Embedding the Constructive and the Classical in the Intuitionistic Continuum

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Outline:

1. Intuitionistic vs. classical logic and arithmetic
2. Intuitionistic, constructive and classical Baire space ("the continuum," ω^ω with extensional equality):
   - Brouwer's choice sequences (infinitely proceeding sequences)
   - Constructive sequences and Brouwer's lawlike sequences (hypothetically or actually completely determined)
   - Classical sequences (arbitrary, completely determined)
3. Embedding the constructive and classical continua in the intuitionistic continuum via a 3-sorted system **FIRM**, using logic and language to separate the recursive, constructive, classical and intuitionistic components of the continuum.
4. Kreisel's intensionally lawless sequences, an extensional alternative, and a translation theorem
5. A definably well ordered subset $(\mathcal{R}, \prec_{\mathcal{R}})$ of the continuum, with the consistency of **FIRM** assuming $\mathcal{R}$ is countable
6. Genericity, precursors, and summary of the unified picture

Intuitionistic vs. classical logic and arithmetic:

1. $\neg$, &, $\vee$, $\rightarrow$, $\forall$ and $\exists$ are all primitive in pure intuitionistic logic, though $\neg$ and $\vee$ are definable in intuitionistic arithmetic.
2. Intuitionistic propositional logic replaces the classical laws $(A \vee \neg A)$ and $(\neg\neg A \rightarrow A)$ by $(\neg A \rightarrow (A \rightarrow B))$.
3. $\neg\forall x \neg A(x) \leftrightarrow \neg\neg\exists x A(x)$ and $\neg\exists x \neg A(x) \leftrightarrow \forall x \neg\neg A(x)$ are valid intuitionistically, but the double negations cannot be eliminated. Intuitionistically, $\neg\neg\exists$ and $\forall\neg\neg$ express the classical existential and universal quantifiers. (P. Kraus)
4. *Intuitionistic first-order arithmetic* **IA** has the same mathematical axioms as classical (Peano) arithmetic **PA**: axioms for $=$, 0, $'$, $+$, $\cdot$, and the mathematical induction schema. Formally, **IA** is a proper subsystem of **PA**, but the negative translations showed that "*the system of intuitionistic arithmetic and number theory is only apparently narrower than the classical one, and in truth contains it, albeit with a somewhat deviant interpretation.*" (Gödel 1933)

How Intuitionistic Logic Affects Consistency:

Intuitionistic propositional logic proves $\neg\neg(A \vee \neg A)$, so *every consistent* formal system based on intuitionistic logic is *consistent with every sentence* (closed formula) of the form $A \vee \neg A$. Brouwer wrote: "*Consequently, the theorems which are usually considered* as proved *in mathematics, ought to be divided into those that are* true *and those that are* non-contradictory."

Intuitionistic predicate logic does *not* prove $\neg\neg\forall x(A(x) \vee \neg A(x))$.

If intuitionistic arithmetic **IA** is consistent, then every arithmetical sentence of the form $\forall x(A(x) \vee \neg A(x))$ is consistent with **IA** because **IA** is contained in **PA**.

But if $A(x)$ is $\exists z T(x, x, z)$, expressing "the computation of $\{x\}(x)$ converges," then $\neg\forall x(A(x) \vee \neg A(x))$ is also consistent with **IA** (and in fact *true* in Russian recursive mathematics).

Intuitionistic logic permits divergent mathematical views.

"The Continuum": The points of the linear continuum can be represented by infinite sequences of natural numbers. The collection ω^ω of all such sequences with the finite initial segment topology will be called "*the continuum*" from now on.

▷ *Brouwer's intuitionistic continuum* consists of infinitely proceeding sequences or "*choice sequences*" α of natural numbers, generated by more or less freely choosing one integer after another. At each stage, the chooser may *or may not* specify restrictions (consistent with those already made) on future choices. It follows that $\neg\forall\alpha(\forall x(\alpha(x) = 0) \vee \neg\forall x(\alpha(x) = 0))$ is *intuitionistically true*.

▷ Brouwer called "*lawlike*" any choice sequence all of whose values are determined in advance according to some fixed law.

▷ Reviewing Bishop's fundamental work, Myhill wrote about the *constructive continuum*: "An important difference [from Brouwer's continuum] is that the notion of 'free choice sequence' is dropped and the only sequences used are lawlike." (JSL 1970)

▷ *The classical continuum* consists of all possible infinite sequences of natural numbers, each considered to be completely determined. $\forall\alpha(A(\alpha) \vee \neg A(\alpha))$ is *classically true* for classical sequences α.

So the intuitionistic continuum seems incompatible with the classical continuum. Still, there are areas of agreement, including

1. 2-sorted primitive recursive arithmetic $\mathbf{IA}_1$,
2. the axiom of countable choice AC_{01}, and
3. induction up to a countable ordinal (bar induction BI!).

The constructive continuum may be thought of as a proper subset of the intuitionistic continuum, satisfying (1) and (2).

▷ In "Foundations of Intuitionistic Mathematics" (1965) Kleene and Vesley formalized the common core **B** ((1), (2) and (3)) using intuitionistic logic with two sorts of variables. The intuitionistic $\mathbf{FIM} = \mathbf{B} + \mathrm{CC}_{11}$, where CC_{11} expresses continuous choice. The classical theory $\mathbf{C} = \mathbf{B} + (A \vee \neg A)$ is inconsistent with **FIM**, but Kleene proved that **FIM** is consistent relative to **B**.

Brouwer distinguished three infinite cardinalities:

1. *denumerably infinite*, as the natural numbers.
2. *denumerably unfinished*, when "each element can be individually realized, and ... for every denumerably infinite subset there exists an element not belonging to this subset." (footnote to "Intuitionism and Formalism" [1912])
3. *nondenumerable*, as the intuitionistic continuum.

▷ The class of all lawlike sequences is denumerably unfinished. If $b_0, b_1, b_2, \ldots$ is a lawlike sequence of lawlike sequences, the sequence $b^*(n) = b_n(n) + 1$ is lawlike and differs from every b_n.

▷ The classical sequences are individually realized and have no classical enumeration. *The classical continuum can be treated as denumerably unfinished by considering a countable model!*

▷ Kleene proved that **FIM** + **PA** is consistent if **C** is. In fact the arithmetical LEM is equivalent over $\mathbf{IA}_1 + \mathrm{AC}_{00}!$ to classical bar induction for arithmetical predicates. How about the *lawlike* LEM?

Embedding Theorem. *(a)* A unified axiomatic theory **FIRM** of the continuum extends Kleene's **FIM** in a language with a third sort of variable over lawlike sequences. **FIRM** proves that every lawlike sequence is extensionally equal to a choice sequence, but not conversely. The subsystem $\mathbf{FIM}^+$ of **FIRM** without lawlike sequence variables is **FIM** with Kleene's CC_{11} strengthened to Troelstra's GC_{11}. The subsystem **R** of **FIRM** without choice sequence variables is a notational variant of the classical system **C**.

(b) The subsystem **IR** of **R** with intuitionistic logic expresses Bishop's constructive mathematics **BCM**. Russian recursive mathematics $\mathbf{RRM} = \mathbf{IR} + \mathrm{ECT} + \mathrm{MP}$ is consistent relative to **IR**.

(c) Assuming a certain definably well ordered subset $\mathcal{R}$ of the continuum is countable, a classical realizability interpretation establishes the consistency of an extension $\mathbf{FIRM}(\prec)$ of **FIRM** in which $\prec$ well orders the lawlike sequences.

Why can't we assume all lawlike sequences are recursive?

▷ Kleene's formal language had variables $x, y, z, \ldots$ over numbers and $\alpha, \beta, \gamma, \ldots$ over choice sequences, but no special variables $a, b, c, \ldots$ over lawlike sequences. All the lawlike sequences he needed were recursive.

▷ Brouwer seems to have expressed no opinion on Church's Thesis, although it is likely that he was aware of it.

▷ Primitive recursive sequences are lawlike, so recursive sequences are lawlike by the *comprehension axiom*:

$\mathrm{AC}_{00}^{R}!\quad \forall x \exists ! y A(x, y) \rightarrow \exists b \forall x A(x, b(x))$

for $A(x, y)$ with only number and lawlike sequence variables, where

$$\exists ! y B(y) \equiv \exists y B(y) \ \&\ \forall x \forall y (B(x) \ \&\ B(y) \rightarrow x = y).$$

In **R** with classical logic, by $\mathrm{AC}_{00}^{R}!$ *all classical analytic functions* (with sequence quantifiers ranging over the lawlike part of the continuum) are lawlike. *So why aren't all sequences lawlike*?

In "*Lawless sequences of natural numbers*," *Comp. Math.* (1968), Kreisel described a system **LS** of axioms for numbers $m, n \ldots$, lawlike sequences $b, c, \ldots$ and *intensionally "lawless"* sequences $\alpha, \beta, \ldots$ in which *"the simplest kind of restriction on restrictions is made, namely some finite initial segment of values is prescribed, and, beyond this, no restriction is to be made"*.

- Equality ($=$ identity) of lawless sequences is decidable, and distinct lawless sequences are independent.
- Every neighborhood contains a lawless sequence.
- The axiom of open data holds: If $A(\alpha)$ where α is lawless, then $A(\beta)$ for all lawless β in some neighborhood of α.
- Strong effective continuous choice holds. If $\forall\alpha\exists b A(\alpha, b)$ then for some lawlike b, e: b codes a sequence $\{\lambda x.b(\langle n, x\rangle)\}_{n\in\omega}$ of sequences, e codes a continuous function defined on *all* choice sequences, and $\forall\alpha A(\alpha, \lambda x.b(\langle e(\alpha), x\rangle))$.

Troelstra, in “Choice Sequences: A Chapter of Intuitionistic Mathematics”(1977) and in Chapter 12 of “Constructivism in Mathematics: An Introduction” (Troelstra and van Dalen, 1988), analyzed and corrected the axioms of **LS**. To justify strong effective continuous choice, Troelstra formulated

The Extension Principle: Every function defined (and continuous) on all the lawless sequences has a continuous total extension.

He noted that identity is the *only* lawlike operation under which the class of lawless sequences is closed, suggested that lawlike sequence variables *may* be interpreted as ranging over “the classical universe of sequences,” and gave a detailed proof of Kreisel’s

Translation Theorem: Every formula E of **LS** without free lawless sequence variables can be translated uniformly into an equivalent formula $\tau(E)$ with only number and lawlike sequence variables, so “lawless sequences can be regarded as a figure of speech.”

Relatively Lawless Sequences: an Extensional Alternative

In 1987-1996 I developed a system **RLS** of axioms for numbers, lawlike sequences $a, b, \ldots, h$ and choice sequences $\alpha, \beta, \ldots$ extending Kleene's **B**. An arbitrary choice sequence α is *defined* to be "*R-lawless*" (lawless relative to the class R of lawlike sequences) if *every lawlike predictor correctly predicts* α *somewhere*:

$$RLS(\alpha) \equiv \forall b(Pred(b) \rightarrow \exists x \alpha \in \overline{\alpha}(x) * b(\overline{\alpha}(x))).$$

$Pred(b) \equiv \forall n(Seq(n) \rightarrow Seq(b(n)))$ where $Seq(n)$ says n codes a finite sequence of length $lh(n)$; $u * v$ codes the concatenation of sequences coded by u and v; and $\alpha \in u$ abbreviates $u = \overline{\alpha}(lh(u))$ where $\overline{\alpha}(0) = \langle\,\rangle$ and $\overline{\alpha}(x+1) = \langle \alpha(0)+1, \ldots, \alpha(x)+1 \rangle$.

Extensional equality between arbitrary R-lawless sequences α, β is *not* assumed to be decidable. Two R-lawless sequences α and β are *independent* if and only if their *merge* $[\alpha, \beta]$ is R-lawless, where $[\alpha, \beta](2n) = \alpha(n)$ and $[\alpha, \beta](2n+1) = \beta(n)$. (cf. Fourman 1982)

RLS has logical axioms and rules for all three sorts of quantifiers and an inductive definition of *term* and *functor*. R-terms and R-functors are those without choice sequence variables.

The new mathematical axioms of **RLS** include two *density axioms*:

RLS1. $\forall w(Seq(w) \to \exists\alpha[RLS(\alpha) \ \& \ \alpha \in w])$,

RLS2. $\forall w(Seq(w) \to \forall\alpha[RLS(\alpha) \to \exists\beta[RLS([\alpha,\beta]) \ \& \ \beta \in w]])$.

Definition. A formula is *restricted* if its choice sequence quantifiers all vary over relatively independent R-lawless sequences, so $\forall\alpha(RLS([\alpha,\beta]) \to B(\alpha,\beta))$ and $\exists\alpha(RLS([\alpha,\beta]) \ \& \ B(\alpha,\beta))$ are restricted if $B(\alpha,\beta)$ is restricted and has no choice sequence variables free other than α,β.

For $A(x,y)$, $A(x,b)$ restricted, with no free occurrences of choice sequence variables, **RLS** has the *lawlike comprehension axiom*

$AC^R_{00}!$ $\forall x\exists!yA(x,y) \to \exists b\forall xA(x,b(x))$, or equivalently

$AC^R_{01}!$ $\forall x\exists!bA(x,b) \to \exists b\forall xA(x,\lambda y.b(\langle x,y\rangle))$.

For the axiom schemas of *open data*

RLS3. $\forall\alpha[RLS(\alpha) \rightarrow (A(\alpha) \rightarrow$
$\exists w(Seq(w) \mathbin{\&} \alpha \in w \mathbin{\&} \forall\beta[RLS(\beta) \rightarrow (\beta \in w \rightarrow A(\beta))]))]$

and *effective continuous choice for R-lawless sequences*

RLS4. $\forall\alpha[RLS(\alpha) \rightarrow \exists b A(\alpha, b)] \rightarrow$
$\exists e \exists b \forall\alpha[RLS(\alpha) \rightarrow \exists n(e(\alpha) = n \mathbin{\&} A(\alpha, \lambda x.b(\langle n, x\rangle)))]$

and the *restricted law of excluded middle*

RLEM. $\forall\alpha[RLS(\alpha) \rightarrow A(\alpha) \vee \neg A(\alpha)]$

the $A(\alpha)$ and $A(\alpha, b)$ must be restricted, with no choice sequence variables free but α. The LEM *for formulas with only number and lawlike sequence variables* follows from RLEM by RLS1.

Remark. RLS1, $AC^R_{00}!$ and RLS4 entail *lawlike countable choice*:

AC^R_{01}. $\forall x \exists b A(x, b) \rightarrow \exists b \forall x A(x, \lambda y.b(\langle x, y\rangle))$

for $A(x, b)$ restricted, with no choice sequence variables free.

AC^R_{01} entails $AC^R_{00}!$, and RLS3, AC^R_{01} and RLEM entail RLS4.

RLS proves:

- $\forall a \exists ! \beta (\forall x\ a(x) = \beta(x))$. Every lawlike sequence is (extensionally) equal to an arbitrary choice sequence.
- $\forall \alpha [RLS(\alpha) \rightarrow \neg \exists b (\forall x\ b(x) = \alpha(x))]$. No R-lawless sequence is equal to a lawlike sequence.
- Independent R-lawless sequences are unequal.
- The R-lawless sequences are closed under prefixing an arbitrary finite sequence of natural numbers.
- If α is R-lawless and b is a lawlike injection with lawlike range, then $\alpha \circ b$ is R-lawless.
- The R-lawless sequences are dense in the continuum.
- Troelstra's extension principle fails. Every R-lawless sequence contains a (first) 1 but the constant 0 sequence doesn't.

RLS does *not* prove that equality between arbitrary R-lawless sequences α, β is decidable.

Theorem 1. Every restricted formula E with no arbitrary choice sequence variables free is equivalent in **RLS** to a formula $\varphi(E)$ with only number and lawlike sequence variables.

Proof: Instead of the constant K_0 Troelstra used to represent the class of lawlike codes of continuous *total* functions, we *define*

$$J_0(e) \equiv \forall u[Seq(u) \;\&\; \forall n < lh(u)(e(\overline{u}(n)) = 0) \rightarrow \exists v(Seq(v) \;\&\; e(u * v) > 0)].$$

Then **RLS** proves $\forall e(J_0(e) \leftrightarrow \forall \alpha[RLS(\alpha) \rightarrow e(\alpha)\downarrow])$ and $\forall \alpha[RLS(\alpha) \leftrightarrow \forall e(J_0(e) \rightarrow e(\alpha)\downarrow)]$, so the conclusion of effective continuous choice for R-lawless sequences can be rewritten

$$\exists e \exists b(J_0(e) \,\&\, \forall n \forall \alpha[RLS(\alpha) \rightarrow (e(\alpha) = n \rightarrow A(\alpha, \lambda x.b(\langle n, x\rangle)))]).$$

As in Troelstra's proof for **LS**, open data converts existential R-lawless quantifiers to universal ones. Subformulas of the form $\forall \alpha[RLS(\alpha) \rightarrow A(\alpha)]$ are translated in terms of simpler ones. For prime, lawlike $P(a)$, **RLS** proves $\forall \alpha[RLS(\alpha) \rightarrow P(\alpha)] \leftrightarrow \forall a P(a)$.

Definition. **R** is the subsystem of **RLS** obtained by restricting the language to number and lawlike sequence variables, omitting RLS1-4, replacing $\mathrm{AC}^R_{00}!$ by AC^R_{01}, and replacing RLEM by LEM. **IR** is the intuitionistic subsystem of **R**, without the LEM.

Remark. **IR**, with intuitionistic logic and countable choice, represents Bishop's constructive mathematics. For $B(w)$ and $A(w)$ without lawlike sequence variables, Brouwer's *bar theorem* is

$$\text{BI!}\quad \forall\alpha\exists!xB(\overline{\alpha}(x))\ \&\ \forall w(Seq(w)\ \&\ B(w)\rightarrow A(w))\ \&\ \forall w(Seq(w)\ \&\ \forall sA(w*\langle s+1\rangle)\rightarrow A(w))\rightarrow A(\langle\,\rangle).$$

BI (like BI! but omitting the !) conflicts with **FIM**, but for $B(w)$, $A(w)$ without choice sequence variables, **R** proves

$$\text{BI}^R.\quad \forall a\exists xB(\overline{a}(x))\ \&\ \forall w(Seq(w)\ \&\ B(w)\rightarrow A(w))\ \&\ \forall w(Seq(w)\ \&\ \forall sA(w*\langle s+1\rangle)\rightarrow A(w))\rightarrow A(\langle\,\rangle).$$

So **R** is a notational variant of the classical theory **C**.

Definition. **RLS**($\prec$) is the system resulting from **RLS** by extending the language to include prime formulas $u \prec v$ where u, v are functors, and adding axioms W0-W5:

W0. $\alpha = \beta \,\&\, \alpha \prec \gamma \to \beta \prec \gamma$.

$\beta = \gamma \,\&\, \alpha \prec \beta \to \alpha \prec \gamma$.

W1. $\forall a \forall b[a \prec b \to \neg b \prec a]$.

W2. $\forall a \forall b \forall c[a \prec b \,\&\, b \prec c \to a \prec c]$.

W3. $\forall a \forall b[a \prec b \vee a = b \vee b \prec a]$.

W4. $\forall a[\forall b(b \prec a \to A(b)) \to A(a)] \to \forall a A(a)$,

where $A(a)$ is any restricted formula, with no choice sequence variables free, in which b is free for a.

W5. $\alpha \prec \beta \to \neg \forall a \forall b \neg(\alpha = a \,\&\, \beta = b)$.

Remark. For $A(a)$ restricted, with no choice sequence variables free, **RLS**($\prec$) proves

$$\exists a A(a) \leftrightarrow \exists ! a(A(a) \,\&\, \forall b(b \prec a \to \neg A(b))).$$

Definition. **FIM**$^+$ is **FIM** with CC_{11} strengthened to GC_{11}.
FIRM is the common extension of **FIM**$^+$ and **RLS** in the 3-sorted language. **FIRM**$(\prec)$ similarly extends **FIM**$^+$ and **RLS**$(\prec)$.

Theorem 2. Iterating definability, quantifying over numbers and lawlike and independent R-lawless sequences, eventually yields a *definably well ordered subset* $(\mathcal{R}, \prec_{\mathcal{R}})$ of the classical continuum. *Assuming $\mathcal{R}$ is countable "from the outside,"*

(a) There is a classical model $\mathcal{M}(\prec_R)$ of **RLS**$(\prec)$, with $\mathcal{R}$ as the class of lawlike sequences.

(b) The class $\mathcal{R}LS$ of $\mathcal{R}$-lawless sequences of the model is disjoint from $\mathcal{R}$ and is Baire comeager in ω^ω, with classical measure 0.

(c) A classical realizability interpretation establishes the consistency of **FIRM**$(\prec)$ and hence of **FIRM**.

We outline the inductive definition of $(\mathcal{R}, \prec_{\mathcal{R}})$ and define $\mathcal{M}(\prec_R)$. For details see "Iterated definability, lawless sequences, and Brouwer's continuum," `http://www.math.ucla.edu/~joan/` .

Definition. If $F(a_0, \ldots, a_{k-1}) \equiv \forall x \exists ! y E(x, y, a_0, \ldots, a_{k-1})$ is a restricted formula where x, y are all the distinct number variables free in E, and the distinct lawlike sequence variables $a_0, \ldots, a_{k-1}$ are all the variables free in F in order of first free occurrence, and if $A \subset \omega^\omega$, $\prec_A$ wellorders A, $\varphi \in \omega^\omega$ and $\psi_0, \ldots, \psi_{k-1} \in A$, then *$E$ defines φ over A from $\psi_0, \ldots, \psi_{k-1}$* if and only if when lawlike sequence variables range over A and choice sequence variables over ω^ω, $\prec$ is interpreted by $\prec_A$, and $a_0, \ldots, a_{k-1}$ by $\psi_0, \ldots, \psi_{k-1}$:

(i) F is true, and

(ii) for all $x, y \in \omega$: $\varphi(x) = y$ if and only if $E(\mathbf{x}, \mathbf{y})$ is true

Definition. **Def($A, \prec_A$)** is the class of all $\varphi \in \omega^\omega$ which are defined over $(A, \prec_A)$ by some E from some $\psi_0, \ldots, \psi_{k-1}$ in A.

Observe that $A \subseteq \mathrm{Def}(A, \prec_A)$, since $a(x) = y$ defines every $\varphi \in A$ over A from itself. We have to extend $\prec_A$ to a well ordering $\prec_A^*$ of $\mathrm{Def}(A, \prec_A)$ so the process can be iterated.

The classical model $\mathcal{M}(\prec_{\mathcal{R}})$ of **RLS**$(\prec)$*:*

An R-*formula* has no arbitrary choice sequence variables free. Let $E_0(x,y), E_1(x,y), \ldots$ enumerate all restricted R-formulas in the language $\mathcal{L}(\prec)$ containing free no number variables but x, y, where $E_0(x,y) \equiv a(x) = y$. For each i, let $F_i \equiv \forall x \exists ! y E_i(x,y)$.

For $\varphi, \theta \in \mathrm{Def}(A, \prec_A)$, set $\varphi \prec_A^* \theta$ if and only if $\Delta_A(\varphi) < \Delta_A(\theta)$ where $\Delta_A(\varphi)$ is the smallest tuple $(i, \psi_0, \ldots, \psi_{k-1})$ in the lexicographic ordering $<$ of $\omega \cup \bigcup_{k>o}(\omega \times A^k)$ determined by $<$ on ω and $\prec_A$ on A such that E_i defines φ over A from $\psi_0, \ldots, \psi_{k-1}$.

If $\varphi \in A$ then $\Delta_A(\varphi) = (0, \varphi)$, so $\prec_A$ is an initial segment of $\prec_A^*$.

Define $R_0 = \phi$, $\prec_0 = \phi$, $R_{\zeta+1} = \mathrm{Def}(R_\zeta, \prec_\zeta)$, $\prec_{\zeta+1} = \prec_\zeta^*$, and at limit ordinals take unions.

By cardinality considerations there is a least ordinal η_0 such that $R_{\eta_0} = R_{\eta_0+1}$. Let $\mathcal{R} = R_{\eta_0}$ and $\prec_{\mathcal{R}} = \prec_{\eta_0}$. $\mathcal{M}(\prec_{\mathcal{R}})$ is the natural classical model in which lawlike sequence variables range over $\mathcal{R}$.

Key lemmas for the proof that if $\mathcal{R}$ is countable then $\mathcal{M}(\prec_{\mathcal{R}})$ is a classical model of **RLS**($\prec$) with $\mathcal{R}$ as the lawlike sequences:

Lemma 1. If $\mathcal{R}$ is countable then

(i) There is an $\mathcal{R}$-lawless sequence, and

(ii) If α is $\mathcal{R}$-lawless there is a sequence β such that $[\alpha, \beta]$ is $\mathcal{R}$-lawless.

Lemma 2. If α is $\mathcal{R}$-lawless, so are $\langle n+1 \rangle * \alpha$ for every natural number n and $\alpha \circ g$ for every injection $g \in \mathcal{R}$ whose range can be enumerated by an element of $\mathcal{R}$.

Lemma 3. If $A(\alpha)$ satisfies the axiom RLS3 of open data in $\mathcal{M}(\prec_{\mathcal{R}})$, so does $\neg A(\alpha)$.

▷ *We now appeal to the classical set-theoretic assumption that every definably well ordered subset of* ω^ω *is countable.*

▷ This assumption is provably consistent with classical **ZFC** if **ZFC** + 'there exists an inaccessible' is consistent. (Levy 1968)

$\mathcal{R}$ and $\prec_{\mathcal{R}}$ are definable over ω^ω with closure ordinal η_0. Let $\chi : \omega \times \omega \to \{0,1\}$ code a well ordering of type η_0 and let $\Gamma : \omega \to \mathcal{R}$ be a bijection witnessing simultaneously the countability of $\mathcal{R}$ and (via χ) the order of generation of its elements, so that for each $n, m \in \omega$:

$$\Gamma(n) \prec_{\mathcal{R}} \Gamma(m) \Leftrightarrow \chi(n,m) = 1.$$

A Γ-*interpretation* $\mathit{\Psi}$ of a list $\Psi = x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_k, a_1, \ldots, a_m$ of distinct variables is any choice of n numbers, k elements of ω^ω and m numbers $r_1, \ldots, r_m$. Then $\Gamma(\mathit{\Psi})$ is the corresponding list of n numbers, k sequences and m elements $\Gamma(r_1), \ldots, \Gamma(r_m)$ of $\mathcal{R}$.

Lemma 4. To each list Ψ of distinct *number and lawlike sequence* variables and each restricted R-formula $A(x,y)$ containing free at most Ψ, x, y where $x, y, a \notin \Psi$, there is a partial function $\xi_A(\mathit{\Psi})$ so that for each Γ-interpretation $\mathit{\Psi}$ of Ψ: If $\forall x \exists ! y A(x,y)$ is true-$\Gamma(\mathit{\Psi})$ then $\xi_A(\mathit{\Psi})$ is defined and $\forall x A(x, a(x))$ is true-$\Gamma(\mathit{\Psi}, \xi_A(\mathit{\Psi}))$.

The Γ*-realizability interpretation of* **FIRM**$(\prec)$: For $\pi \in \omega^\omega$, E a formula of $\mathcal{L}(\prec)$ with at most the distinct variables Ψ free, and Ψ a Γ-interpretation of Ψ, define π **Γ-realizes-Ψ** E:

- π Γ-realizes-Ψ a prime formula P, if P is true-$\Gamma(\Psi)$.
- π Γ-realizes-Ψ A & B, if $(\pi)_0$ $(= \lambda n.(\pi(n))_0)$ Γ-realizes-Ψ A and $(\pi)_1$ Γ-realizes-Ψ B.
- π Γ-realizes-Ψ $A \vee B$, if $(\pi(0))_0 = 0$ and $(\pi)_1$ Γ-realizes-Ψ A, or $(\pi(0))_0 \neq 0$ and $(\pi)_1$ Γ-realizes-Ψ B.
- π Γ-realizes-Ψ $A \to B$, if, if σ Γ-realizes-Ψ A, then $\pi[\sigma]$ Γ-realizes-Ψ B.
- π Γ-realizes-Ψ $\neg A$, if π Γ-realizes-Ψ $A \to 1 = 0$.
- π Γ-realizes-Ψ $\forall x A(x)$, if $\pi[x]$ $(= \pi[\lambda n.x])$ Γ-realizes-Ψ, x $A(x)$ for each $x \in \omega$.
- π Γ-realizes-Ψ $\exists x A(x)$, if $(\pi)_1$ Γ-realizes-$\Psi, (\pi(0))_0$ $A(x)$.
- π Γ-realizes-Ψ $\forall a A(a)$, if $\pi[r]$ Γ-realizes-Ψ, r $A(a)$ for each $r \in \omega$.

- π Γ-realizes-Ψ $\exists a A(a)$, if $(\pi)_1$ Γ-realizes-$\Psi, (\pi(0))_0$ $A(a)$.
- π Γ-realizes-Ψ $\forall \alpha A(\alpha)$, if $\pi[\alpha]$ Γ-realizes-Ψ, α $A(\alpha)$ for each $\alpha \in \omega^\omega$.
- π Γ-realizes-Ψ $\exists \alpha A(\alpha)$, if $(\pi)_1$ Γ-realizes-$\Psi, (\pi)_0$ $A(\alpha)$.

Lemma 5. To each almost negative formula E of $\mathcal{L}[\prec]$ with at most Ψ free there is a function $\varepsilon_E(\Psi) = \lambda t.\varepsilon_E(\Psi, t)$ partial recursive in Γ so that for each Γ-interpretation Ψ of Ψ:

(i) If E is Γ-realized-Ψ then E is true-Ψ, and

(ii) E is true-Ψ if and only if $\varepsilon_E(\Psi)$ Γ-realizes-Ψ E.

Lemma 6. To each list $\Psi = x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_k, a_1, \ldots, a_m$ and each restricted formula E of $\mathcal{L}(\prec)$ with at most Ψ free there is a continuous partial function $\zeta_E(\Psi)$ so that for each Γ-interpretation Ψ of Ψ with $[\alpha_1, \ldots, \alpha_k] \in \mathcal{R}LS$:

(i) If E is Γ-realized then E is true-Γ(Ψ), and

(ii) E is true-Γ(Ψ) if and only if $\zeta_E(\Psi)$ Γ-realizes-Ψ E.

Theorem 2 (c) (restated). Assuming Γ enumerates $\mathcal{R}$ as above, every closed theorem of **FIRM**($\prec$) is Γ-realized by some function and hence **FIRM**($\prec$) and **FIRM** are consistent.

Corollary. The constructive (**IR**) and classical (**R**) continuua can consistently be viewed as the lawlike part of Brouwer's continuum with intuitionistic or classical logic, respectively.

Theorem 3. If $\mathcal{R}$ is countable, the $\mathcal{R}$-lawless sequences are all the generic sequences with respect to properties of finite sequences of natural numbers which are definable over $(\omega, \mathcal{R}, \omega^\omega)$ by restricted R-formulas with parameters from $\omega, \mathcal{R}$.

Dragalin (1974, in Russian), van Dalen (1978, "An interpretation of intuitionistic analysis") and Fourman (1982, "Notions of choice sequence") all suggested modeling lawless by generic sequences. For the approach described here, Theorem 3 was an afterthought.

In a nutshell:

- *R-lawless* and *random* are orthogonal concepts, since a random sequence of natural numbers should possess certain regularity properties (e.g. the percentage of even numbers in its nth initial segment should approach .50 as n increases) while an R-lawless sequence will possess none.
- Brouwer's choice sequences satisfy the bar theorem, countable choice and continuous choice.
- Bishop's constructive sequences satisfy countable choice.
- The classical sequences satisfy the bar theorem and countable choice, but not continuous choice.
- The R-lawless sequences satisfy open data and restricted continuous choice, but not the restricted bar theorem.
- The recursive sequences satisfy recursive countable choice but not the bar or fan theorem.

Some References:

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6. Moschovakis, J. R., "A classical view of the intuitionistic continuum," *APAL* 81 (1996), 9-24.
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Details of the 3-sorted language $\mathcal{L}(\prec)$:

Variables with or without subscripts, also used as metavariables:

- $i, j, k, \ldots, p, q, x, y, z$ over natural numbers,
- a, b, c, d, e, g, h over lawlike sequences,
- $\alpha, \beta, \gamma, \ldots$ over arbitrary choice sequences.

Constants $\mathrm{f}_0, \ldots, \mathrm{f}_p$ for primitive recursive functions and functionals:

$\mathrm{f}_0 = 0$, $\mathrm{f}_1 = {}'$ (successor), $\mathrm{f}_2 = +$, $\mathrm{f}_3 = \cdot$, $\mathrm{f}_4 = \exp$, ...

Binary predicate constants $=$, $\prec$. Church's λ. Parentheses (,).

Metavariables s, t over *terms* and u, v over *functors*, defined by

- Sequence variables and unary function constants are functors.
- Number variables and $\mathrm{f}_i(u_1, \ldots, u_{k_i}, t_1, \ldots, t_{m_i})$ are terms.
- $(u)(t)$ is a term if u is a functor and t is a term.
- $\lambda n.t$ is a functor if t is a term.

Prime formulas: $s = t$ (s, t terms) and $u \prec v$ (u, v functors).

Logical symbols: $\&$, $\vee, \rightarrow, \neg$ and $\forall, \exists$ (3 sorts).

Axioms and rules for 3-sorted intuitionistic predicate logic. The two sorts of sequence quantifiers are distinguished by

- $C \to A(b)$ / $C \to \forall b A(b)$ if b is not free in C.
- $\forall b A(b) \to A(u)$ if u is an R-functor free for b in $A(b)$.
- $A(u) \to \exists b A(b)$ if u is an R-functor free for b in $A(b)$.
- $A(b) \to C$ / $\exists b A(b) \to C$ if b is not free in C.

(An R-*functor* contains no arbitrary choice sequence variables.)

- $C \to A(\beta)$ / $C \to \forall \beta A(\beta)$ if β is not free in C.
- $\forall \beta A(\beta) \to A(u)$ if u is any functor free for β in $A(\beta)$.
- $A(u) \to \exists \beta A(\beta)$ if u is any functor free for β in $A(\beta)$.
- $A(\beta) \to C$ / $\exists \beta A(\beta) \to C$ if β is not free in C.

Note that $\forall a \exists ! \beta (\forall x\ a(x) = \beta(x))$ is a theorem. ($! =$ unique)
Equality Axioms assert that $=$ is an equivalence relation and

- $x = y \to \alpha(x) = \alpha(y)$ (so also $x = y \to a(x) = a(y)$).

Axioms for 3-sorted intuitionistic number theory:

- Mathematical induction extended to $\mathcal{L}(\prec)$.
- Defining equations for the primitive recursive function and functional constants $\mathrm{f}_i(\alpha_1, \ldots, \alpha_{k_i}, x_1, \ldots, x_{m_i})$.
- $(\lambda x.s(x))(t) = s(t)$ for $s(x)$, t terms with t free for x in $s(x)$.

Coding finite sequences (after Kleene):

- $\langle x_0, \ldots, x_k \rangle = \Pi_0^k p_i^{x_i}$ where p_i is the *ith* prime with $p_0 = 2$.
- $(y)_i$ is the exponent of p_i in the prime factorization of y.
- $Seq(y)$ abbreviates $\forall i < lh(y)\ (y)_i > 0$ where $lh(y)$ is the number of nonzero exponents in the prime factorization of y.
- If $Seq(y)$ and $Seq(z)$ then $y * z$ codes the concatenation.
- $\overline{\alpha}(0) = 1 = \langle\ \rangle$ and $\overline{\alpha}(n+1) = \langle \alpha(0)+1, \ldots, \alpha(n)+1 \rangle$.
- $\alpha \in w$ abbreviates $\overline{\alpha}(lh(w)) = w$, and $\overline{\alpha} = \{\overline{\alpha}(n) : n \in \omega\}$.
- If $\alpha \in w$ and $n < lh(w)$ then $w \restriction n = \overline{\alpha}(n)$.

Countable choice:

▷ AC_{01}. $\forall x \exists \alpha A(x, \alpha) \to \exists \beta \forall x A(x, \lambda y.\beta(\langle x, y\rangle))$.

Lawlike comprehension, for $A(x, b)$ restricted and with only number and lawlike sequence variables free:

▷ $\mathrm{AC}^R_{00}!$ $\forall x \exists ! y A(x, y) \to \exists b \forall x A(x, b(x))$.

Brouwer's Bar Theorem:

▷ BI! $\forall \alpha \exists ! x B(\overline{\alpha}(x))$ & $\forall w(Seq(w)$ & $B(w) \to A(w))$ &
$\forall w(Seq(w)$ & $\forall s\, A(w * \langle s + 1\rangle) \to A(w)) \to A(\langle\, \rangle)$.

Generalized Continuous Choice, for $A(\alpha)$ almost negative:

▷ GC_{11}. $\forall \alpha[A(\alpha) \to \exists \beta B(\alpha, \beta)] \to$
$\exists \sigma \forall \alpha[A(\alpha) \to \sigma[\alpha]\downarrow$ & $\forall \beta(\sigma[\alpha] = \beta \to A(\alpha, \beta))]$.

$\sigma[\alpha]\downarrow$ abbreviates $\forall x \exists y \sigma[\alpha](x) = y$ and $\sigma[\alpha](x) = y$ abbreviates $\exists z[\sigma(\langle x + 1\rangle * \overline{\alpha}(z)) = y + 1$ & $\forall n < z\, \sigma(\langle x + 1\rangle * \overline{\alpha}(n)) = 0]$.

Continuous Choice CC_{11} is like GC_{11} but without the $A(\alpha) \to$.