive mathematicians accept Markov's principle and Church's Thesis; Markov's Rule and the Church-Kleene Rule are generally admissible for formal systems for all varieties of constructive mathematics (cf. [4],[33],[13]) Applied proof theorists use constructive and semi-constructive interpretations and translations to extract computational information from classical proofs ([15],[1]).

Recent vigorous development of constructive reverse mathematics establishes precise equivalences, over weak neutral axiomatic systems with intuitionistic logic, between constructively meaningful principles and versions of classical and constructive mathematical theorems. A refinement of classical reverse mathematics [28] results if these principles and theorems are required to be classically sound.

Intuitionistic analysis and constructive recursive mathematics, unlike Bishop's constructive mathematics, include some principles which conflict with classical logic, although some of their constructive consequences are classically sound. From a neutral viewpoint, classically sound consequences of Brouwer's [bar and fan theorems and] continuity principles express common features of intuitionistic and classical Baire space; classically sound consequences of [Markov's principle and] Church's thesis are properties of recursive sequences; and the Gentzen negative interpretation makes faithful copies of true classical analysis and classical recursive mathematics which are separately consistent with intuitionistic principles.

This article explores the classical consequences of formal systems, based on two-sorted intuitionistic logic, which extend intuitionistic first-order arithmetic by adding variables and terms for infinite sequences of natural numbers together with axioms for Bishop constructive analysis, intuitionistic analysis or constructive recursive mathematics. Fundamental work by Kleene, Vesley, Troelstra, Ishihara and Veldman is featured, but many others have contributed richly to this subject.

2. Challenges and Possibilities

The first problem is to find a coherent definition of the classical content of a constructive formal system **S** based on intuitionistic logic. The classical equivalences $\exists x A(x) \leftrightarrow \neg\forall x \neg A(x)$ and $(A \vee B) \leftrightarrow \neg(\neg A \ \& \ \neg B)$ clearly indicate that the classical content of a formula E includes its Gentzen negative interpretation E^g, and the classical content of **S** includes $\mathbf{S}^g \equiv_{\mathrm{df}} \{E^g : \mathbf{S} \vdash E\}$.

If **S** is a formal system based on intuitionistic logic, let us follow Kleene in defining $\mathbf{S}^\circ \equiv_{\mathrm{df}} \mathbf{S} + (\neg\neg A \to A)$. If prime formulas P are stable in **S** under double negation (that is, if $\mathbf{S} \vdash (\neg\neg P \to P)$ when P is prime), then the negative interpretation of **S** includes all theorems of **S** not involving $\exists$ or $\vee$. If $\mathbf{S}^\circ$ is consistent, so is $\mathbf{S} \cup \mathbf{S}^g$ (which does *not* prove $(\neg\neg A \to A)$ by intuitionistic logic).

It may happen, as for intuitionistic arithmetic, that $\mathbf{S}^g \subseteq \mathbf{S}$, so **S** includes its classical content. However, this is not the case even for weak constructive systems like Troelstra's **EL** (cf. [30]) or Veldman's **BIM** (cf. [36]) which assume countable choice for quantifier-free relations.

2.1. **Definition.** If **S** is a formal system based on intuitionistic logic, then (as in [25]) the *minimum classical extension* $\mathbf{S}^{+g}$ *of* **S** is the *closure under intuitionistic logic of* $\mathbf{S} \cup \mathbf{S}^g$. If, in addition, $\mathbf{S}^\circ$ is consistent, then $\mathbf{S}^{+g}$ is also the *(maximum) classical content of* **S**.

2.2. **What to do if $\mathbf{S}^\circ$ is inconsistent.** Formal systems **S** for Brouwer's intuitionistic analysis or constructive recursive mathematics which refute instances of $(\neg\neg A \to A)$ are more challenging. If the mathematical axioms of **S** are of the form $\Gamma_1 + \Gamma_2$, where Γ_1 are classically sound (in the sense of holding for a classical model $\mathcal{M}^\circ$ which corresponds by convention to the intended interpretation $\mathcal{M}$ of **S**) and Γ_2 are not, **S** may prove properties of $\mathcal{M}^\circ$ which do not follow from Γ_1 by classical logic. One possibility is to strengthen the classically acceptable axioms of **S** and weaken the classically false ones, improving the modularity of the axiomatization without changing the theory. This is the focus of Sections 3 - 5.

A separate question is whether any notion of maximum classical content of an intuitionistic system **S** makes sense, in the case that $\mathbf{S} + (\neg\neg A \to A) \vdash 0 = 1$. Section 6 presents a possible solution, including a recent result by Vafeiadou.

Section 7 proposes axiomatizing constructive recursive mathematics using the language of analysis with intuitionistic logic. A basic system, consistent with the axiom Veldman [36] calls "Kleene's alternative (to the fan theorem)," contains its negative interpretation and has a classical omega-model $\mathcal{K}^\circ$ in which the type-1 objects are all recursive sequences. The negative interpretation of the basic system is consistent with Kleene's intuitionistic analysis.

3. Axiomatizing Brouwer and Bishop

Brouwer and Bishop worked informally, but they gave plenty of clues about how to build formal systems, based on two-sorted intuitionistic predicate logic with variables for natural numbers and infinite sequences of natural numbers, extending Heyting arithmetic, with mathematical induction for all formulas of the language, recursive comprehension and countable and dependent choice. Negative integers and rational numbers are easily coded by pairs of natural numbers. Real number generators are Cauchy sequences of rationals ([2] and [14]) or nested sequences of rational intervals ([38]). With addition, subtraction, multiplication, and positive ordering $<$ of reals r, s defined so that the rationals are dense in $<$, $\mathcal{R}$ is an Archimidean partially ordered field and a complete, separable metric space with $\rho(r, s) \equiv_{\mathrm{df}} |r - s|$. Trichotomy fails but $<$ is *co-transitive*:

$$(r < s) \to (r < t) \vee (t < s).$$

Apartness (positive difference) $\#$ is defined by $(r \# s) \equiv_{\mathrm{df}} (r < s) \vee (s < r)$, and $(r \leq s) \equiv_{\mathrm{df}} \neg(s < r)$, so with $(r = s) \equiv_{\mathrm{df}} (r \leq s) \;\&\; (s \leq r)$ it follows by intuitionistic logic that $\neg(r \# s) \leftrightarrow (r = s)$.

3.1. **Kleene's and Vesley's Foundations of Intuitionistic Mathematics.** Intuitionistic two-sorted arithmetic, with full induction and an expandable finite list of constants and axioms for primitive recursive functions and functionals, is the base over which Kleene and Vesley [14] axiomatized intuitionistic analysis. Kleene clarified Brouwer's principles of bar induction and continuous choice, weakening the former and strengthening the latter, and proved that the full intuitionistic system **I** is consistent relative to a classically sound subsystem **B** with countable choice and bar induction. Vesley developed in **I** a significant portion of Brouwer's mathematical analysis, in sufficient detail to convince the classical mathematical community of its formal correctness. Two years later Bishop published his treatise [2] on a less controversial version of constructive mathematics, consistent with the classical,[1] and others followed.

In [13] Kleene formalized his relative consistency proof, based on a recursive function-realizability interpretation, over a weaker subsystem **M** with countable choice AC_{01} (defined in §3.5 below) replaced by countable comprehension $\mathrm{AC}_{00}!$. Formalization of a stronger interpretation, combining function-realizability with truth, led to the corollary that **I** and all subsystems **S** of **I** extending **M** satisfy the following form of Church's Rule.

Church-Kleene Rule: For every closed formula E of the form $\exists\alpha \mathrm{A}(\alpha)$ such that $\mathbf{S} \vdash \mathrm{E}$, a number e can be found so that $\mathbf{S} \vdash \exists\alpha(\forall \mathrm{x}(\alpha(\mathrm{x}) \simeq \{\mathbf{e}\}(\mathrm{x})) \;\&\; \mathrm{A}(\alpha))$, while $\forall \mathrm{x} \exists \mathrm{y}(\{\mathbf{e}\}(\mathrm{x}) \simeq \mathrm{y})$ is provable in a classically sound subsystem of **S**.

It is a curious fact that while Brouwer and Bishop did not restrict the continuum to its recursive points, every point **I** or **B** can prove to exist is recursive.

3.2. **Three parallel extensions of primitive recursive arithmetic.** In [30] Troelstra observed that Kleene's arguments in [13] used countable comprehension only for formulas which are "quantifier-free" (contain no sequence quantifiers and only bounded number quantifiers) and therefore satisfy the law of excluded middle over primitive recursive arithmetic. He defined a two-sorted extension **EL** (for "elementary analysis") of primitive recursive arithmetic with intuitionistic logic,

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[1] Bishop agreed with Brouwer on the use of intuitionistic logic and unrestricted mathematical induction. However, by *defining* a continuous function to be "one that is uniformly continuous on compact intervals" ([2] p. x) and restricting attention to these, Bishop sidestepped the notion of pointwise continuity and with it, Brouwer's fan and bar theorems and continuous choice principle.

full induction and quantifier-free countable choice as his base for developing the constructive theory of partial recursive functions.[2]

Veldman's **BIM** (for "basic intuitionistic mathematics," cf. [36]) has a parallel structure, as does the subsystem **IRA** (for "intuitionistic recursive analysis," cf. [24]) of Kleene's **B** described in §3.5 below. In [34] Vafeiadou made a detailed comparison of weak constructive systems such as these, showing that their differences are essentially definitional so metamathematical results over any one of **EL**, **BIM** or **IRA** generally hold over the others.

3.3. **A weaker system $\mathbf{IA}_1$ which proves its own negative interpretation.** $\mathbf{IA}_1$ is a formal system for intuitionistic two-sorted arithmetic, with variables and metavariables $\mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{m}, \mathrm{n}, \ldots, \mathrm{y}, \mathrm{z}, \mathrm{m}_0, \mathrm{n}_0, \mathrm{m}_1, \ldots$ of type 0 and $\alpha, \beta, \ldots$ of type 1, constants and axioms for 0, successor $'$, $+$, $\cdot$, and a sufficient selection of other primitive recursive functions and functionals.[3] Church's lambda makes it possible to form terms $\lambda \mathrm{x}.\mathrm{s}$ of type 1 from terms s of type 0, and the λ-reduction axiom schema $(\lambda \mathrm{x}.\mathrm{s}(\mathrm{x}))(\mathrm{t}) = \mathrm{s}(\mathrm{t})$ is assumed, along with $(\mathrm{x} = \mathrm{y} \to \alpha(\mathrm{x}) = \alpha(\mathrm{y}))$ and the axiom schema of mathematical induction for all formulas of the language.

The prime formulas of $\mathbf{IA}_1$ are equations $\mathrm{s} = \mathrm{t}$ where s, t are terms of type 0. $\mathbf{IA}_1 \vdash \forall \mathrm{x} \forall \mathrm{y}(\mathrm{x} = \mathrm{y} \vee \neg(\mathrm{x} = \mathrm{y}))$, so prime formulas are (equivalent to) their negative interpretations. Equality at type 1 is defined extensionally: $\alpha = \beta$ may abbreviate $\forall \mathrm{x}(\alpha(\mathrm{x}) = \beta(\mathrm{x}))$ but $\mathbf{IA}_1 \nvdash \forall \alpha \forall \beta(\alpha = \beta \vee \neg(\alpha = \beta))$. Adding quantifier-free countable choice to $\mathbf{IA}_1$ produces **IRA**; removing it from **EL** or **BIM** results in a subsystem essentially equivalent to $\mathbf{IA}_1$.

3.4. **Notational conventions.** It doesn't really matter which primitive recursive functions are used to code and decode pairs and sequences of integers, but some conventions are necessary. Kleene coded pairs by $\langle n, m \rangle = 2^n \cdot 3^m$ and finite sequences by $\langle n_0, n_1, \ldots, n_k \rangle = \Pi_{j=0}^{k} p_j^{n_j}$ where p_j is the jth prime, and denoted by $(n)_j$ the exponent of the jth prime in the prime factorization of n. Respecting his notation, let $\langle\,\rangle^+ \equiv_{\mathrm{df}} 1$ and $\langle n_0, n_1, \ldots, n_k \rangle^+ \equiv_{\mathrm{df}} \langle n_0 + 1, n_1 + 1, \ldots, n_k + 1 \rangle$ so $\overline{\alpha}(k+1) = \langle \alpha(0), \ldots, \alpha(k) \rangle^+$ represents the $(k+1)$st finite initial segment of α, and $\overline{\alpha}(0) = 1$. If w is such a *sequence number* (abbreviated by $Seq(w)$) then $lh(w)$ is the length of the sequence w codes, and its jth element is $(w)_j - 1$ if $j < lh(w)$. If w, v are sequence numbers then $w * v$ codes their concatenation.

3.5. **Axioms of countable choice and countable comprehension.** Brouwer and Bishop used countable and dependent choice routinely.[4] As their axiom schema of countable choice ($^{\mathrm{x}}$2.1 in [14]) Kleene and Vesley assumed

$$\mathrm{AC}_{01}. \quad \forall \mathrm{x} \exists \alpha \mathrm{A}(\mathrm{x}, \alpha) \to \exists \beta \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\beta(\langle \mathrm{x}, \mathrm{y} \rangle))$$

for all formulas of the language, with free variables of both types allowed, and with conditions on the distinguished variables guaranteeing that the substitution of $\lambda \mathrm{y}.\beta(\langle \mathrm{x}, \mathrm{y} \rangle)$ for α in $\mathrm{A}(\mathrm{x}, \alpha)$ is free. While AC_{01} may be assumed to hold in constructive analysis, they showed that for all but one of their applications it could be replaced by

$$\mathrm{AC}_{00}. \quad \forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \to \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x}))$$

[2] Kreisel's and Troelstra's **EL** in [16] includes Kleene's countable comprehension axiom, but [33] agrees with [30]. Cf. [32]

[3] Kleene's list f_0 - f_{26}, repeated with definitions in [24], is intended to be expanded as needed.

[4] A referee correctly observed that the use of choice in constructive mathematics indicates that some data is missing, and that Richman [27] works in Bishop-style mathematics without choice.

(*2.2 in [14]), which is equivalent over $\mathbf{IA}_1$ to dependent choice for numbers

$$\mathrm{DC}_0.\ \forall x\exists y A(x,y) \to \forall x\exists\alpha[\alpha(0)=x \;\&\; \forall n A(\alpha(n),\alpha(n+1))].$$

With intuitionistic logic *countable comprehension* or "unique choice"

$$\mathrm{AC}_{00}!.\quad \forall x\exists! y A(x,y) \to \exists\alpha\forall x A(x,\alpha(x))$$

is weaker than AC_{00} over $\mathbf{IA}_1$ by [39] although they are equivalent over $\mathbf{IA}_1^{\circ}$. Here $\exists! y A(x,y)$ abbreviates $\exists y A(x,y) \;\&\; \forall y\forall z(A(x,y) \;\&\; A(x,z) \to y=z)$ and similarly, $\exists!\alpha A(x,\alpha)$ abbreviates $\exists\alpha A(x,y) \;\&\; \forall\alpha\forall\beta(A(x,\alpha) \;\&\; A(x,\beta) \to \alpha=\beta)$. AC_{01} is apparently a stronger axiom than AC_{00}, but $\mathbf{IA}_1 + \mathrm{AC}_{00}! \vdash \mathrm{AC}_{01}!$ by [20].

QF-AC_{00} (quantifier-free countable choice) restricts AC_{00} to formulas $A(x,y)$ containing no sequence quantifiers, and only bounded numerical quantifiers. $\mathbf{IA}_1$ has a classical omega-model with the primitive recursive functions as its type-1 objects, but $\mathbf{IRA} \equiv_{\mathrm{df}} \mathbf{IA}_1$ + QF-AC_{00} proves the existence of general recursive functions and can be axiomatized equivalently by adding to $\mathbf{IA}_1$ a single axiom

$$\forall\rho[\forall x\exists y\, \rho(\langle x,y\rangle)=0 \to \exists\alpha\forall x\, \rho(\langle x,\alpha(x)\rangle)=0]$$

asserting that the universe of sequences is closed under unbounded effective search. Veldman chooses this axiom, rather than the schema QF-AC_{00}, for his **BIM**.

3.6. **Brouwer's bar and fan theorems.** Brouwer believed he could justify a principle of backwards induction on finite sequences of natural numbers, his "bar theorem." Kleene formulated this principle as a schema

$$\mathrm{BI}^{\circ}.\quad \forall\alpha\exists x R(\overline{\alpha}(x)) \;\&\; \forall w(\mathrm{Seq}(w) \;\&\; R(w) \to A(w)) \;\&\; \forall w(\mathrm{Seq}(w) \;\&\; \forall s A(w * \langle s+1\rangle) \to A(w)) \to A(1)$$

and showed that, while BI° is classically equivalent to AC_{00}, from Brouwer's viewpoint it is too strong.[5] Kleene added the hypothesis $\forall w[\mathrm{Seq}(w) \to R(w) \vee \neg R(w)]$ and adopted the resulting *detachable bar theorem* BI_{d} as an axiom schema (ˣ26.3a in [14]) to express Brouwer's bar theorem in **B** and **I**.[6]

Brouwer used bar induction to prove his "fan theorem," which allowed him to establish that (if every total function is pointwise continuous, then) every function completely defined on the closed unit interval is uniformly continuous. Brouwer's *full fan theorem* ([14] *27.9) is simply

$$\mathrm{FT}.\quad \forall\alpha_{B(\alpha)}\exists x\, R(\overline{\alpha}(x)) \to \exists n\forall\alpha_{B(\alpha)}\exists x \le n\, R(\overline{\alpha}(x)),$$

where $B(\alpha)$ abbreviates $\forall x\, \alpha(x) \le \beta(\overline{\alpha}(x))$; a special case, the *binary fan theorem* with $B(\alpha) \equiv \forall x \alpha(x) \le 1$, is equally strong over **IRA**. FT proves a corresponding principle of *fan induction*, with no restriction on the predicates (cf. [36]).

Kleene considered four intuitionistically correct versions (ˣ26.3a-d) of the bar induction schema with different restrictions on the basis predicate $R(w)$. Three are equivalent over **IRA**, but one (ˣ26.3b) assumes that a characteristic function for the basis exists. This version of bar induction,

$$\mathrm{BI}_1.\quad \forall\alpha\exists x\rho(\overline{\alpha}(x))=0 \;\&\; \forall w(\mathrm{Seq}(w) \;\&\; \rho(w)=0 \to A(w)) \;\&\; \forall w(\mathrm{Seq}(w) \;\&\; \forall s A(w * \langle s+1\rangle) \to A(w)) \to A(1),$$

[5]In fact, **IRA** + $\mathrm{BI}^{\circ} \vdash$ WLPO (*27.23 in [14]).

[6]Over $\mathbf{IA}_1 + \mathrm{AC}_{00}!$ (but not over **IRA**) this restriction is equivalent to requiring $R(w)$ to be quantifier-free.

is weaker than $\mathrm{BI_d}$ in the absence of countable comprehension. Solovay proved (cf. [23]) that arithmetical countable choice $\mathrm{AC_{00}^{Ar}}$ ($\mathrm{AC_{00}}$ restricted to $\mathrm{A(x,y)}$ without sequence quantifiers) can be negatively interpreted in $\mathbf{IA}_1 + \mathrm{BI}_1 + \mathrm{MP}_1$, where

$$\mathrm{MP}_1. \quad \forall\alpha[\neg\neg\exists x\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0].$$

The version of the binary fan theorem corresponding to BI_1 is

$$\mathrm{FT}_1. \quad \forall\alpha_{B(\alpha)}\exists x\, \rho(\overline{\alpha}(x)) = 0 \rightarrow \exists n\forall\alpha_{B(\alpha)}\exists x \leq n\, \rho(\overline{\alpha}(x)) = 0.$$

In Theorem 9.6 and Corollary 9.8 of [36], Veldman has compiled a long list of theorems of intuitionistic mathematics equivalent to FT_1 over his minimal formal system **BIM** (hence over **IRA**). In particular, FT_1 is equivalent to the version of FT with $\mathrm{R(w)} \equiv \exists y(\beta(y) = w + 1)$ which he calls the *enumerable fan theorem*.

The *monotone bar theorem* (*27.13 in [14]) makes it possible to combine the bar and inductive predicates, and may be succinctly stated (following Veldman) as

$$\mathrm{BI_{mon}}. \quad \forall\alpha\exists x A(\overline{\alpha}(x)) \;\&\; \forall w(\mathrm{Seq}(w) \rightarrow (A(w) \leftrightarrow \forall n A(w * \langle n\rangle^+)) \rightarrow \forall w(\mathrm{Seq}(w) \rightarrow A(w)).$$

Veldman's close analysis of Brouwer's writing led him to the conclusion that Brouwer sometimes assumed a monotone bar, but sometimes fell into the error of trying to justify full bar induction, which conflicts with his continuity principles as Kleene observed in [14].

3.7. **Brouwer's continuous choice and weak continuity principles.** The continuous choice principle Kleene called "Brouwer's Principle for Functions"

$$\mathrm{CC_{11}}. \quad \forall\alpha\exists\beta A(\alpha,\beta) \rightarrow \exists\sigma\forall\alpha[\forall x\exists! y\sigma(\langle x+1\rangle * \overline{\alpha}(y)) > 0 \;\& \; \forall\beta(\forall x\exists y\sigma(\langle x+1\rangle * \overline{\alpha}(y)) = \beta(x) + 1 \rightarrow A(\alpha,\beta))]$$

is ${}^{x}27.1$ in [14]. The full formal system for intuitionistic analysis Kleene and Vesley presented in [14] is $\mathbf{I} \equiv_{\mathrm{df}} \mathbf{IA}_1 + \mathrm{AC_{01}} + \mathrm{BI_d} + \mathrm{CC_{11}}$, sometimes referred to by the acronym **FIM**.

The subsystem $\mathbf{I}^-$ of $\mathbf{I}$ in which $\mathrm{CC_{11}}$ is replaced by "Brouwer's Principle for Numbers":

$$\mathrm{CC_{10}}. \quad \forall\alpha\exists x A(\alpha,x) \rightarrow \exists\sigma\forall\alpha\exists y\exists x[\sigma(\overline{\alpha}(y)) = (x+1) \;\& \; \forall z(\sigma(\overline{\alpha}(z)) > 0 \rightarrow y = z) \;\&\; A(\alpha,x)]$$

(cf. *27.2 in [14]) is adequate to formalize most of Brouwer's intuitionistic mathematics. $\mathbf{I}^-$ is consistent with Kripke's Schema [26] while $\mathbf{I}$ is not.

Kleene also formulated the principle (*27.15 in [14]) known as "weak continuity for numbers":

$$\text{WC-N}. \; \forall\alpha\exists x A(\alpha,x) \rightarrow \forall\alpha\exists y\exists x\forall\gamma(\overline{\gamma}(y) = \overline{\alpha}(y) \rightarrow A(\gamma,x)).$$

Dummett [6] realized that, in the presence of countable choice, $\mathbf{I}^-$ can also be axiomatized by replacing $\mathrm{BI_d}$ by $\mathrm{BI_{mon}}$ and replacing "Brouwer's Principle for Numbers" by WC-N. These changes would strengthen the classically acceptable axioms and weaken continuous choice.

3.8. **Stronger neutral subsystems of intuitionistic analysis.** Kleene chose to distinguish the neutral subsystem $\mathbf{B} \equiv_{\mathrm{df}} \mathbf{IA}_1 + \mathrm{AC}_{01} + \mathbf{BI}_{\mathrm{d}}$ of intuitionistic analysis, so $\mathbf{I} \equiv_{\mathrm{df}} \mathbf{B} + \mathrm{CC}_{11}$. There is no doubt that $\mathbf{B}$ is classically sound, because $(\mathbf{IA}_1 + \mathrm{AC}_{00})^\circ \vdash \mathrm{BI}^\circ$ (by *26.1° in [14]) and $\mathbf{IA}_1 + \mathrm{AC}_{01} \vdash \mathrm{AC}_{00}$ (by *2.2 in [14]). Thus $\mathbf{B}^\circ = (\mathbf{IA}_1 + \mathrm{AC}_{01})^\circ$, and a classical mathematician can understand $\mathbf{B}$ as the intuitionistic version of classical analysis with countable choice.

There are stronger neutral subsystems of $\mathbf{I}$, however. The variant $\mathbf{B}'$ of $\mathbf{B}$ with $\mathrm{BI}_{\mathrm{mon}}$ as an axiom schema in place of BI_{d} is classically sound and lies strictly between $\mathbf{B}$ and $\mathbf{I}$. Veldman takes monotone bar induction as an axiom expressing Brouwer's intent.

Kleene used neighborhood functions to code moduli of continuity and values of continuous functions. Troelstra formulated a *neighborhood function principle*:

$$\mathrm{NFP.}\ \forall\alpha\exists \mathrm{x}\mathrm{A}(\overline{\alpha}(\mathrm{x})) \rightarrow \exists\sigma\forall\alpha[\exists!\mathrm{x}\sigma(\overline{\alpha}(\mathrm{x})) > 0\ \&\ \forall \mathrm{x}\forall \mathrm{y}(\sigma(\overline{\alpha}(\mathrm{x})) = \mathrm{y}+1 \rightarrow \mathrm{A}(\overline{\alpha}(\mathrm{y})))],$$

a classically sound choice principle, and derived it from CC_{10}. A monotone version

$$\mathrm{NFP}_{\mathrm{mon}}.\ \forall\alpha[\exists \mathrm{x}\mathrm{A}(\overline{\alpha}(\mathrm{x}))\ \&\ \forall \mathrm{x}(\mathrm{A}(\overline{\alpha}(\mathrm{x})) \rightarrow \mathrm{A}(\overline{\alpha}(\mathrm{x}+1)))] \rightarrow \exists\sigma\forall\alpha[\exists \mathrm{x}\sigma(\overline{\alpha}(\mathrm{x})) = 0\ \&\ \forall \mathrm{x}(\sigma(\overline{\alpha}(\mathrm{x})) = 0 \rightarrow \mathrm{A}(\overline{\alpha}(\mathrm{x})))]$$

is interderivable with $\mathrm{BI}_{\mathrm{mon}}$ over $\mathbf{B}$ (cf. [24]), so $\mathbf{I}^- = \mathbf{B} + \mathrm{NFP}_{\mathrm{mon}} + \mathrm{WC\text{-}N}$.

The schema of *dependent choice for sequences*

$$\mathrm{DC}_1.\ \forall\alpha\exists\beta\mathrm{A}(\alpha,\beta) \rightarrow \forall\alpha\exists\delta[(\delta)_0 = \alpha\ \&\ \forall \mathrm{n}\mathrm{A}((\delta)_{\mathrm{n}}, (\delta)_{\mathrm{n}+1})]$$

(where $(\delta)_{\mathrm{n}}$ abbreviates $\lambda \mathrm{x}.\delta(\langle \mathrm{n}, \mathrm{x}\rangle)$)[7] is not obviously provable in $\mathbf{B}$ or even in $\mathbf{B}^\circ$, but the following argument shows that $\mathbf{B} + \mathrm{DC}_1$ is a classically sound subsystem of $\mathbf{I}$. Kleene could have strengthened AC_{01} to DC_1, but he did not.

3.9. **Theorem.**

(1) $\mathbf{IA}_1 + \mathrm{AC}_{01}! + \mathrm{CC}_{11} \vdash \mathrm{DC}_1$.
(2) $\mathbf{IA}_1 + \mathrm{DC}_1 \vdash \mathrm{AC}_{01}$.

Proof. (1) Assume **(a)** $\forall\alpha\exists\beta\mathrm{A}(\alpha,\beta)$. By CC_{11} there is a σ satisfying **(b)** $\forall\alpha[\exists!\beta\{\sigma\}[\alpha] \simeq \beta\ \&\ \forall\beta(\{\sigma\}[\alpha] \simeq \beta \rightarrow \mathrm{A}(\alpha,\beta))]$. Fix α. We will show there is a ζ so that **(c)** $\forall \mathrm{n}[((\zeta)_{\mathrm{n}})_0 = \alpha\ \&\ \forall \mathrm{i} < \mathrm{n}[\mathrm{A}(((\zeta)_{\mathrm{n}})_{\mathrm{i}}, ((\zeta)_{\mathrm{n}})_{\mathrm{i}+1})\ \&\ ((\zeta)_{\mathrm{n}})_{\mathrm{i}} = ((\zeta)_{\mathrm{n}+1})_{\mathrm{i}}]]$. From ζ we can define γ so that **(d)** $\forall \mathrm{n}[(\gamma)_{\mathrm{n}} = ((\zeta)_{\mathrm{n}+2})_{\mathrm{n}}]$ and it will follow that $(\gamma)_0 = \alpha$ and for all n: $(\gamma)_{\mathrm{n}+1} = ((\zeta)_{\mathrm{n}+3})_{\mathrm{n}+1} = ((\zeta)_{\mathrm{n}+2})_{\mathrm{n}+1}$ so $\mathrm{A}((\gamma)_{\mathrm{n}}, (\gamma)_{\mathrm{n}+1})$.

Toward (c), prove by induction: $\forall \mathrm{n}\exists!\delta[(\delta)_0 = \alpha\ \&\ \forall \mathrm{i} < \mathrm{n}((\delta)_{\mathrm{i}+1} \simeq \{\sigma\}[(\delta)_{\mathrm{i}}])\ \&\ \forall \mathrm{x}[\neg(\mathrm{x} = \langle(\mathrm{x})_0, (\mathrm{x})_1\rangle\ \&\ (\mathrm{x})_0 \leq \mathrm{n}) \rightarrow \delta(\mathrm{x}) = 0]]$. Then by $\mathrm{AC}_{01}!$ $\exists\zeta\forall \mathrm{n}[((\zeta)_{\mathrm{n}})_0 = \alpha\ \&\ \forall \mathrm{i} < \mathrm{n}((\zeta)_{\mathrm{n}})_{\mathrm{i}+1} \simeq \{\sigma\}[((\zeta)_{\mathrm{n}})_{\mathrm{i}}]]$, and (b) completes the argument.[8]

(2) Assume **(a)** $\forall \mathrm{n}\exists\alpha\mathrm{A}(\mathrm{n},\alpha)$. We want to show $\exists\beta\forall \mathrm{n}\mathrm{A}(\mathrm{n}, (\beta)_{\mathrm{n}})$. From (a) we conclude **(b)** $\forall\alpha\exists\beta[\beta(0) = \alpha(0) + 1\ \&\ \mathrm{A}(\alpha(0), \lambda \mathrm{x}.\beta(\mathrm{x}+1))]$, and then DC_1 gives **(c)** $\exists\gamma[(\gamma)_0 = \lambda \mathrm{t}.0\ \&\ \forall \mathrm{n}[(\gamma)_{\mathrm{n}+1}(0) = (\gamma)_{\mathrm{n}}(0) + 1\ \&\ \mathrm{A}((\gamma)_{\mathrm{n}}(0), \lambda \mathrm{x}.(\gamma)_{\mathrm{n}+1}(\mathrm{x}+1))]]$. For any such γ, **(d)** $\forall \mathrm{n}\,[(\gamma)_{\mathrm{n}}(0) = \mathrm{n}\ \&\ \mathrm{A}(\mathrm{n}, \lambda \mathrm{x}.(\gamma)_{\mathrm{n}+1}(\mathrm{x}+1))]$ holds by induction; so **(e)** $\forall \mathrm{n}\mathrm{A}(\mathrm{n}, (\lambda \mathrm{y}.(\gamma)_{(\mathrm{y})_0+1}((\mathrm{y})_1 + 1))_{\mathrm{n}})$; so **(f)** $\exists\beta\forall \mathrm{n}\mathrm{A}(\mathrm{n}, (\beta)_{\mathrm{n}})$. □

[7] This abbreviation, which conflicts with Kleene's definition of $(\delta)_n$, is adopted here in order to save space and improve readability.

[8] The logic of partial terms is *not* involved essentially in this argument because the informal expression $\{\sigma\}[\alpha]$, which helps to clarify the proof, always designates a fully defined sequence β satisfying $\forall \mathrm{x}\forall \mathrm{y}[\beta(\mathrm{x}) = \mathrm{y} \leftrightarrow \exists \mathrm{z}[\sigma(\langle \mathrm{x}\rangle^+ * \overline{\alpha}(\mathrm{z})) = \mathrm{y}+1\ \&\ \forall \mathrm{n} < \mathrm{z}\,\sigma(\langle \mathrm{x}\rangle^+ * \overline{\alpha}(\mathrm{n})) = 0]]$.

3.10. **Bar induction of type one.** A year after the publication of [14] by Kleene and Vesley, Howard and Kreisel [8] corrected a typo (discovered by Kleene) in the conclusion of the formulation, in Section 6.3 of [29], of Spector's axiom schema of bar induction of type one; call the corrected version "SBI_1." In a minor variant $\mathbf{H}$ of $\mathbf{IA}_1$ they derived SBI_1 from CC_{10} and the special case $\mathrm{BI}_{\mathrm{QF}}$ of BI in which R(w) is required to be quantifier-free. Their proof also shows that $\mathbf{IA}_1 + \mathrm{BI}_1 + \mathrm{CC}_{10} \vdash \mathrm{SBI}_1$.

In Appendix 1 of [8] they proved that DC_1 is equivalent to SBI_1 over $\mathbf{H}^\circ$, and asked (Problem 9 of the appendix) whether DC_1 is derivable from AC_{01} over $\mathbf{H}$ or $\mathbf{H}^\circ$. More than thirty years later, analyzing the classical content of countable and dependent choice, Berardi, Bezem and Coquand [1] did not know if this question was still open. As far as I know it can still be asked over $\mathbf{IA}_1$ and $\mathbf{IA}_1^\circ$ today.

3.11. **Bishop's initial constructive analysis as a subsystem of I.** Bridges and Richman describe Bishop's constructive mathematics ("BISH" in [4]) as the common core of intuitionistic, classical and constructive recursive mathematics. Given the fact that Bishop accepted countable and dependent choice but rejected bar induction and even the binary fan theorem, by the previous result a reasonable axiomatization of Bishop's initial constructive analysis is $\mathbf{IA}_1 + \mathrm{DC}_1$. However, see the next section.

4. Examples of Classically True Theorems of Intuitionistic Analysis

4.1. **Vesley's mathematical examples.** The last two chapters of [14] clarified and criticized Brouwer's development of the real numbers. In Chapter III ("The Intuitionistic Continuum"), using Heyting's informal exposition [7] and Brouwer's [5] as his primary sources, Vesley developed formally in $\mathbf{I}$ a significant part of intuitionistic real analysis including Brouwer's uniform continuity theorem.

Following Heyting, Vesley coded real numbers by Cauchy sequences of rational numbers. Properties of real numbers are properties of these sequences respecting coincidence, an equivalence relation; Brouwer's "point cores" (equivalence classes with respect to coincidence) are treated indirectly. For his formal treatment Vesley defined a *real number generator* (r.n.g.) to be any infinite sequence α consisting of the successive numerators of a Cauchy convergent sequence of dual fractions, and defined the *coincidence* relation on the set $\mathcal{R}$ of r.n.g.s formally by

$$(\alpha \overset{\circ}{=} \beta) \ \equiv_{\mathrm{df}} \ \forall \mathrm{k} \exists \mathrm{x} \forall \mathrm{y} 2^{\mathrm{k}} |\alpha(\mathrm{x}+\mathrm{y}) - \beta(\mathrm{x}+\mathrm{y})| < 2^{\mathrm{x}+\mathrm{y}}.$$

Apartness is defined by $(\alpha \# \beta) \equiv_{\mathrm{df}} \exists \mathrm{k} \exists \mathrm{x} \forall \mathrm{y} 2^{\mathrm{k}} |\alpha(\mathrm{x}+\mathrm{y}) - \beta(\mathrm{x}+\mathrm{y})| \geq 2^{\mathrm{x}+\mathrm{y}}$, and the *positive ordering* by $(\alpha <\!\circ\, \beta) \equiv_{\mathrm{df}} \exists \mathrm{k} \exists \mathrm{x} \forall \mathrm{y} 2^{\mathrm{k}} (\beta(\mathrm{x}+\mathrm{y}) \dot{-} \alpha(\mathrm{x}+\mathrm{y})) \geq 2^{\mathrm{x}+\mathrm{y}}$.

Vesley proved that with these definitions the r.n.g.s satisfy the properties Bishop later singled out as basic for constructive real numbers. Brouwer considered other relations between real numbers as well; Vesley analyzed them in detail and applied a continuity principle to prove that sharp difference $\neq_s$ is equivalent to apartness. In Chapter IV Kleene derived equivalences and differences among Brouwer's many relations between reals.

The subset $\mathcal{R}'$ of $\mathcal{R}$ consists of the *canonical real number generators* (c.r.n.g.s) α satisfying $\forall \mathrm{x} |2\alpha(\mathrm{x}) - \alpha(\mathrm{x}+1)| \leq 1$. Vesley used AC_{00} to prove that every r.n.g. coincides with a c.r.n.g., simplifying the spread representation of Brouwer's continuum which (with the fan theorem and CC_{10}) makes it obvious that every function defined on a closed real interval is uniformly continuous. He developed the metric structure of $\mathcal{R}$ in detail and verified formally that the intuitionistic continuum is separable, connected and dense in itself as Brouwer claimed in [5].

4.2. **Troelstra's list of mathematical examples.** Chapter 6 of [31] presents a list of theorems of intuitionistic analysis, including versions of several of the ones formalized by Vesley. Most are classically correct, or can be made so by inserting the additional hypothesis "[continuous]" where the intuitionistic proof appeals to a classically unsound continuity principle *and* the theorem would fail classically without this qualification.[9] Among them are

(1) Every [continuous] real-valued function on the closed unit continuum is uniformly continuous.
(2) A [continuous] mapping from a complete metric space to a separable metric space is sequentially continuous.
(3) A sequentially continuous mapping from a separable metric space to a metric space is continuous.
(4) A real-valued function of a real variable which is differentiable at x has a sequential derivative at x.
(5) A [continuous] real-valued function which has a sequential derivative at x is differentiable at x.
(6) Lindelöf's Theorem: Every indexed open covering of a complete, separable metric space has a countable subcovering.
(7) The Heine-Borel Theorem: Every indexed open covering of a closed real interval has a finitely indexed subcovering.
(8) A sequence of real-valued functions defined on $[0, 1]$ which converges pointwise for every $x \in [0, 1]$ converges uniformly on $[0, 1]$.
(9) Riemann's Permutation Theorems: If $\{x_n\}_{n\in\mathbb{N}}$ is an infinite sequence of real numbers, then $\Sigma_{n=0}^{\infty}|x_n|$ converges if and only if $\Sigma_{n=0}^{\infty}x_{\sigma(n)}$ converges for each permutation σ of $\mathbb{N}$; and if $\Sigma_{n=0}^{\infty}|x_n|$ diverges but $\Sigma_{n=0}^{\infty}x_n$ converges, then for each extended real number x there is a permutation σ of $\mathbb{N}$ such that $\Sigma_{n=0}^{\infty}x_{\sigma(n)} = x$.

4.3. **Veldman's mathematical examples.** Veldman's treatise "Intuitionism: An Inspiration" [38], which has just appeared, is a very readable and authentic introduction to intuitionism, written with a classically trained reader in mind, giving a clear presentation of the constructive real numbers $\mathcal{R}$ in Section 6 and an engaging justification of intuitionistic logic (and language) in Section 8. His examples of classically correct intuitionistic theorems include

(1) Euclid's Theorem: Given any finite list of prime numbers, another prime number can be found which is not on the list.
(2) Cantor's Theorem: Given any infinite sequence of real numbers, another real number can be found which is apart from every number on the list.
(3) The approximate intermediate value theorem.
(4) The negative intermediate value theorem: If f is a [pointwise continuous] function from $[0, 1]$ to $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$, it is impossible that $\forall x \in [0, 1](f(x) \# 0)$.
(5) The uniform continuity theorem: Every [pointwise continuous] function from $[0, 1]$ to $\mathcal{R}$ is uniformly continuous on $[0, 1]$.

[9] I am indebted to Wim Veldman for reinforcing this suggestion, as a way to demystify Brouwer.

(6) The Intuitionistic Determinacy Theorem: If Player I is able to [continuously] defeat every possible strategy for Player II in a two-person game on $2^{\mathbb{N}}$, then Player I has a winning strategy in the game.
(7) The Principle of Open Induction on $[0, 1]$: If $\mathcal{G}$ is an open subset of $[0, 1]$ which is *progressive* in $[0, 1]$ (in the sense that if $[0, x) \subseteq \mathcal{G}$ then $x \in \mathcal{G}$), then $\mathcal{G} = [0, 1]$.
(8) Dickson's Lemma.
(9) An Intuitionistic Ramsey Theorem, and the Clopen Ramsey Theorem.
(10) An intuitionistic version of Cantor's Uniqueness Theorem (cf. [37]).

4.4. **Examples from Bishop constructivists.** Just as intuitionists continue to interpret Brouwer, constructivists of the Bishop school study principles which are classically, recursively and intuitionistically correct and have desirable constructive consequences. Two examples are "weak Markov's principle," rendered informally by $\forall s \in \mathcal{R}(\forall r \in \mathcal{R}(\neg\neg(0 < r) \vee \neg\neg(r < s)) \to 0 < s)$ or formally over $\mathbf{IA}_1$ by

$$\text{WMP.}\ \forall\alpha(\forall\beta(\neg\forall \mathrm{n}(\beta(\mathrm{n}) = 0) \vee \neg\forall \mathrm{n}(\beta(\mathrm{n}) = 0 \to \alpha(\mathrm{n}) = 0)) \to \exists \mathrm{n}(\alpha(\mathrm{n}) \neq 0)),$$

and Ishihara's boundedness principle

$$\text{BD-N.}\ \ \forall\beta[\forall\alpha\exists \mathrm{n}\forall \mathrm{m} \geq \mathrm{n}(\beta(\alpha(\mathrm{m})) < \mathrm{m}) \to \exists \mathrm{m}\forall \mathrm{n}(\beta(\mathrm{n}) \leq \mathrm{m})].$$

Neither is provable in $\mathbf{IA}_1$ + DC_1, but $\mathbf{IA}_1$ + DC_1 + WC-N $\vdash$ WMP (cf. [10]) and $\mathbf{IA}_1$ + DC_1 + $\mathrm{FT_d}$ $\vdash$ BD-N (cf. [11]). In $\mathbf{IA}_1$ + DC_1 + WMP + BD-N (but not in $\mathbf{IA}_1$ + DC_1) a Bishop constructivist can prove (cf. [9], [3])

(1) Every mapping of a complete metric space into a metric space is strongly extensional.
(2) Every sequentially nondiscontinuous mapping of a complete metric space into a metric space is sequentially continuous.
(3) Every sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous.
(4) Uniform sequential continuity is equivalent to uniform continuity for mappings from metric spaces to metric spaces.

5. Replacing choice by comprehension

Kleene's constructive interpretation of the quantifier combination $\forall\exists$ led him to incorporate choice principles into his axioms for intuitionistic mathematics instead of separating comprehension from choice. This approach simplified the proof that his function-realizability interpretation is sound for his formal system **I**.

Vesley and I were interested in weakening choice assumptions where possible. My aim was to replace choice by comprehension, for philosophical reasons[10] and because countable and continuous comprehension for sequences are provable over $\mathbf{IA}_1$ from the corresponding principles for numbers. Kleene's strong axiom schema of continuous choice consciously extends Brouwer's known assumptions; Vesley showed that in the context of Kleene's **I** the countable choice axiom schema AC_{01} can be weakened to AC_{00}, and his argument ([14] p. 88) shows that $\mathrm{AC}_{00}!$ suffices.

[10]From Moschovakis [20] (p. 17): "By a statement of the form $(a)(\exists b)A(a, b)$, an intuitionist presumably means that he has an algorithm $\mathcal{A}$ which, from any a, will at some point in its operation first produce a b for which $A(a, b)$. Thus he could equally well assert $(a)(\exists! b)\{(b$ is the first output of the algorithm $\mathcal{A}$ for the input $a)$ & $A(a, b)\}$.".

5.1. **Remark.** The unique existential quantifier, naturally defined by[11]

$$\exists! \mathrm{x} \mathrm{A}(\mathrm{x}) \ \equiv_{\mathrm{df}} \ \exists \mathrm{x} \mathrm{A}(\mathrm{x}) \ \& \ \forall \mathrm{x} \forall \mathrm{y}(\mathrm{A}(\mathrm{x}) \ \& \ \mathrm{A}(\mathrm{y}) \rightarrow \mathrm{x} = \mathrm{y}),$$

is powerful and convenient in constructive systems based on intuitionistic predicate logic with equality; but unlike ordinary quantifiers, unique existential quantifiers in a finite sequence generally cannot be reordered without changing the meaning.

Part (2) of the next theorem is a logical principle characterizing those cases in which the order is unimportant. For each formula $\mathrm{A}(\mathrm{x},\mathrm{y})$, let $\exists!(\mathrm{x},\mathrm{y})\mathrm{A}(\mathrm{x},\mathrm{y})$ abbreviate $\exists \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \forall \mathrm{x} \forall \mathrm{y} \forall \mathrm{z} \forall \mathrm{u}(\mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \mathrm{A}(\mathrm{z},\mathrm{u}) \rightarrow \mathrm{x} = \mathrm{z} \ \& \ \mathrm{y} = \mathrm{u})$.

5.2. **Theorem.** Intuitionistic predicate logic with equality proves

(1) $\forall \mathrm{x} \forall \mathrm{y}((\mathrm{x} = \mathrm{y}) \vee \neg(\mathrm{x} = \mathrm{y})) \ \& \ \exists! \mathrm{x} \mathrm{A}(\mathrm{x}) \rightarrow \forall \mathrm{y}(\mathrm{A}(\mathrm{y}) \vee \neg \mathrm{A}(\mathrm{y}))$.
(2) $\exists!(\mathrm{x},\mathrm{y})\mathrm{A}(\mathrm{x},\mathrm{y}) \leftrightarrow (\exists! \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \exists \mathrm{x} \exists! \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}))$.
(3) $\exists!(\mathrm{x},\mathrm{y})\mathrm{A}(\mathrm{x},\mathrm{y}) \leftrightarrow \exists!(\mathrm{y},\mathrm{x})\mathrm{A}(\mathrm{x},\mathrm{y})$.
(4) $\exists!(\mathrm{x},\mathrm{y})\mathrm{A}(\mathrm{x},\mathrm{y}) \rightarrow (\exists! \mathrm{x} \exists! \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \exists! \mathrm{y} \exists! \mathrm{x} \mathrm{A}(\mathrm{x},\mathrm{y}))$.
(5) $\exists! \mathrm{x} \mathrm{A}(\mathrm{x}) \leftrightarrow \exists \mathrm{x}(\mathrm{A}(\mathrm{x}) \ \& \ \forall \mathrm{y}(\mathrm{A}(\mathrm{y}) \rightarrow \mathrm{x} = \mathrm{y}))$.

Proof. (1): Assume the hypotheses (**a**) $\forall \mathrm{x} \forall \mathrm{y}((\mathrm{x} = \mathrm{y}) \vee \neg(\mathrm{x} = \mathrm{y}))$, (**b**) $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$ and (**c**) $\forall \mathrm{x} \forall \mathrm{y}(\mathrm{A}(\mathrm{x}) \ \& \ \mathrm{A}(\mathrm{y}) \rightarrow \mathrm{x} = \mathrm{y})$. For $\exists$-elimination from (b) assume (**d**) $\mathrm{A}(\mathrm{x})$. Then (**e**) $((\mathrm{x} = \mathrm{y}) \vee \neg(\mathrm{x} = \mathrm{y}))$ by (a), and (**f**) $(\mathrm{A}(\mathrm{y}) \rightarrow \mathrm{x} = \mathrm{y})$ by (d) and (c), so (**g**) $(\neg(\mathrm{x} = \mathrm{y}) \rightarrow \neg \mathrm{A}(\mathrm{y}))$ by contraposition. Substituting equals for equals in (d) then gives (**h**) $(\mathrm{x} = \mathrm{y} \rightarrow \mathrm{A}(\mathrm{y}))$, so (**j**) $(\mathrm{A}(\mathrm{y}) \vee \neg \mathrm{A}(\mathrm{y}))$ by (e), (g) and (h); hence (**j**) $\forall \mathrm{y}(\mathrm{A}(\mathrm{y}) \vee \neg \mathrm{A}(\mathrm{y}))$ follows from (a), (b) and (c) after $\exists \mathrm{x}$-elimination on (d).

(2$\rightarrow$): Assume (**a**) $\exists!(\mathrm{x},\mathrm{y})\mathrm{A}(\mathrm{x},\mathrm{y})$, so after &-eliminations (**b**) $\exists \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$ and (**c**) $\forall \mathrm{x} \forall \mathrm{y} \forall \mathrm{z} \forall \mathrm{u}(\mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \mathrm{A}(\mathrm{z},\mathrm{u}) \rightarrow \mathrm{x} = \mathrm{z} \ \& \ \mathrm{y} = \mathrm{u})$. Assume (**d**) $\exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$ for $\exists$-elimination from (b) ; then (c) entails (**e**) $\forall \mathrm{x} \forall \mathrm{z}(\exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \exists \mathrm{y} \mathrm{A}(\mathrm{z},\mathrm{y}) \rightarrow \mathrm{x} = \mathrm{z})$, which with (d) gives (**f**) $\exists! \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$. For $\exists \mathrm{y}$-elimination from (d) assume (**g**) $\mathrm{A}(\mathrm{x},\mathrm{y})$; then (c) gives $\forall \mathrm{y} \forall \mathrm{u}(\mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \mathrm{A}(\mathrm{x},\mathrm{u}) \rightarrow \mathrm{y} = \mathrm{u})$, so (**h**) $\exists! \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$ and therefore (**i**) $\exists \mathrm{x} \exists! \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$. Conclude (**j**) $\exists! \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \exists \mathrm{x} \exists! \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$ from (f) and (i), and then use $\exists \mathrm{y}$- elimination on (g) followed by $\exists \mathrm{x}$-elimination on (d).

(2$\leftarrow$): Assume (**a**) $\exists! \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$, (**b**) $\exists \mathrm{x} \exists! \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$ and (**c**) $\mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \mathrm{A}(\mathrm{z},\mathrm{u})$. Then (**d**) $\exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \exists \mathrm{y} \mathrm{A}(\mathrm{z},\mathrm{y})$ so (**e**) $\mathrm{x} = \mathrm{z}$ by (a). For $\exists$-elimination from (b), assume (**f**) $\exists! \mathrm{y} \mathrm{A}(\mathrm{v},\mathrm{y})$. Then $\exists \mathrm{y} \mathrm{A}(\mathrm{v},\mathrm{y})$, so $\mathrm{x} = \mathrm{v}$ by (a) and (c), so $\mathrm{v} = \mathrm{z}$ by (e), so (**g**) $\mathrm{A}(\mathrm{v},\mathrm{y}) \ \& \ \mathrm{A}(\mathrm{v},\mathrm{u})$ by (c) with the substitution property of equality, and then (**h**) $\mathrm{y} = \mathrm{u}$ follows by (f). Finally, using &-introduction on (e) and (h), $\exists \mathrm{v}$-elimination discharging (f), $\rightarrow$-introduction discharging (c), and $\forall$-introductions: (**j**) $\forall \mathrm{x} \forall \mathrm{y} \forall \mathrm{z} \forall \mathrm{u}(\mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \mathrm{A}(\mathrm{z},\mathrm{u}) \rightarrow \mathrm{x} = \mathrm{z} \ \& \ \mathrm{y} = \mathrm{u})$. Obviously $\exists \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y})$ follows from (a), so (**k**) $\exists!(\mathrm{x},\mathrm{y})\mathrm{A}(\mathrm{x},\mathrm{y})$.

(3) and (5) follow from the definitions, and the proof of (4) from (2) and (3) is straightforward.[12] □

5.3. **Corollary.** $\mathbf{IA}_1$ proves

(1) $\exists! \beta \mathrm{A}(\beta) \leftrightarrow \forall \mathrm{x} \exists! \mathrm{y} \exists \beta(\mathrm{A}(\beta) \ \& \ \beta(\mathrm{x}) = \mathrm{y})$.
(2) $\exists! \mathrm{z}[\mathrm{z} = \langle (\mathrm{z})_0, (\mathrm{z})_1 \rangle \ \& \ \mathrm{A}((\mathrm{z})_0, (\mathrm{z})_1)] \leftrightarrow (\exists! \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}) \ \& \ \exists \mathrm{x} \exists! \mathrm{y} \mathrm{A}(\mathrm{x},\mathrm{y}))$.
(3) $\exists! \gamma \mathrm{A}(\lambda \mathrm{t}.\gamma(2\mathrm{t}), \lambda \mathrm{t}.\gamma(2\mathrm{t}+1)) \leftrightarrow (\exists! \alpha \exists \beta \mathrm{A}(\alpha,\beta) \ \& \ \exists \alpha \exists! \beta \mathrm{A}(\alpha,\beta))$.

[11] Bishop constructivists often work with a constructively stronger uniqueness condition, but this meaning of $\exists! x A(x)$ goes back at least to Kleene [12].

[12] The converse of (4) fails intuitionistically and classically. An arithmetical counterexample is $A(x,y) \leftrightarrow [(x = 2 \rightarrow y = 3) \ \& \ (y = 4 \rightarrow x = 5)]$.

Proof. (1$\rightarrow$): Assume **(a)** $\exists!\beta A(\beta)$. The aim is to prove $\forall x\exists! y\exists\beta C(\beta, x, y)$, where $C(\beta, x, y)$ abbreviates $A(\beta)\ \&\ \beta(x) = y$. By Theorem 5.2(5), (a) is equivalent to **(a′)** $\exists\beta[A(\beta)\ \&\ \forall\gamma(A(\gamma) \rightarrow \forall x(\beta(x) = \gamma(x)))]$, so **(b)** $\exists\beta[C(\beta, x, \beta(x))\ \&\ \forall\gamma\forall z(C(\gamma, x, z) \rightarrow \beta(x) = z)]$, so **(c)** $\exists\beta\exists y[C(\beta, x, y)\ \&\ \forall z(\exists\gamma C(\gamma, x, z) \rightarrow y = z)]$ or equivalently **(d)** $\exists y[\exists\beta C(\beta, x, y)\ \&\ \forall z(\exists\gamma C(\gamma, x, z) \rightarrow y = z)]$, or equivalently $\exists! y\exists\beta C(\beta, x, y)$. An $\forall$x-introduction completes the proof.

(1$\leftarrow$): Assume **(a)** $\forall x\exists! y\exists\beta(A(\beta)\ \&\ \beta(x) = y)$. By Theorem 5.2(5), **(b)** $\exists\beta A(\beta)$ and **(c)** $\forall x\forall y\forall z[(\exists\beta A(\beta)\ \&\ \beta(x) = y)\ \&\ \exists\gamma(A(\gamma)\ \&\ \gamma(x) = z) \rightarrow y = z]$, and so **(c′)** $\forall\beta\forall\gamma\forall x\forall y\forall z[(A(\beta)\ \&\ \beta(x) = y)\ \&\ (A(\gamma)\ \&\ \gamma(x) = z) \rightarrow y = z]$. This entails **(d)** $\forall\beta\forall\gamma\forall x[A(\beta)\ \&\ A(\gamma) \rightarrow \beta(x) = \gamma(x)]$, so **(e)** $\forall\beta\forall\gamma[A(\beta)\ \&\ A(\gamma) \rightarrow \beta = \gamma]$, which with (b) gives $\exists!\beta A(\beta)$.

(2) and (3) are immediate applications of Theorem 5.2(2). □

5.4. **Remark.** If the hypothesis of CC_{10} is strengthened to $\forall\alpha\exists! x A(\alpha, x)$ then the resulting *continuous comprehension* principle $CC_{10}!$ is equivalent to $CC_{11}!$. Definable functions from $\mathbb{N}^\mathbb{N}$ or $\mathbb{N}$ to $\mathbb{N}$, $\mathbb{N}\times\mathbb{N}$, $\mathbb{N}^\mathbb{N}$ and $\mathbb{N}^\mathbb{N}\times\mathbb{N}^\mathbb{N}$ can be studied constructively in the subsystem $\mathbf{B!} \equiv_{df} \mathbf{IA}_1 + AC_{00}! + BI_1$ of $\mathbf{B}$, where detachable subsets of $\mathbb{N}$ have characteristic functions. In the subsystem $\mathbf{I!} \equiv_{df} \mathbf{B!} + CC_{10}!$ of $\mathbf{I}^-$, definable functions even have moduli of continuity.

5.5. **Theorem.**

(1) $\mathbf{IA}_1 + AC_{00}! \vdash AC_{01}!$, and hence $\mathbf{B!} \vdash AC_{01}!$

(2) $\mathbf{I!} \vdash CC_{11}!$

(3) $\mathbf{B} \subseteq \mathbf{B!} + DC_1 \subsetneq \mathbf{I!} + DC_1 \subsetneq \mathbf{I}^- + DC_1 \subsetneq \mathbf{I}$.

Proof. (1) My original proof in [20], apparently not well known, is summarized here. Assume **(a)** $\forall x\exists!\alpha A(x, \alpha)$. Then **(b)** $\forall x\forall y\exists! z\exists\alpha(A(x, \alpha)\ \&\ \alpha(y) = z)$ by Corollary 5.3(1), so **(c)** $\forall x\exists! z\exists\alpha(A((x)_0, \alpha)\ \&\ \alpha((x)_1) = z)$, and hence by $AC_{00}!$: **(d)** $\exists\beta\forall x\exists\alpha(A((x)_0, \alpha)\ \&\ \alpha((x)_1) = \beta(x))$. For $\exists\beta$-elimination from (d) assume **(e)** $\forall x\exists\alpha(A((x)_0, \alpha)\ \&\ \alpha((x)_1) = \beta(x))$, so **(f)** $\forall x\forall y\exists\alpha(A(x, \alpha)\ \&\ \alpha(y) = \beta(\langle x, y\rangle))$ and also (using (b)) $\forall x\forall y\forall\alpha(A(x, \alpha) \rightarrow \alpha(y) = \beta(\langle x, y\rangle))$. From this we conclude **(g)** $\forall x\forall\alpha(A(x, \alpha) \rightarrow \alpha = \lambda y.\beta(\langle x, y\rangle))$, and thus $\forall x A(x, \lambda y.\beta(\langle x, y\rangle))$ by (a). Then **(h)** $\exists\beta\forall x A(x, \lambda x.\beta(\langle x, y\rangle))$ by $\exists\beta$-introduction, discharging (e).

(2) was proved by Kleene; a hint is on p. 89 of [14]. His proof, rewritten by him to use $AC_{01}!$ in place of AC_{00}, was included in [20] at his request.

(3) The inclusions follow from the definitions, Theorem 3.9 and parts (1),(2) of this theorem. For the inequalities: $\mathbf{B!} + DC_1$ is consistent with classical logic but $\mathbf{I!} + DC_1$ is not. $\mathbf{I!} + DC_1$ is consistent with Kripke's Schema by [22], and the model presented there does not satisfy $\mathbf{I}^-$ by [17]. Krol's model [18] for $\mathbf{I}^-$ satisfies Kripke's Schema, which is inconsistent with $\mathbf{I}$ as Kripke showed (cf. [26]). □

6. Classical Content of I Relative to $\mathcal{N}^\circ$

Omega-models of classically sound constructive systems may not interest an intuitionist, but should be significant for Bishop constructivists, who work in the common core of intuitionistic, classical and constructive recursive mathematics. Classical omega-models, with the standard natural numbers as type-0 objects but restrictions on the type-1 objects, make it easy to separate subsystems of **B**.

Kleene proved in [14] that $\mathbf{IRA} \nvdash FT_1$ because the recursive sequences do not provide a classical omega-model of $\mathbf{IRA} + FT_1$ but the arithmetical sequences do; this distinction is exploited in [36]. Even the hyperarithmetical sequences fail to satisfy BI_1, so BI_1 is stronger than FT_1 over **IRA**. Classical omega-models of **B** contain sequences from all levels of the projective hierarchy, although Veldman [35] proved that in **I** the projective hierarchy collapses at Σ^1_2.

If **S** is a formal system based on intuitionistic logic such that $\mathbf{S}^\circ \vdash 0 = 1$, then $\mathbf{S}^g$ is also inconsistent.[13] However, if $\mathbf{S}'$ is a subsystem of **S** such that $\mathbf{S}'^\circ$ is consistent then $(\mathbf{S}'^\circ)^g$ is part of the classical content of **S**, and $\mathbf{S} \cup (\mathbf{S}'^\circ)^g$ may be consistent.

6.1. **Definition.** If **S** is a formal system extending $\mathbf{IA}_1$ such that $\mathbf{S}^\circ \vdash 0 = 1$, and if **S** has a subsystem with a classical model $\mathcal{M}^\circ$, the *maximum subtheory* $\mathbf{S}_{\mathcal{M}^\circ}$ *of* **S** *relative to* $\mathcal{M}^\circ$ is the set of all theorems of **S** which are (classically) true in $\mathcal{M}^\circ$. The *minimum classical extension* $\mathbf{S}^{+g}_{\mathcal{M}^\circ}$ *of* **S** *relative to* $\mathcal{M}^\circ$ is the closure under intuitionistic logic of $\mathbf{S} \cup (\mathbf{S}_{\mathcal{M}^\circ})^g$. The *maximum classical content of* **S** *relative to* $\mathcal{M}^\circ$ is the closure under intuitionistic logic of $\mathbf{S}_{\mathcal{M}^\circ} \cup (\mathrm{S}_{\mathcal{M}^\circ})^g$.

6.2. **Theorem.** (G. Vafeiadou, in [25]) The minimum classical extension $\mathbf{I}^{+g}_{\mathcal{N}^\circ}$ of **I** relative to classical Baire space $\mathcal{N}^\circ$ (a classical omega-model of **B**) is consistent and includes the negative interpretation $(\mathrm{T}_{\mathcal{N}^\circ})^g$ of true classical analysis.

Proof. Let F be any sentence (e.g. $\forall\alpha[\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0]$) which has the properties $\mathbf{I} \vdash \neg\mathrm{F}$ and $\mathbf{B} \vdash \mathrm{F}^g$, so $\mathbf{B}^\circ \vdash \mathrm{F}$. Then $\mathbf{I} \vdash (\neg\mathrm{F} \vee \mathrm{E})$ for every sentence E; but $(\neg\mathrm{F} \vee \mathrm{E})$ is true in $\mathcal{N}^\circ$ if and only if E is true in $\mathcal{N}^\circ$, if and only if $\mathrm{E}^g \in (\mathrm{T}_{\mathcal{N}^\circ})^g$. Moreover, $\mathbf{B} \vdash (\neg\mathrm{F} \vee \mathrm{E})^g \leftrightarrow \mathrm{E}^g$, so $(\mathrm{T}_{\mathcal{N}^\circ})^g \subseteq \mathbf{I}^{+g}_{\mathcal{N}^\circ}$. If $\mathcal{N}^\circ$ is classical Baire space, then $(\mathrm{T}_{\mathcal{N}^\circ})^g$ is consistent with **I** by Lemma 8.4a, Theorem 9.3 and Corollary 9.4 in [14], so $\mathbf{I}^{+g}_{\mathcal{N}^\circ}$ is consistent also. □

This result is quite subtle. Lemma 8.4a in [14] says that negative statements which are true in $\mathcal{N}^\circ$ are recursively realizable; therefore they are consistent with **I** by Theorem 9.3 and Corollary 9.4. The consistency of $\mathbf{I}^{+g}_{\mathcal{N}^\circ}$ follows immediately.

7. Another Context: Constructive Recursive Mathematics

7.1. **Axiomatizing the recursive model.** Troelstra and van Dalen [33] propose to axiomatize constructive recursive mathematics, up to and including the Kreisel-Lacombe-Shoenfield-Tsejtin Theorem, by **HA** + ECT_0 + MP_0 where ECT_0 is Troelstra's "extended Church's Thesis" (cf. [30]) and MP_0 is an arithmetical form of Markov's Principle. By Kleene's number-realizability, **HA** + ECT_0 + MP_0 is consistent relative to its classically sound subtheory **HA** + MP_0; but ECT_0 is inconsistent with classical arithmetic $\mathbf{HA}^\circ$. This is a problem because Russian recursive analysis is consistent with classical logic (although Markov's Principle is the only nonconstructive logical principle it actually uses.)

For a classically sound axiomatization of constructive recursive mathematics using intuitionistic logic, the two-sorted language is ideal. **IRA** ensures that all recursive sequences are available and makes it possible to develop the theory of recursive partial functions. Markov's Principle takes the form MP_1, and Church's Thesis (with no parameters allowed) becomes simply

$$\mathrm{CT}_1.\quad \forall\alpha \mathrm{GR}(\alpha)$$

when $\mathrm{GR}(\alpha) \equiv_{\mathrm{df}} \exists e[\forall x\exists y \mathrm{T}(e, x, y)\ \&\ \forall x\forall y(\mathrm{T}(e, x, y) \rightarrow \mathrm{U}(y) = \alpha(x))]$.

[13]The definitions of E^g and $\mathbf{S}^g$, and of $\mathbf{S}^{+g}$ when $\mathbf{S}^\circ$ is consistent, are in Section 2.

7.2. **Proposal.** $\mathbf{GR} \equiv_{\mathrm{df}} \mathbf{IRA} + \mathrm{MP}_1 + \mathrm{CT}_1$ is an appropriate basic axiomatic system for constructive recursive analysis, based on intuitionistic logic. It has an omega-model $\mathcal{K}^\circ$ in which recursive sequences are the type-1 objects, so is classically sound.

The axiom CT_1 fails in $\mathbf{B}^\circ$ by Lemma 9.8 of [14], and is refutable in $\mathbf{I}$ using CC_{10}, but its negative interpretation is consistent with $\mathbf{I}$ by [21].[14] In fact, $\mathbf{GR}$ contains its negative interpretation. Consider the restricted double-negation-shift principle

$$\Sigma^0_1\text{-DNS}_0.\ \forall \mathrm{x} \neg\neg \exists \mathrm{y} \alpha(\langle \mathrm{x}, \mathrm{y} \rangle) = 0 \to \neg\neg \forall \mathrm{x} \exists \mathrm{y} \alpha(\langle \mathrm{x}, \mathrm{y} \rangle) = 0,$$

a relatively weak consequence of MP_1 which suffices over $\mathbf{IA}_1$ to prove the negative interpretation of $\mathrm{QF\text{-}AC}_{00}$.

7.3. **Theorem.**

(1) $\mathbf{IA}_1^{+g} = \mathbf{IA}_1$, so $\mathbf{IA}_1 + \mathrm{MP}_1$ is negatively interpretable in $\mathbf{IA}_1$.
(2) $\mathbf{IRA}^{+g} = \mathbf{IRA} + \Sigma^0_1\text{-DNS}_0$.
(3) $\mathbf{IA}_1 + \mathrm{CT}_1 \vdash \forall \alpha \neg\neg \mathrm{GR}(\alpha)$.
(4) $\mathbf{IA}_1 + \Sigma^0_1\text{-DNS}_0 \vdash (\forall \alpha \neg\neg \mathrm{GR}(\alpha)) \leftrightarrow (\mathrm{CT}_1)^g)$.
(5) $\mathbf{GR}^{+g} = \mathbf{GR} + \Sigma^0_1\text{-DNS}_0 + \forall \alpha \neg\neg \mathrm{GR}(\alpha) = \mathbf{GR}$.

Proof. Corollary 1 (c),(d) in [25] justify (1),(2) respectively, since the negative interpretation of MP_1 is provable by intuitionistic logic. (3) is trivial. (4) is by intuitionistic logic with the fact that $\mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{y})$ is quantifier-free, so equivalent in $\mathbf{IA}_1$ to its double negation. (5) follows immediately from (1)-(4). □

7.4. **Corollary.** $\mathbf{I} + \neg \mathrm{MP}_1$ is consistent with the negative interpretation of $\mathbf{GR}$.

Proof. The proof in [21] that $\forall \alpha \neg\neg \mathrm{GR}(\alpha)$ is consistent with $\mathbf{I} + \neg \mathrm{MP}_1$ is by ^{G}realizability. $\Sigma^0_1\text{-DNS}_0$ is ^{G}realizable, so $\mathbf{I} + \Sigma^0_1\text{-DNS}_0 + \neg \mathrm{MP}_1 + \forall \alpha \neg\neg \mathrm{GR}(\alpha)$ is also consistent; and $\mathbf{GR}^g \subseteq \mathbf{IRA} + \Sigma^0_1\text{-DNS}_0 + \forall \alpha \neg\neg \mathrm{GR}(\alpha)$ by Theorem 7.3. □

7.5. **Remark.** Properties of recursive sequences can easily be formulated and studied over $\mathbf{GR}$. Kleene's recursive counterexample to the fan theorem is the basis for the axiom Veldman [36] calls "Kleene's Alternative:"

$$\mathrm{KA?}\quad \exists \beta [\forall \alpha_{\mathrm{B}(\alpha)} \exists \mathrm{n} [\beta(\overline{\alpha}(\mathrm{n})) = 1\ \&\ \forall \mathrm{m} \exists \alpha_{\mathrm{B}(\alpha)} \forall \mathrm{n} (\overline{\alpha}(\mathrm{n}) \leq \mathrm{m} \to \beta(\overline{\alpha}(\mathrm{n})) \neq 1)]]$$

(where $\mathrm{B}(\alpha) \equiv \forall \mathrm{x} \alpha(\mathrm{x}) \leq 1$). One might ask whether or not $\mathbf{GR} \vdash \mathrm{KA?}$

7.6. **Adding dependent choice.** Although DC_1 evidently fails in the classical omega-model of $\mathbf{GR}$ with recursive sequences as the type-1 objects, $\mathbf{GR} + \mathrm{DC}_1$ may well be consistent. Kleene and Nelson's 1945-realizability can be adapted to the two-sorted language by interpreting sequence variables by gödel numbers of general recursive functions and using the fact that every functor and every term of the language represents a primitive recursive function of its free variables. The only number which can realize a prime formula $\mathrm{s}(\mathrm{x}, \alpha) = \mathrm{t}(\mathrm{y}, \beta)$ when $\mathrm{x}, \mathrm{y}, \alpha, \beta$ are interpreted by n, m, f, g is the code $\langle s(n, \{f\}), t(m, \{g\}) \rangle$ for the pair of values, and only under the condition that $s(n, \{f\}) = t(m, \{g\})$.

Assuming this argument is correct, $\mathbf{GR} + \mathrm{DC}_1$ is a good formal system for constructive recursive mathematics and $\mathbf{IA}_1 + \mathrm{DC}_1$ (= $\mathbf{IRA} + \mathrm{DC}_1$) is a common subsystem of the formal systems for classical, intuitionistic, constructive recursive, and Bishop constructive analysis. Moreover, $\mathbf{IA}_1 + \mathrm{WMP} + \mathrm{DC}_1$ is a common subsystem of the first three which is consistent with the fourth.

[14] In contrast, the negative interpretation of CC_{11} is inconsistent with $\mathbf{I}$ and with $\mathbf{B}^\circ$.

8. Acknowledgements

I am grateful to Wim Veldman for writing "Intuitionism: An Inspiration?" and for helping me find a focus for this article; to Juan Aguilera, Fedor Pakhomov and Andreas Weiermann for their invitation to contribute to this special issue; to the organizers of CIE 2021 for their invitation which led to joint work with Garyfallia Vafeiadou, resulting in her Theorem 6.2; to Yiannis Moschovakis for asking the question which inspired Theorem 5.2(2); and to Alice Power for editorial patience. I thank two anonymous referees for many good suggestions, most of which I have tried to follow; but all remaining errors are my own.

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