HIERARCHIES IN

INTUITIONISTIC ARITHMETIC

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Gjuletchica, December 2002

Intuitionistic First Order Arithmetic HA

$\mathcal{L}$: individual variables $v_0, v_1, \ldots$ and constant 0; relation constant $=$; function constants S, $+$, $\cdot$.

Definition. 0 and v_i are *terms*. If $\mathbf{s}$ and $\mathbf{t}$ are *terms* so are $\mathbf{s}+\mathbf{t}$ and $\mathbf{s}\cdot\mathbf{t}$. *Prime formulas* are all equations $\mathbf{s}=\mathbf{t}$ where $\mathbf{s},\mathbf{t}$ are terms. If A, B are *formulas* and x is a variable then $A \wedge B$, $A \vee B$, $A \rightarrow B$, $\neg A$, $\forall x A$ and $\exists x A$ are *formulas*.

Axioms: $\neg Sx = 0$, $Sx = Sy \leftrightarrow x = y$,

$$x = y \rightarrow (x = z \rightarrow y = z),$$

equations defining $+$ and $\cdot$ recursively; induction

$$A(0) \;\wedge\; \forall x(A(x) \rightarrow A(Sx)) \rightarrow A(x).$$

Axioms and rules of intuitionistic predicate logic, like classical first order logic but with the axiom

$$\neg A \rightarrow (A \rightarrow B)$$

replacing $\neg\neg A \rightarrow A$ (so **HA** $\nvdash A \vee \neg A$).

Definition. *Markov's Principle* $\mathsf{MP}_{\mathsf{PR}}$ is

$$\neg\forall x\neg R(x) \to \exists x R(x)$$

where $R(x)$ must be primitive recursive.

Definition. *Church's Thesis* CT_0 is

$$\forall x\exists y A(x,y) \to \exists e\forall x\exists w[T(e,x,w) \wedge A(x,U(w))],$$

where $T(e,x,w)$ expresses "w is the least Gödel number of a computation of a value for $\{e\}(x)$" and $U(w)$ is that value.

Theorem.(Nelson) **HA** + $\mathsf{MP}_{\mathsf{PR}}$ + CT_0 is consistent.

Definition. *Classical Peano Arithmetic* is

$$\textbf{PA} \equiv_{df} \textbf{HA} + (\neg\neg A \to A).$$

Proposition. **PA** $\vdash$ $\mathsf{MP}_{\mathsf{PR}}$.

Proposition. **PA** + CT_0 is inconsistent. Hence **HA** + $\mathsf{MP}_{\mathsf{PR}}$ $\nvdash A \vee \neg A$.

The Standard Arithmetical Hierarchy

Definition. The levels Π^0_n, Σ^0_n and Δ^0_n of the *arithmetical hierarchy* are defined as follows.

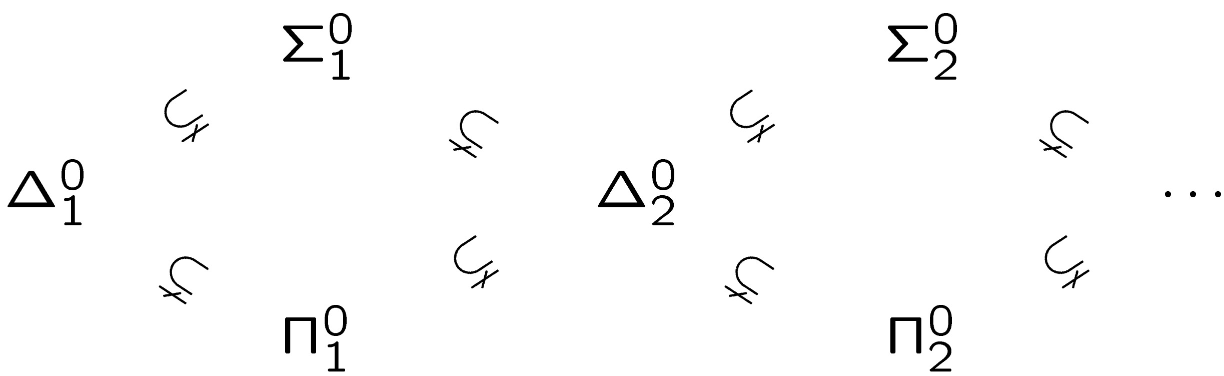
A relation $R(x)$ is Π^0_1 if and only if $R(x)$ is expressible in the form $\forall y P(x,y)$ where $P(x.y)$ is recursive; Σ^0_1 if and only if $R(x)$ can be expressed in the form $\exists y Q(x,y)$ where $Q(x,y)$ is recursive; Δ^0_1 if and only if $R(x)$ is both Σ^0_1 and Π^0_1.

For $n > 1$, a relation $R(x)$ is Π^0_n if and only if it can be expressed in the form $\forall y P(x,y)$ where $P(x,y)$ is Σ^0_{n-1}; $R(x)$ is Σ^0_n if and only if it is expressible as $\exists y Q(x,y)$ where $Q(x,y)$ is Π^0_{n-1}; and for all $n > 0$:

$$\Delta^0_n = \Pi^0_n \cap \Sigma^0_n.$$

Proposition. In **HA** + $\mathrm{MP}_{\mathrm{PR}}$ and in **PA**: Every Δ^0_1 relation is recursive, and conversely, every recursive relation is Δ^0_1.

Proposition. In **PA** every relation $R(x)$ belongs to some level of the standard arithmetical hierarchy. Moreover, each level is different:



Proposition. In **HA** + CT_0, the arithmetical hierarchy collapses at Σ^0_3.

Proof. In **HA**: $\Sigma^0_n \cup \Pi^0_n \subsetneq \Delta^0_{n+1}$ for every n, and adjacent quantifiers of the same kind can be contracted by primitive recursive pairing. So it is enough to show that in **HA** + CT_0:

(i) $\Pi^0_3 \subseteq \Sigma^0_3$, and

(ii) $\Pi^0_4 \subseteq \Sigma^0_3$.

For (i) observe that in **HA** + CT_0 the Π^0_3 relation $\forall y \exists z \forall w P(x,y,z,w)$ is equivalent to each of the following:

(a) $\exists e \forall y \exists z [T(e,x,y,z) \wedge \forall w P(x,y,U(z),w)]$,

(b) $\exists e [\forall y \exists z T(e,x,y,z)$

$$\wedge \forall y \forall z [T(e,x,y,z) \rightarrow \forall w P(x,y,U(z),w)]],$$

(c) $\exists e \forall y [\exists z T(e,x,y,z)$

$$\wedge \forall z \forall w [T(e,x,y,z) \rightarrow P(x,y,U(z),w)]],$$

(d) $\exists e \forall y \forall z \forall w \exists v [T(e,x,y,v)$

$$\wedge [T(e,x,y,z) \rightarrow P(x,y,U(z),w)]].$$

Since $T(e,x,y,v)$ and $P(x,y,U(z),w)$ are primitive recursive, after contracting like quantifiers (d) will be Σ^0_3.

The proof of (ii) is similar.

Definition. An arithmetical theory $\mathcal{T}$ is closed under **Kleene's Rule** if, whenever $\forall x \exists y A(x,y)$ is closed and $\mathcal{T} \vdash \forall x \exists y A(x,y)$, then for some number e:

$$\mathcal{T} \vdash \forall x \exists y [T(\mathbf{e}, x, y) \wedge A(x, U(y))].$$

Theorem.(Kleene) If $\mathcal{T}$ is **HA**, **HA** + $\mathrm{MP}_{\mathrm{PR}}$, **HA** + CT_0 or **HA** + $\mathrm{MP}_{\mathrm{PR}}$ + CT_0, then $\mathcal{T}$ is closed under Kleene's Rule.

Definition. A formula $A(x)$ is *decidable in* a theory $\mathcal{T}$ if

$$\mathcal{T} \vdash \forall x [A(x) \vee \neg A(x)].$$

Similary for $A(x_1, \ldots, x_n)$.

Proposition. If $\mathcal{T}$ contains **HA** and is closed under Kleene's Rule, then $A(x_1, \ldots, x_n)$ is decidable in $\mathcal{T}$ if and only if $A(x_1, \ldots, x_n)$ is recursive, provably in $\mathcal{T}$.

Definition. A formula $A(x)$ is *stable in* a theory $\mathcal{T}$ if

$$\mathcal{T} \vdash \forall x[\neg\neg A(x) \rightarrow A(x)].$$

Similarly for $A(x_1, \ldots, x_n)$.

Remark. Decidability implies stability, but not conversely. For example, every Π^0_1 relation is stable in **HA** because every recursive relation is stable and $\neg\neg\forall x A(x) \rightarrow \forall x \neg\neg A(x)$ holds in intuitionistic logic. But the Π^0_1 relation $\forall y \neg T(x, x, y)$ is not recursive, and hence not decidable in **HA** or even in **HA** + $\mathsf{MP}_{\mathsf{PR}}$ + CT_0.

Note. Even when there is (classically) a recursive decision procedure, we may not know what it is. For example, consider

$$B(x) \equiv \forall y[y > x \;\wedge\; Pr(y) \rightarrow \neg Pr(y + 2)]$$

where $Pr(y)$ expresses "y is prime." $B(x)$ cannot be nonrecursive. Its Gödel number is ?

Definition. The *classical quantifiers* $\dot{\exists}$, $\dot{\forall}$ are

$$\dot{\exists} \equiv_{Df} \neg\neg\exists \quad \text{and} \quad \dot{\forall} \equiv_{Df} \forall\neg\neg.$$

Note. **HA** $\vdash \dot{\exists}xA(x) \leftrightarrow \neg\forall x\neg A(x) \leftrightarrow \neg\neg\dot{\exists}xA(x)$ and **HA** $\vdash \dot{\forall}xA(x) \leftrightarrow \neg\exists x\neg A(x) \leftrightarrow \neg\neg\dot{\forall}xA(x)$.

Definition. The levels of the *classical arithmetical hierarchy* are defined using the classical quantifiers. A relation $R(x)$ is $\dot{\Pi}^0_1$ if it is expressible as $\dot{\forall}yP(x,y)$ for some recursive $P(x,y)$; $R(x)$ is $\dot{\Sigma}^0_1$ if it is $\dot{\exists}yP(x,y)$ for some recursive $P(x,y)$. For $n > 1$: $R(x)$ is $\dot{\Pi}^0_n$ if it can be expressed as $\dot{\forall}yP(x,y)$ where $P(x,y)$ is $\dot{\Sigma}^0_{n-1}$; $R(x)$ is $\dot{\Sigma}^0_n$ if it is expressible as $\dot{\exists}yQ(x,y)$ where $Q(x,y)$ is $\dot{\Pi}^0_{n-1}$. For all $n > 0$:

$$\dot{\Delta}^0_n = \dot{\Pi}^0_n \cap \dot{\Sigma}^0_n.$$

Proposition. In **HA**, $\dot{\Pi}^0_1 = \Pi^0_1$. In **HA** + $\mathrm{MP}_{\mathrm{PR}}$ also $\dot{\Sigma}^0_1 = \Sigma^0_1$, hence $\dot{\Delta}^0_1 = \Delta^0_1$ (= recursive) and $\dot{\Pi}^0_2 = \Pi^0_2$.

Proposition. Every relation which is $\dot{\Sigma}^0_n$ or $\dot{\Pi}^0_n$ for *any* $n > 0$ is stable.

Proposition. In **HA**, and in every consistent extension of **HA** (including **HA** + $\mathsf{MP}_{\mathsf{PR}}$ + CT_0), every level of the classical arithmetical hierarchy contains new relations.

$$\begin{array}{ccccccccc}
 & & \dot\Sigma^0_1 & & & & \dot\Sigma^0_2 & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
\dot\Delta^0_1 & & & & \dot\Delta^0_2 & & & & \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
 & & \dot\Pi^0_1 & & & & \dot\Pi^0_2 & &
\end{array}$$

Proof. Replace the quantifiers in the complete Π^0_n and complete Σ^0_n relations, given by Kleene's normal form theorem for **PA**, by classical quantifiers to get stable, complete $\dot\Pi^0_n$ and $\dot\Sigma^0_n$ relations for **HA**. The classical diagonal arguments by contradiction work because of stability.

Remark. Classically, there is no difference between this hierarchy and the standard arithmetical hierarchy. Intuitionistically they are very different, and neither contains all relations.

Proposition. The relation

$$C(x) \equiv_{Df} \exists y \forall z \neg T(x,y,z) \vee \neg \exists y \forall z \neg T(x,y,z)$$

is not stable in **HA** or in any consistent extension $\mathcal{T}$ of **HA** satisfying Kleene's Rule. So $C(x)$ is not in the classical arithmetical hierarchy.

Proof. If it were, since **HA** $\vdash \forall x \neg\neg C(x)$ we would have $\mathcal{T} \vdash \forall x C(x)$ so there would be a recursive decision procedure for $\exists y \forall z \neg T(x,y,z)$, which is impossible.

Question. Is there a consistent extension of **HA** which satisfies Kleene's Rule, and admits some sort of *total* arithmetical hierarchy which does not collapse?

Answer. Yes. First let $\mathbf{HA}^\bullet \equiv_{Df} \mathbf{HA} + \mathsf{MP}_{\mathsf{PR}}$. In $\mathbf{HA}^\bullet$, every Π^0_2 relation is stable, and

$$\dot{\Pi}^0_1 = \Pi^0_1, \quad \dot{\Sigma}^0_1 = \Sigma^0_1, \quad \dot{\Pi}^0_2 = \Pi^0_2.$$

Even in **HA**: $\wedge$, $\rightarrow$, $\neg$ and $\forall$ preserve stability.

Definition. The *classical extension of Church's Thesis* $\mathrm{ECT}^{\bullet}$ is

$$\forall x[A(x) \to \exists y B(x,y)] \\ \to \exists e \forall x[A(x) \to \exists w[T(e,x,w) \wedge B(x, U(w))]],$$

for any *classical* $A(x)$ (belonging to the classical arithmetical hierarchy).

Theorem. (essentially Troelstra) The theory $\mathbf{HA}^{\bullet} + \mathrm{ECT}^{\bullet}$ is consistent and obeys Kleene's Rule. Moreover, every relation $R(x)$ has a corresponding classical relation $R^{\bullet}(x,y)$ such that

(i) $\mathbf{HA}^{\bullet} + \mathrm{ECT}^{\bullet} \vdash \forall x[R(x) \leftrightarrow \exists y R^{\bullet}(x,y)]$.

(ii) $\mathbf{HA}^{\bullet} + \mathrm{ECT}^{\bullet} \vdash R(\mathbf{t}) \Leftrightarrow \mathbf{HA}^{\bullet} \vdash \exists y R^{\bullet}(\mathbf{t}, y)$.
($\mathbf{t}$ is any term free for x in $\exists y R^{\bullet}(x,y)$.)

Remark. In $\mathbf{HA}^{\bullet} + \mathrm{ECT}^{\bullet}$, every stable relation is classical, since $\neg\neg\exists y R^{\bullet}(x,y)$ is classical.

The Extended Intuitionistic Hierarchy

Definition. The *extended intuitionistic arithmetical hierarchy* is defined as follows for $n \geq 1$: The relation $R(x)$ is $\Sigma^0(\dot{\Sigma}^0_n)$ if and only if $R(x)$ is expressible as $\exists y B(x,y)$ where $B(x,y)$ is $\dot{\Sigma}^0_n$; and $R(x)$ is $\Sigma^0(\dot{\Pi}^0_n)$ if and only if it can be expressed as $\exists y B(x,y)$ where $B(x,y)$ is $\dot{\Pi}^0_n$.

Proposition. In **HA** (or any consistent extension of **HA**), for every $n \geq 1$: $\Sigma^0(\dot{\Pi}^0_n) \not\subseteq \Sigma^0(\dot{\Sigma}^0_n)$.

Proof. ($n = 1$) Let $R(x)$ be $\exists y \dot{\forall} z \neg T(x,x,y,z)$. If $R(x) \leftrightarrow \exists u \dot{\exists} v Q(x,u,v)$ with a recursive $Q(x,u,v)$, then using primitive recursive pairing and projection with intuitionistic logic,

$$\begin{aligned} \neg\neg R(x) &\leftrightarrow \dot{\exists} u \dot{\exists} v Q(x,u,v) \\ &\leftrightarrow \dot{\exists} w Q(x,(w)_0,(w)_1). \end{aligned}$$

But $\dot{\exists} w Q(x,(w)_0,(w)_1)$ is $\dot{\Sigma}^0_1$, while $\neg\neg R(x)$ is complete $\dot{\Sigma}^0_2$. The proof for $n > 1$ is similar.

Proposition. In $\mathbf{HA}^\bullet + \mathsf{ECT}^\bullet$, for every $n \geq 2$: $\Sigma^0(\dot{\Sigma}^0_n) \not\subseteq \Sigma^0(\dot{\Pi}^0_n)$.

Proof.($n = 2$) Let $D(x, y) \equiv \dot{\exists} z \dot{\forall} w \neg T(x, x, y, z, w)$ and $\mathcal{T}$ be $\mathbf{HA}^\bullet + \mathsf{ECT}^\bullet$. Suppose for contradiction that $\mathcal{T} \vdash \forall x[\exists y D(x, y) \leftrightarrow \exists u \dot{\forall} v \dot{\exists} t Q(x, u, v, t)]$ with $Q(x, u, v, t)$ primitive recursive, so also $\mathcal{T} \vdash$

(a) $\forall x \forall y[D(x, y) \rightarrow \exists u \forall v \dot{\exists} t Q(x, u, v, t)]$,

(b) $\exists e \forall x \forall y[D(x, y) \rightarrow [\exists u T(e, x, y, u)$

$\wedge\ \forall w(T(e, x, y, w) \rightarrow \forall v \dot{\exists} t Q(x, U(w), v, t))]]$,

or equivalently by $\mathsf{MP}_{\mathsf{PR}}$:

(c) $\exists e \forall x \forall y[D(x, y) \rightarrow [\dot{\exists} u T(e, x, y, u)$

$\wedge\ \forall w(T(e, x, y, w) \rightarrow \forall v \dot{\exists} t Q(x, U(w), v, t))]]$.

By Kleene's Rule there is a number e so that

(d) $\forall x \forall y[D(x, y) \rightarrow [\dot{\exists} u T(\mathbf{e}, x, y, u)$

$\wedge\ \forall w(T(\mathbf{e}, x, y, w) \rightarrow \forall v \dot{\exists} t Q(x, U(w), v, t))]]$

where the right hand side is $\dot{\Pi}^0_2$, so for some g by the normal form theorem with Kleene's Rule:

(e) $\forall x \forall y [D(x,y) \rightarrow \forall z \dot{\exists} w T(\mathbf{g}, x, y, z, w)]$ where

(f) $\forall x \forall y [\forall z \dot{\exists} w T(\mathbf{g}, x, y, z, w) \leftrightarrow$

$$\forall w \forall v \dot{\exists} u \dot{\exists} t [T(\mathbf{e}, x, y, u) \wedge$$

$$(T(\mathbf{e}, x, y, w) \rightarrow Q(x, U(w), v, t))]].$$

By (e) with the definition of $D(x,y)$:

(g) $\forall y \neg D(\mathbf{g}, y)$ and so $\forall z \dot{\exists} w T(\mathbf{g}, \mathbf{g}, y, z, w)$.

Treating $\dot{\exists} u$ first on the right hand side of (c),

(h) $\forall x \forall y [\forall z \dot{\exists} w T(\mathbf{g}, x, y, z, w) \leftrightarrow$

$$\dot{\exists} u [T(\mathbf{e}, x, y, u) \wedge \forall v \dot{\exists} t Q(x, U(u), v, t)]]$$

so (g) gives $\dot{\exists} u [T(\mathbf{e}, \mathbf{g}, y, u) \wedge \forall v \dot{\exists} t Q(\mathbf{g}, U(u), v, t)]$, so $\dot{\exists} y D(\mathbf{g}, y)$ by hypothesis, contradicting (g).

The proof for $n > 2$ is similar. Thus the classical arithmetical hierarchy does not collapse in any consistent extension of $\mathbf{HA}^\bullet + \mathrm{ECT}^\bullet$ satisfying Kleene's Rule.

Definition. A formula $A(x)$ is (or describes) a *Church domain* for a theory $\mathcal{T}$ if, whenever

$$\mathcal{T} \vdash \forall x[A(x) \rightarrow \exists y B(x,y)],$$

then $\mathcal{T} \vdash \exists e \forall x[A(x) \rightarrow \exists w[T(e,x,w) \wedge B(x,U(w)]]$.

Theorem. In $\mathbf{HA}^\bullet + \mathrm{ECT}^\bullet$:

(a) The extended intuitionistic hierarchy is total, and each level contains new relations.

(b) $\Sigma^0(\dot{\Sigma}^0_1) = \Sigma^0_1$.

(c) $\Sigma^0(\dot{\Pi}^0_1) = \Sigma^0_2$.

(d) $\Sigma(\dot{\Pi}^0_2) = \Sigma^0_3$ (so $\Sigma(\dot{\Pi}^0_2)$ contains the entire standard arithmetical hierarchy).

(e) $\dot{\Sigma}^0_n \subsetneq \Sigma^0(\dot{\Sigma}^0_n)$ and $\dot{\Pi}^0_n \subsetneq \Sigma^0(\dot{\Pi}^0_n)$, so the extended intuitionistic hierarchy subsumes the classical hierarchy.

(f) Every Church domain is classical.