# A Translation Theorem For Restricted $R$-Formulas

Joan Rand Moschovakis
*Occidental College (Emerita) and $\mu\pi\lambda\alpha$*
joan.rand@gmail.com

**Abstract**

The three-sorted formal system **RLS** described in [4] is like **RLS($\prec$)** in [5] but without the $\prec$. **IRLS** is a strictly intuitionistic subsystem of **RLS**. This note gives a natural, syntactically defined translation $\varphi$ mapping each restricted formula $E$ with only number and lawlike sequence variables free, to a formula $\varphi(E)$ containing only number and lawlike sequence variables, such that **IRLS** proves $E \leftrightarrow \varphi(E)$. If $E$ contains no choice sequence variables then $\varphi(E)$ is $E$.

# 1 The systems RLS, IRLS, R, IR and C

## 1.1 A three-sorted language $\mathcal{L}$

The language, extending the two-sorted language of [2] and [1], contains three sorts of variables with or without subscripts, also used as metavariables:

$i, j, k, l, m, n, w, x, y, z$ over natural numbers,

$a, b, c, d, e, g, h$ over lawlike sequences,

$\alpha, \beta, \gamma, \ldots$ over arbitrary choice sequences;

finitely many constants $\mathrm{f}_0$ $(= 0)$, $\mathrm{f}_1$ $(= {}')$ (successor), $\mathrm{f}_2$ $(= +)$, $\mathrm{f}_3$ $(= \cdot)$, $\mathrm{f}_4$ $(= exp)$, $\mathrm{f}_5$, $\ldots$, $\mathrm{f}_p$ for primitive recursive functions and functionals; the binary predicate constant $=$ (between terms); Church's $\lambda$ denoting function abstraction; parentheses (,) denoting function application; and the logical symbols $\&$ , $\vee, \rightarrow, \neg$ and quantifiers $\forall, \exists$ over each sort of variable.

I thank Sean Walsh and Kai Wehmeyer of UC Irvine, and the organizers of the 2014 Chiemsee Summer School, for giving me new opportunities to talk about this subject, resulting in this theorem. I am also very grateful to an anonymous referee whose careful reading led to many improvements.

*Terms* (of type 0) and *functors* (of type 1) are defined inductively. Number variables and 0 are terms. Sequence variables of both sorts, and unary function constants, are functors. If $\mathrm{f}_i$ is a $k_i, m_i$-ary function constant, $u_1, \ldots, u_{k_i}$ are functors and $t_1, \ldots, t_{m_i}$ are terms, then $\mathrm{f}_i(u_1, \ldots, u_{k_i}, t_1, \ldots, t_{m_i})$ is a term. If $u$ is a functor and $t$ is a term then $(u)(t)$ (also written $u(t)$) is a term. If $t$ is a term and $x$ is a number variable then $\lambda x(t)$ (also written $\lambda x.t$) is a functor.

*Prime formulas* are of the form $s = t$ where $s, t$ are terms. If $u, v$ are functors then $u = v$ abbreviates $\forall x\,(u(x) = v(x))$. Composite formulas are formed as usual, with parentheses determining scopes.

Terms and functors with no occurrences of arbitrary choice sequence variables are *R-terms* and *R-functors* respectively. Formulas with no *free* occurrences of arbitrary choice sequence variables are *R-formulas*.

## 1.2 The logical axioms and rules

The logical basis is intuitionistic three-sorted predicate logic, extending the rules and axiom schemas in [2] to formulas, terms and functors of $\mathcal{L}$ as defined above, with new rules and axiom schemas for lawlike sequence variables and $R$-functors:

$9^R$. $C \to A(b)$ / $C \to \forall b A(b)$ if $b$ is not free in $C$.

$10^R$. $\forall b A(b) \to A(u)$ if $u$ is an $R$-functor free for $b$ in $A(b)$.

$11^R$. $A(u) \to \exists b A(b)$ if $u$ is an $R$-functor free for $b$ in $A(b)$.

$12^R$. $A(b) \to C$ / $\exists b A(b) \to C$ if $b$ is not free in $C$.

## 1.3 Axioms for 3-sorted intuitionistic number theory

Equality axioms assert that $=$ is an equivalence relation and $x = y \to \alpha(x) = \alpha(y)$, so $\forall x(a(x) = a(x))$ is provable (since lawlike sequence variables are functors), so $\forall a \exists \beta \forall x(a(x) = \beta(x))$ follows by the instance $\forall x(a(x) = \gamma(x)) \to \exists \beta \forall x(a(x) = \beta(x))$ of axiom schema $11^F$ from [2]. Just as Brouwer's infinitely proceeding sequences include all the sharp arrows, every lawlike sequence is (equal to) a choice sequence.

By a similar argument, if $u$ is an $R$-functor in which the variable $b$ does not occur then $\exists b \forall x(b(x) = u(x))$ is provable, so every $R$-functor denotes a lawlike sequence.

For terms $r(x), t$ with $t$ free for $x$ in $r(x)$, the $\lambda$-reduction axiom schema is

$$(\lambda x.r(x))(t) = r(t),$$

where $r(t)$ is the result of substituting $t$ for all free occurrences of $x$ in $r(x)$.

The mathematical axioms include the assertions that 0 (= $f_0$) is not a successor and the successor function (= $f_1$) is one-to-one, the defining equations for the primitive recursive function and functional constants $f_2, \ldots, f_p$ ([2], [1]) and the mathematical induction schema extended to $\mathcal{L}$. For the countable axiom of choice

$$\text{AC}_{01}. \qquad \forall x \exists \alpha A(x, \alpha) \to \exists \alpha \forall x A(x, \lambda y.\alpha(2^x \cdot 3^y))$$

the $x$ must be distinct from $y$, and free for $\alpha$ in $A(x, \alpha)$.

Finite sequences are coded primitive recursively as in [2], so $\langle x_0, \ldots, x_k \rangle = \Pi_0^k p_i^k$ where $p_i$ is the *ith* prime with $p_0 = 2$, and $(y)_i$ is the exponent of $p_i$ in the prime factorization of $y$. Let $Seq(y)$ abbreviate $\forall i < lh(y)((y)_i > 0)$ where $lh(y)$ is the number of nonzero exponents in the prime factorization of $y$. The empty sequence is coded by $\langle\,\rangle = 1$, and if $k \geq 0$ then $\langle x_0 + 1, \ldots, x_k + 1 \rangle$ codes the finite sequence $(x_0, \ldots, x_k)$. If $Seq(y)$ and $Seq(z)$ then $y * z$ codes the concatenation of the sequences coded by $y$ and $z$.

The finite initial segment of length $n$ of a choice sequence $\alpha$ is coded by $\overline{\alpha}(n)$, where $\overline{\alpha}(0) = 1$ and $\overline{\alpha}(n+1) = \langle \alpha(0) + 1, \ldots, \alpha(n) + 1 \rangle$. Other useful abbreviations are $\alpha \in w$ for $\overline{\alpha}(lh(w)) = w$, $w \sqsubseteq y$ for $Seq(y)$ & $\forall i < lh(w)((w)_i = (y)_i)$, and $w \sqsubset y$ for $w \sqsubseteq y$ & $lh(y) > lh(w)$. If $Seq(w)$ then $w * \alpha = \beta$ where $\beta \in w$ and $\beta(lh(w) + n) = \alpha(n)$; if $\neg Seq(w)$ then $w * \alpha = \alpha$. Note that $w * \alpha$ is a functor and $w * a$ is an $R$-functor.

## 1.4 Bar induction

Kleene formulated Brouwer's "bar theorem" as an axiom schema, in four versions which are all equivalent using $\text{AC}_{01}$ (or even $\text{AC}_{00}!$), and included it in his basic system **B**. The version we assume (now for the three-sorted language) is[1]

$$\text{BI!} \quad \forall \alpha \exists ! x R(\overline{\alpha}(x)) \ \& \ \forall w(Seq(w) \ \& \ R(w) \to A(w)) \\ \& \ \forall w(Seq(w) \ \& \ \forall n A(w * \langle n \rangle) \to A(w)) \to A(1).$$

This schema (for the two-sorted language without lawlike sequence variables) completed Kleene's *basic system* **B**, which is neutral in the sense that it is correct both intuitionistically and classically.

## 1.5 $R$-lawless sequences, restricted quantification and lawlike comprehension

Intuitively, a lawless sequence should not be predictable by any lawlike process, but this negative condition is not enough to satisfy Kreisel's axioms. Instead, call

[1]In general, $\exists ! x A(x)$ abbreviates $\exists x A(x)$ & $\forall x \forall y (A(x)$ & $A(y) \to x = y)$.

a choice sequence $\beta$ a *predictor* if $\beta$ maps finite sequence codes to finite sequence codes, and call a choice sequence $\alpha$ *R-lawless* if every lawlike predictor correctly predicts $\alpha$ somewhere. Formally, $RLS(\alpha)$ abbreviates

$$\forall b[Pred(b) \rightarrow \exists x\ \alpha \in \overline{\alpha}(x) * b(\overline{\alpha}(x))],$$

where $Pred(b)$ abbreviates $\forall w(Seq(w) \rightarrow Seq(b(w)))$.

Since each prediction affects only finitely many values, this positive condition leaves room for (indeed, insures) plenty of chaotic behavior *if there are only countably many lawlike predictors.* The usual diagonal argument guarantees that there is no lawlike enumeration of the lawlike sequences, but a classical model with countably many lawlike sequences is described in [5].

Troelstra's extension principle, which claims that every continuous partial function defined on all lawless sequences has a continuous total extension, fails for $R$-lawless sequences, since $\forall\alpha[RLS(\alpha) \rightarrow \exists n\alpha(n) = 1]$ but the function assigning to each $R$-lawless $\alpha$ the least $n$ such that $\alpha(n) = 1$ cannot be extended continuously to all choice sequences. And while Kreisel and Troelstra considered any two distinct lawless sequences to be independent, a stronger condition for independence is needed here.

Two $R$-lawless sequences $\alpha, \beta$ will be called *independent* if their fair merge $[\alpha, \beta]$ is lawless, and similarly for $\alpha_0, \ldots, \alpha_k$ where $[\alpha_0, \ldots, \alpha_k]((k+1)n + i) = \alpha_i(n)$ for $0 \leq i \leq k$ and all $n$. This natural notion of independence for lawless sequences was proposed by M. Fourman at the Brouwer Centenary Conference in 1981.

The class of *restricted* formulas is defined inductively: Each formula $E$ with no arbitrary choice sequence quantifiers is *restricted.* If $A$ is *restricted* and contains free no arbitrary choice sequence variables other than $\alpha$, then $\forall\alpha[RLS(\alpha) \rightarrow A]$ and $\exists\alpha[RLS(\alpha)\ \&\ A]$ are *restricted.* If $k > 0$ and $A$ is *restricted* with no arbitrary choice sequence variables other than $\alpha_0, \ldots, \alpha_k$ occurring free, then for $i = 0, \ldots, k$ the formulas $\forall\alpha_i[RLS([\alpha_0, \ldots, \alpha_k]) \rightarrow A]$ and $\exists\alpha_i[RLS([\alpha_0, \ldots, \alpha_k])\ \&\ A]$ are *restricted.* No other formulas are *restricted.*

There is a lawlike function-comprehension schema

$$\mathrm{AC}^R_{00}! \qquad \forall x \exists! y A(x, y) \rightarrow \exists b \forall x A(x, b(x))$$

where $A(x, y)$ is any restricted $R$-formula and $b$ is free for $y$ in $A(x, y)$. By this axiom, the lawlike sequences are closed under "recursive in."[2]

[2]While a restricted formula can have free occurrences of arbitrary choice sequence variables, a restricted $R$-formula cannot. If $A(x, y)$ is a restricted $R$-formula, the informal abbreviation $\mu y A(x, y)$ may be allowed under either of the assumptions $\exists! y A(x, y)$ or $\exists y(A(x, y)\ \&\ \forall z < y \neg A(x, z))$.

For restricted $R$-formulas $A(x,a)$ the lawlike comprehension schema entails

$$\mathrm{AC}^{R}_{01}! \qquad \forall x \exists ! a A(x,a) \rightarrow \exists b \forall x A(x, \lambda y. b(2^x \cdot 3^y)),$$

with the obvious conditions on the variables.

## 1.6 Axioms for $R$-lawless sequences

These are Kreisel's and Troelstra's axioms from [3] and [7], adapted to Kleene's convention for coding continuous functions, with inequality of lawless sequences replaced by independence. There are two *density axioms*:

$$\text{RLS1.} \qquad \forall w(Seq(w) \rightarrow \exists \alpha[RLS(\alpha) \;\&\; \alpha \in w]),$$

$$\text{RLS2.} \qquad \forall w(Seq(w) \rightarrow \forall \alpha[RLS(\alpha) \rightarrow \exists \beta[RLS([\alpha,\beta]) \;\&\; \beta \in w]]).$$

Kreisel's principle of *open data* is stated as follows, on condition that $A(\alpha)$ is restricted and has no other arbitrary choice sequence variables free, and $\beta$ is free for $\alpha$ in $A(\alpha)$:

$$\text{RLS3.} \quad \forall \alpha[RLS(\alpha) \rightarrow (A(\alpha) \rightarrow \exists w(Seq(w) \;\&\; \alpha \in w \;\&\; \forall \beta[RLS(\beta) \rightarrow (\beta \in w \rightarrow A(\beta))]))].$$

*Effective continuous choice for lawless sequences* is the schema

$$\text{RLS4.} \quad \forall \alpha[RLS(\alpha) \rightarrow \exists b A(\alpha,b)] \rightarrow \exists e \exists b \forall \alpha[RLS(\alpha) \rightarrow \exists ! y e(\overline{\alpha}(y)) > 0 \;\&\; \forall y (e(\overline{\alpha}(y)) > 0 \rightarrow A(\alpha, \lambda x.\, b(\langle e(\overline{\alpha}(y)) \dot{-} 1, x \rangle)))]$$

where $A(\alpha,b)$ is restricted with no arbitrary choice sequence variables but $\alpha$ free, and $e, y, \alpha$ are free for $b$ in $A(\alpha,b)$.[3]

## 1.7 The restricted law of excluded middle

For $A(\alpha)$ restricted, with no choice sequence variables free except possibly $\alpha$, **RLS** also has the axiom schema

$$\text{RLEM.} \qquad \forall \alpha[RLS(\alpha) \rightarrow A(\alpha) \vee \neg A(\alpha)].$$

By an easy argument, RLS3 and the restricted LEM entail the following principle of *closed data* with the same restrictions on $A(\alpha)$ as for RLS3:

$$\text{RLS5.} \quad \forall \alpha[RLS(\alpha) \rightarrow (\forall w(\alpha \in w \rightarrow \exists \beta[RLS(\beta) \;\&\; \beta \in w \;\&\; A(\beta)]) \rightarrow A(\alpha))].$$

[3] In general, $e(\alpha) \simeq n$ abbreviates $\exists x(e(\overline{\alpha}(x)) = n+1 \;\&\; \forall y < x e(\overline{\alpha}(y)) = 0)$.

In a strictly intuitionistic system without RLEM, RLS5 may or may not be taken as an additional axiom schema. With RLS1, RLEM entails the law of excluded middle for all formulas with only lawlike and number variables. Observe that RLS1, RLS2, and all instances of RLS3, RLS4, RLEM and RLS5 are restricted $R$-formulas.

## 1.8 Five axiomatic systems

In addition to Kleene's basic formal system **B** for neutral analysis we consider five other formal systems. All but one are consistent with full intuitionistic analysis **FIM** as formalized in [2].[4]

**IRLS** extends **B** to the three-sorted language and adds axioms RLS1,2 and axiom schemas $\mathrm{AC}_{00}^{R}!$ and RLS3,4. **IRLS** expresses a strictly intuitionistic theory of lawlike and relatively lawless sequences in the context of full intuitionistic analysis.

**RLS** is the three-sorted semi-intuitionistic system **IRLS** + RLEM.[5]

*Lawlike classical analysis* **R** is the two-sorted subsystem of **RLS** obtained by restricting the language to number and lawlike sequence variables, omitting RLS1-4 and BI!, replacing RLEM by $A \vee \neg A$ for formulas of the two-sorted language, replacing $\mathrm{AC}_{01}$ and $\mathrm{AC}_{00}^{R}!$ by $\mathrm{AC}_{01}^{R}$ (like $\mathrm{AC}_{01}^{R}!$ but without the !) for formulas of the two-sorted language, and restating the equality axioms and primitive recursive definitions of function constants using lawlike instead of arbitrary choice sequence variables.

*Constructive analysis* **IR** is the two-sorted intuitionistic subsystem of **R** obtained by omitting $A \vee \neg A$. Note that **IR** has no version of bar induction.

*Classical analysis* **C** is the two-sorted system obtained from Kleene's **B** by strengthening the logic to classical logic. A lawlike version $\mathrm{BI}!^{R}$ of BI!, with lawlike sequence variables replacing arbitrary choice sequence variables, is provable in **R**. Thus **C** and **R** are notational variants, as are **B** and **IR** + $\mathrm{BI}!^{R}$.

## 1.9 Closure properties of $RLS$: Lemma

*The three-sorted subsystem* **IRS** *of* **IRLS** *obtained by omitting* RLS1-4, *but retaining* $\mathrm{AC}_{00}^{R}!$, *proves*

(i) $\forall\alpha[RLS(\alpha) \leftrightarrow \forall w(Seq(w) \rightarrow RLS(w * \alpha))]$.

[4] The relative consistency of a common extension of **RLS** and **FIM** is established in [5] under the assumption that a definably well-ordered subset of $\omega^{\omega}$ is countable.

[5] The translation theorem will show that **RLS** can also be axiomatized by **IRLS** plus the law of excluded middle for strictly lawlike formulas, so **RLS** is indeed semi-intuitionistic.

(ii) $\forall b(\forall x \forall y(b(x) = b(y) \to x = y) \,\&\, \forall y(\exists x\, b(x) = y \;\vee\; \neg\exists x\, b(x) = y) \to \forall\alpha[RLS(\alpha) \to RLS(\alpha \circ b)])$.

(iii) $\forall b(Pred(b) \to \forall n \forall\alpha[RLS(\alpha) \to \exists m(m \geq n \,\&\, \alpha \in \overline{\alpha}(m) * b(\overline{\alpha}(m)))])$.

*Proofs.* This is a formal version of Lemma 2 of [5]. For (i $\to$) assume $Seq(w)$ and $Pred(b)$. Then $\forall x \exists! y((Seq(x) \to y = b(w * x)) \,\&\, (\neg Seq(x) \to y = 0))$, so by $\mathrm{AC}^R_{00}!$ there is a $c$ such that $Pred(c)$ and $\forall x(Seq(x) \to c(x) = b(w * x))$, so if $RLS(\alpha)$ then $\exists z\, \alpha \in \overline{\alpha}(z) * c(\overline{\alpha}(z))$ and hence $\exists z(w * \alpha \in \overline{w * \alpha}(lh(w) + z) * b(\overline{w * \alpha}(lh(w) + z)))$. For (i $\leftarrow$) take $w = 1 = \langle\,\rangle$, so $w * \alpha = \alpha$. The proofs of (ii), (iii) similarly formalize the proofs of (ii), (iii) of Lemma 2 of [5]. □

### 1.10 Axioms RLS1-3 reconsidered

Lemma 1.9 guarantees that the following schemas RLS1′, RLS2′ and RLS3′ are equivalent over **IRS** to RLS1, RLS2 and RLS3 respectively.

RLS1′. $$\exists\alpha RLS(\alpha)$$

RLS2′. $$\forall\alpha[RLS(\alpha) \leftrightarrow \exists\beta RLS([\alpha, \beta])]$$

RLS3′. $$\forall\alpha[RLS(\alpha) \to (A(\alpha) \leftrightarrow \exists w(Seq(w) \,\&\, \alpha \in w \,\&\, \forall\beta[RLS(\beta) \to A(w * \beta)]))]$$

under the same conditions on $A(\alpha)$ as for RLS3. The next section suggests a way to simplify RLS4 as well.

## 2 The translation theorem

### 2.1 Theorem

*Every restricted formula $E$ of the three-sorted language with no arbitrary choice sequence variables free is equivalent in* **IRLS** *to a formula $\varphi(E)$ of the two-sorted language with only number and lawlike sequence variables. The mapping $\varphi$ is syntactically defined. If $E$ contains no choice sequence variables then $\varphi(E)$ is $E$.*

The proof is similar to Troelstra's proof of the translation theorem for **LS** into the language without lawless sequence variables (cf. [8], 663ff), with a significant difference. Instead of the constant $K_0$ Troelstra used to represent the class of lawlike codes of continuous *total* functions, we can *define* the condition for $e$ to be a lawlike code of a continuous partial function defined on all the $R$-lawless sequences:

$$J_0(e) \equiv \forall w(Seq(w) \,\&\, \forall n < lh(w)(e(\overline{w}(n)) = 0) \to \exists y(Seq(y) \,\&\, e(w * y) > 0)),$$
$$J_1(e) \equiv J_0(e) \,\&\, \forall w[e(w) > 0 \to Seq(w) \,\&\, \forall y(Seq(y) \to e(w * y) = e(w))].$$

By Lemma 1.9(i) and the next lemma, the conclusion of effective continuous choice for $R$-lawless sequences can be rewritten

$$\exists e \exists b(J_1(e) \ \& \ \forall n \forall w(e(w) = n+1 \rightarrow \forall \alpha[RLS(\alpha) \rightarrow A(w * \alpha, \lambda x.b(\langle n, x \rangle))]])).$$

## 2.2 Lemma

(i) **IRS** + RLS1 *proves* $\forall e(J_0(e) \leftrightarrow \forall \alpha[RLS(\alpha) \rightarrow e(\alpha)\downarrow])$, *and*

(ii) **IRS** *proves* $\forall \alpha[RLS(\alpha) \leftrightarrow \forall e(J_j(e) \rightarrow e(\alpha)\downarrow)]$ *for* $j = 0, 1$,

*where* $e(\alpha)\downarrow$ *abbreviates* $\exists m\,(e(\overline{\alpha}(m)) > 0)$.

*Proofs.* (i) $\rightarrow$: Assume $J_0(e)$. Using $\mathrm{AC}^R_{00}!$ define a lawlike predictor $g$ by

$$g(w) = \begin{cases} \langle\,\rangle & \text{if } \exists y \sqsubseteq w(e(y) > 0) \vee \neg Seq(w), \\ \mu y(Seq(y) \ \& \ e(w * y) > 0) & \text{otherwise.} \end{cases}$$

If $RLS(\alpha)$ then $\alpha \in \overline{\alpha}(n) * g(\overline{\alpha}(n))$ for some $n$, so $e(\alpha)\downarrow$.

(i) $\leftarrow$: Assume $\forall \alpha[RLS(\alpha) \rightarrow e(\alpha)\downarrow]$ and $Seq(w)$. By RLS1 there is an $\alpha \in w$ with $RLS(\alpha)$, so for some $y = \overline{\alpha}(m)$: $e(y) > 0 \ \& \ \forall n < m(e(\overline{\alpha}(n)) = 0)$. If also $\forall n < lh(w)(e(\overline{w}(n)) = 0)$ then $w \sqsubseteq y$, so $y * e(y) = w * z$ where $e(w * z) > 0$.

(ii) $\rightarrow$ follows immediately from (i) $\rightarrow$ (with the fact that $\forall e(J_1(e) \rightarrow J_0(e))$). For (ii) $\leftarrow$: Assume $\forall e(J_1(e) \rightarrow e(\alpha)\downarrow)$ and let $g$ be a lawlike predictor. Define $e$ as follows:

$$e(w) = \begin{cases} 1 & \text{if } Seq(w) \ \& \ \exists y \sqsubseteq w(y * g(y) \sqsubseteq w), \\ 0 & \text{otherwise.} \end{cases}$$

Then $J_1(e)$ holds, so $e(\alpha)\downarrow$, so $g$ correctly predicts $\alpha$ somewhere. □

The proof of the translation theorem depends on Lemmas 1.9, 2.2, and the following sequence of lemmas removing restricted existential quantifiers and reducing restricted $R$-formulas of the form $\forall \alpha[RLS(\alpha) \rightarrow A]$ and $\forall \alpha_i[RLS([\alpha_0, \ldots, \alpha_k]) \rightarrow A]$ to simpler formulas of the same kind. For the case that $A$ is prime the reduction is complete in one step, even in **IRS** + RLS1.

## 2.3 Lemma

*If* $s(\alpha), t(\alpha)$ *are terms with no arbitrary choice variables but* $\alpha$ *free, and* $a$ *is free for* $\alpha$ *in both, then* **IRS** + RLS1 *proves*

(i) $\forall \alpha[RLS(\alpha) \rightarrow s(\alpha) = t(\alpha)] \leftrightarrow \forall a[s(a) = t(a)]$ *and*
(ii) $\exists \alpha[RLS(\alpha) \ \& \ s(\alpha) = t(\alpha)] \leftrightarrow \exists a[s(a) = t(a)]$.

*Proof.* By induction on the complexity of the term $s(\alpha)$ (expressing the value of a primitive recursive function of $\alpha$ and the other free variables) **IRS** proves

$$\forall\alpha\exists x\exists y\forall\beta(\overline{\beta}(x) = \overline{\alpha}(x) \to s(\beta) = y).$$

Only the argument for (i) is completed here since the proof of (ii) is similar.

(i) $\leftarrow$: Assume $\forall a[s(a) = t(a)]$ and $RLS(\alpha)$. Let $x, y, z$ satisfy $\forall\beta(\overline{\beta}(x) = \overline{\alpha}(x) \to s(\beta) = y \ \&\ t(\beta) = z]$ and let $w = \overline{\alpha}(x)$. Then $\forall a[\overline{a}(x) = w \to s(a) = y \ \&\ t(a) = z]$, so $y = z$ since $w * \lambda n.0$ is lawlike by $\mathrm{AC}^R_{00}!$, so $s(\alpha) = t(\alpha)$ since $s(\alpha) = y \ \&\ t(\alpha) = z$. So **IRS** proves $\forall a[s(a) = t(a)] \to \forall\alpha[RLS(\alpha) \to s(\alpha) = t(\alpha)]$.

(i) $\to$: Assume $\forall\alpha[RLS(\alpha) \to s(\alpha) = t(\alpha)]$. Let $x, y, z$ satisfy $\forall\beta[\overline{\beta}(x) = \overline{a}(x) \to s(\beta) = y \ \&\ t(\beta) = z]$, so $s(a) = y \ \&\ t(a) = z$. By RLS1 there is a $\beta$ such that $RLS(\beta) \ \&\ \overline{\beta}(x) = \overline{a}(x)$, so $s(\beta) = t(\beta)$ and $y = z$, so $s(a) = t(a)$. So **IRS** + RLS1 proves $\forall\alpha[RLS(\alpha) \to s(\alpha) = t(\alpha)] \to \forall a[s(a) = t(a)]$. □

## 2.4 Lemma

**IRLS** *proves*

(i) $\forall\alpha[RLS(\alpha) \to A(\alpha) \ \&\ B(\alpha)] \leftrightarrow \forall\alpha[RLS(\alpha) \to A(\alpha)] \ \&\ \forall\alpha[RLS(\alpha) \to B(\alpha)]$,

(ii) $\forall\alpha[RLS(\alpha) \to A(\alpha) \vee B(\alpha)] \leftrightarrow \exists e[J_1(e) \ \&$
$\forall w(e(w) > 0 \to \forall\alpha[RLS(\alpha) \to A(w * \alpha)] \vee \forall\alpha[RLS(\alpha) \to B(w * \alpha)])]$,

(iii) $\forall\alpha[RLS(\alpha) \to (A(\alpha) \to B(\alpha))] \leftrightarrow \forall w(Seq(w) \to$
$(\forall\alpha[RLS(\alpha) \to A(w * \alpha)] \to \forall\alpha[RLS(\alpha) \to B(w * \alpha)]))$,

(iv) $\forall\alpha[RLS(\alpha) \to \neg A(\alpha)] \leftrightarrow \forall w(Seq(w) \to \neg\forall\alpha[RLS(\alpha) \to A(w * \alpha)])$,

*for $A(\alpha), B(\alpha)$ restricted, with no arbitrary choice sequence variables other than $\alpha$ occurring free.*

*Proofs.* (ii) follows from RLS4 using Lemmas 1.9 and 2.2 with the observation

$$\forall e(J_0(e) \to \exists g[J_1(g) \ \& \ \forall x\forall w(g(w) = x + 1 \leftrightarrow \exists y \sqsubseteq w\,(e(y) = x + 1 \ \&\ \forall z \sqsubset y\, e(z) = 0))].$$

Using Lemma 1.9 again: (iii) follows from RLS3, (iv) $\to$ follows from RLS1, and (iv) $\leftarrow$ is an easy consequence of RLS3. □

### 2.5 Lemma

**IRLS** *proves*

$$\exists\alpha[RLS(\alpha)\ \&\ A(\alpha)] \leftrightarrow \exists w(Seq(w)\ \&\ \forall\alpha[RLS(\alpha) \rightarrow A(w * \alpha)])$$

*if* $A(\alpha)$ *is restricted and contains free no arbitrary choice sequence variable but* $\alpha$*.*

*Proof.* Immediate from RLS3. □

### 2.6 Lemma

*For restricted formulas* $A(\alpha, x)$*,* $A(\alpha, b)$ *containing free no arbitrary choice sequence variables other than* $\alpha$*,* **IRLS** *proves*

(i) $\forall\alpha[RLS(\alpha) \rightarrow \exists x A(\alpha, x)] \leftrightarrow \exists e[J_1(e)\ \&$
$\forall w(e(w) > 0 \rightarrow \exists x\forall\alpha[RLS(\alpha) \rightarrow A(w * \alpha, x)])]$,

(ii) $\forall\alpha[RLS(\alpha) \rightarrow \exists b A(\alpha, b)] \leftrightarrow \exists e[J_1(e)\ \&$
$\forall w(e(w) > 0 \rightarrow \exists b\forall\alpha[RLS(\alpha) \rightarrow A(w * \alpha, b)])]$,

(iii) $\forall\alpha[RLS(\alpha) \rightarrow \forall x A(\alpha, x)] \leftrightarrow \forall x\forall\alpha[RLS(\alpha) \rightarrow A(\alpha, x)]$,

(iv) $\forall\alpha[RLS(\alpha) \rightarrow \forall b A(\alpha, b)] \leftrightarrow \forall b\forall\alpha[RLS(\alpha) \rightarrow A(\alpha, b)]$.

*Proofs.* (i) and (ii) are by RLS4; (iii) and (iv) are by predicate logic. □

### 2.7 Lemma

*For* $A(\alpha, \beta)$ *restricted with no arbitrary choice sequence variables free except the distinct variables* $\alpha, \beta$*,* **IRLS** *proves*

(i) $\forall\alpha[RLS(\alpha) \rightarrow \forall\beta[RLS([\alpha, \beta]) \rightarrow A(\alpha, \beta)]] \leftrightarrow \forall\gamma[RLS(\gamma) \rightarrow A([\gamma]_0, [\gamma]_1)]$,

(ii) $\forall\alpha[RLS(\alpha) \rightarrow \exists\beta[RLS([\alpha, \beta])\ \&\ A(\alpha, \beta)]] \leftrightarrow$
$\exists e[J_1(e)\ \&\ \forall y(e(y) > 0 \rightarrow \exists w(Seq(w)\ \&\ \forall\gamma[RLS(\gamma) \rightarrow A(y * [\gamma]_0, w * [\gamma]_1)])))]$,

*where in general* $[\gamma]_0(n) = \gamma(2n)$ *and* $[\gamma]_1(n) = \gamma(2n + 1)$*.*

*Proofs.* (i) is immediate from the definitions (note that $\gamma = [\,[\gamma]_0, [\gamma]_1\,]$) and the fact that $RLS([\alpha, \beta]) \rightarrow RLS(\alpha)$ by Lemma 1.9(ii). (ii) follows from RLS2, RLS3 and the closure properties in Lemma 1.9. □

## 2.8 Proof of the translation theorem

*Definition.* The *index* of a restricted formula $E$ of $\mathcal{L}$ is $I(E) = 2i + j + 2k$ where

1. $i$ is the number of restricted existential sequence quantifiers occurring in $E$,
2. $j$ is the number of restricted universal sequence quantifiers occurring in $E$ and
3. $k$ is the maximum number of logical symbols ($\&, \vee, \rightarrow, \neg, \forall, \exists$) occurring in any part $F$ of a subformula of $E$ of any of the forms $\forall\alpha[RLS(\alpha) \rightarrow F]$, $\exists\alpha[RLS(\alpha) \,\&\, F]$, $\forall\alpha_i[RLS([\alpha_0, \ldots, \alpha_k]) \rightarrow F]$ or $\exists\alpha_i[RLS([\alpha_0, \ldots, \alpha_k]) \,\&\, F]$.

If $C \leftrightarrow D$ denotes any restricted $R$-formula of a type displayed in the statement of any of the Lemmas 2.3, 2.4, 2.5, 2.6 or 2.7, inspection shows that $I(C) > I(D)$.

The lemmas permit the reduction of a given restricted $R$-formula $E$ to a formula $F$ of the two-sorted language, with only number and lawlike sequence variables, such that **IRLS** proves $E \leftrightarrow F$. For a uniform translation, the sequence in which the lemmas are to be applied can be determined uniquely (modulo the renaming of variables) by the logical form of $E$, beginning with the leftmost occurrence of a restricted quantifier. The successive reductions produce a sequence $E_0, \ldots, E_q$ of restricted $R$-formulas with $I(E_i) > I(E_{i+1})$, where $E_0$ is $E$ and $I(E_q) = 0$, so we can define $\varphi(E) = E_q$.

There are two wrinkles which are best illustrated by an example. Suppose $E_i$ is a restricted $R$-formula of the form $\exists\alpha[RLS(\alpha) \,\&\, \forall\beta[RLS([\alpha, \beta]) \rightarrow A(\alpha, \beta)]]$. Lemma 2.5 reduces this to $\exists w[Seq(w) \,\&\, \forall\alpha[RLS(\alpha) \rightarrow \forall\beta[RLS([w * \alpha, \beta]) \rightarrow A(w * \alpha, \beta)]]]$, which is not restricted but can be simplified to $\exists w[Seq(w) \,\&\, \forall\alpha[RLS(\alpha) \rightarrow \forall\beta[RLS([\alpha, \beta]) \rightarrow A(w * \alpha, \beta)]]]$ using Lemma 1.9 once. If needed, repeated uses of Lemma 1.9 reduce $A(w * \alpha, \beta)$ to a restricted formula $A'(w, \alpha, \beta)$. Then $E_{i+1}$ is $\exists w[Seq(w) \,\&\, \forall\alpha[RLS(\alpha) \rightarrow \forall\beta[RLS([\alpha, \beta]) \rightarrow A'(w, \alpha, \beta)]]]$, a restricted $R$-formula with $I(E_{i+1}) < I(E_i)$ since $A(\alpha, \beta)$ and $A'(w, \alpha, \beta)$ have the same number of logical symbols.

By Lemma 2.7(i), in the next step $\forall\alpha[RLS(\alpha) \rightarrow \forall\beta[RLS([\alpha, \beta]) \rightarrow A'(w, \alpha, \beta)]]$ is reduced to $\forall\gamma[RLS(\gamma) \rightarrow A'(w, [\gamma]_0, [\gamma]_1)]$, which may not be restricted. But if for example $A'(w, \alpha, \beta)$ is $\forall\delta[RLS([\alpha, \beta, \delta]) \rightarrow B(w, \alpha, \beta, \delta)]$, Lemma 1.9 reduces $A'(w, [\gamma]_0, [\gamma]_1)$ to $\forall\delta[RLS([\gamma, \delta]) \rightarrow B(w, [\gamma]_0, [\gamma]_1, \delta)]$ and eventually to a restricted formula $\forall\delta[RLS([\gamma, \delta]) \rightarrow B'(w, \gamma, \delta)]$. Then $E_{i+2}$ is $\exists w[Seq(w) \,\&\, \forall\gamma[RLS(\gamma) \rightarrow \forall\delta[RLS([\gamma, \delta]) \rightarrow B'(w, \gamma, \delta)]]]$, a restricted $R$-formula with $I(E_{i+2}) < I(E_{i+1})$. $\square$

## 2.9 Final remarks

Evidently the translation will be unique only up to congruence (renaming of bound variables). While technically RLS1$'$ is not restricted, it is equivalent over **IRS** to

$\exists\alpha[RLS(\alpha)\ \&\ 0 = 0]$, which reduces over **IRS** + RLS1 to $\exists w\forall\alpha[RLS(\alpha) \rightarrow 0 = 0]$ and then to $\exists w\forall a(0 = 0)$, which is equivalent in **IR** to $0 = 0$. RLS2′ permits a similar analysis over **IRS** + RLS2.

Note that the intuitionistic system **IRLS** proves $\mathrm{AC}^R_{01}$; and if $\mathrm{AC}^R_{01}$ replaces $\mathrm{AC}^R_{00}!$ then RLS4 becomes provable from the other axioms of the semi-classical system **RLS**. It follows that if **IRS′** comes from **IRS** by strengthening $\mathrm{AC}^R_{00}!$ to $\mathrm{AC}^R_{01}$, then

$$\mathbf{RLS} = \mathbf{IRS'} + \mathrm{RLS'}\ 1\text{-}3 + \mathrm{RLEM}.$$

Evidently **IRLS** is not a conservative extension of **IRS** or even **IRS′**, since $\exists\alpha\forall a\neg\forall x(\alpha(x) = a(x))$ is provable in **IRLS** but not in **IRS′**. Similarly, **RLS** is not a conservative extension of **IRS′** + RLEM. However, **IRS′** is a conservative extension of its two-sorted subsystem **B**. I am indebted to A. S. Troelstra (cf. [6]) for the hint that prompted these observations.

Two questions remain. Is **IRLS** a conservative extension of (two-sorted) constructive analysis **IR**? Is the semi-classical system **RLS** a conservative extension of two-sorted classical analysis **R**? Since **R** proves lawlike countable choice $\mathrm{AC}^R_{01}$ and a lawlike version $\mathrm{BI}!^R$ of the bar induction schema, the translation lemma suggests that **RLS** may be a conservative extension of **R**, but this is only a conjecture.

## References


[1] S. C. Kleene. *Formalized recursive functionals and formalized realizability.* Number 89 in Memoirs. Amer. Math. Soc., 1969.

[2] S. C. Kleene and R. E. Vesley. *The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions.* North Holland, 1965.

[3] G. Kreisel. Lawless sequences of natural numbers. *Comp. Math.*, 20:222–248, 1968.

[4] J. R. Moschovakis. Embedding the constructive and the classical in the intuitionistic continuum. slides from talk, July 2014. Chiemsee summer school in Proof, Truth and Computation.

[5] J. R. Moschovakis. Iterated definability, lawless sequences and Brouwer's continuum. In L. Horsten and P. Welch, editors, *The Limits of Mathematical Knowledge*, page ?? Oxford, 2016.

[6] A. S. Troelstra. An addendum. *Annals of Math. Logic*, 3:437–439, 1971.

[7] A. S. Troelstra. *Choice Sequences, a Chapter of Intuitionistic Mathematics.* Oxford Logic Guides. Clarendon Press, 1977.

[8] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction, Volumes I and II.* North-Holland, 1988.


Received \jreceived