INTUITIONISTIC ANALYSIS AT THE END OF TIME

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Abstract. Kleene and Vesley's formal system **I** of intuitionistic analysis, with countable choice and a classically false continuity principle, is consistent by [3]. "There are no non-recursive sequences" is consistent with **I** by [4]. Kreisel's "lawless" sequences inspired a model of intuitionistic Baire space as a forcing expansion of a countable ω-model of classical Baire space ([5]). Now Kripke has suggested, as a way of understanding Brouwer's creating subject arguments, that the intuitionistic continuum may be viewed as an expansion *in time* of the classical continuum.

Let $\mathbf{C}^{\circ}$ be the negative interpretation of classical analysis with the axiom of countable choice. Using a classical ω-model of $\mathbf{C}^{\circ}$ we prove that a three-sorted common extension of **I** and $\mathbf{C}^{\circ}$, with an "end of time" axiom asserting that there are no non-classical sequences, is consistent.

1. Introduction

L. E. J. Brouwer accepted Kant's dictum that the intuition of time is *a priori* and lies at the base of all mathematical reasoning. The intuitionistic continuum is composed of *point cores* or equivalence classes of convergent sequences of rational segments or rational numbers. The *reduced continuum* consists of *finished* ("fundamental" or "lawlike") sequences, all of whose values are determined in advance. The *full continuum* includes point cores determined by *unfinished* convergent sequences whose rational values are generated by successive, more or less free, choices.

Abstracting from the continuum to the "universal spread," an intuitionistic version of Baire space, an arbitrary choice sequence α of natural numbers is *potentially infinite*; at any given time, only a finite initial segment of α may have been determined. This intuition justifies (monotone) bar induction and Brouwer's controversial continuity principles. In contrast, as Troelstra observed in [6], the completely determined lawlike sequences could be assumed to obey classical logic.

By [4] Kleene and Vesley's formal system **I** of intuitionistic analysis is consistent with "there are no non-recursive sequences" (but not with "every sequence is recursive"). In [5] we represented the intuitionistic continuum classically as a forcing expansion of a countable ω-model of Brouwer's reduced continuum.

Classical analysis **C**, including the countable axiom of choice, is classically equivalent to its negative translation $\mathbf{C}^{\circ}$. Negative formulas (without $\exists$ or $\vee$) are stable under double negation even with intuitionistic logic. Let $\mathcal{M} = (\omega, \mathcal{C})$ be an ω-model of $\mathbf{C}^{\circ}$. We show by (modified) ${}^{\mathcal{C}}$realizability that **I** and $\mathbf{C}^{\circ}$ are consistent with "there are no non-classical sequences," and with "not every sequence is classical" provided $\mathcal{C} \neq \omega^{\omega}$.

On December 9, 2016, in Amsterdam, Saul Kripke gave a lecture in which he suggested that the intuitionistic continuum could be understood as an expansion, in time, of the classical continuum, depending on the actions of a creating subject. I thank him warmly for the inspiration. I also thank Yiannis Moschovakis for many helpful comments on style.

2. Just the basics

For Brouwer a statement A was in general stronger than its double negation $\neg\neg$A, since intuitionistic negation expresses inconsistency. Thus (A $\rightarrow \neg\neg$A) holds in general, as does ($\neg\neg\neg$A $\rightarrow \neg$A), but not always ($\neg\neg$A $\rightarrow$ A). Even $\neg\neg(\mathrm{A} \vee \mathrm{B}) \rightarrow \neg\neg\mathrm{A} \vee \neg\neg\mathrm{B}$ fails under the constructive interpretation of disjunction; and while $\exists$xA(x) asserts that a witness can be designated, $\neg\neg\exists x A(x)$ says only that $\forall$x$\neg$A(x) is inconsistent.

Classical logic, on the other hand, can be formulated in a negative language with only $\&, \neg, \rightarrow$ and $\forall$, since A $\vee$ B and $\exists$xA(x) are classically equivalent to $\neg(\neg\mathrm{A} \,\&\, \neg\mathrm{B})$ and $\neg\forall$x$\neg$A(x) respectively. The language of classical analysis $\mathbf{C}^\circ$ has two sorts of variables: $\mathrm{i, j, \ldots, p, q, w, x, y, z, i_1, \ldots}$ intended to range over natural numbers, and $\mathrm{a, b, c, d, e, a_1, b_1, c_1, \ldots}$ intended to range over sequences of natural numbers; constants for primitive recursive functions; Church's λ; parentheses, used both to denote function application and also to indicate the scopes of $\&, \neg, \rightarrow$, $\forall$x and $\forall$b in formulas; and equality = between number terms. For ease of reading we sometimes abbreviate

$$\neg(\neg\mathrm{A} \,\&\, \neg\mathrm{B}) \text{ by } \mathrm{A} \overset{\circ}{\vee} \mathrm{B}, \quad \neg\forall\mathrm{x}\neg\mathrm{A(x)} \text{ by } \exists^\circ\mathrm{xA(x)}, \quad \text{and } \neg\forall\mathrm{b}\neg\mathrm{A(b)} \text{ by } \exists^\circ\mathrm{bA(b)}.$$

The Peano axioms are negative in form when the schema of mathematical induction is restricted to formulas of the negative language. The equality axiom $\mathrm{x = y \rightarrow b(x) = b(y)}$ is negative. Primitive recursive functions have negative definitions. The axiom of countable choice can be represented by its negative translation. *Even with intuitionistic logic* the classical law of double negation $\neg\neg$E $\rightarrow$ E holds for formulas E of this language.

The three-sorted axiomatic system **IC** combines Kleene and Vesley's intuitionistic formal system **I**, which has variables $\alpha, \beta, \gamma, \ldots$ ranging over arbitrary choice sequences, with the intuitionistic system resulting from $\mathbf{C}^\circ$ by extending its language and (intuitionistic) logic to include $\vee$ and $\exists$. The only new axiom explicitly connecting the two sorts of sequence variables is $\forall\alpha\neg\forall\mathrm{b}\neg\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$, or equivalently

$$\forall\alpha\neg\neg\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}.$$

The idea is that when mathematical activity has ended and all values of an arbitrary choice sequence α have been specified, *it will turn out* that α coincides with some classical "lawlike" (completely determined) sequence.[1] This correlation cannot be made in advance, as **IC** does not prove $\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$. However, **IC** does prove

$$\forall\mathrm{b}\exists\alpha\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}.$$

Thus every classical lawlike sequence is extensionally equal to a choice sequence, and "at the end of time" intuitionistic and classical Baire space will be indistinguishable.

In order to establish the consistency of **IC** we assume a classical ω-model $\mathcal{M} = (\omega, \mathcal{C})$ of $\mathbf{C}^\circ$ exists. For our modified realizability the ${}^{\mathcal{C}}$realizing objects will belong to the recursively closed set $\mathcal{C}$. Lawlike sequence variables are interpreted by elements of $\mathcal{C}$. All theorems of **IC** are ${}^{\mathcal{C}}$realizable but $0 = 1$ is not, so **IC** is consistent.

Kleene observed (Lemma 8.4a of [3]) that true negative sentences of the language of **I** have primitive recursive realizers. All sentences of the language of $\mathbf{C}^\circ$ which are true in $\mathcal{M}$ are ${}^{\mathcal{C}}$realized by primitive recursive functions, and thus are consistent with **IC**.

[1]This addresses an objection, from a member of the audience after Kripke's talk in Amsterdam, to the effect that the classical continuum is already complete.

3. The formal systems $\mathbf{C}^\circ$, $\mathbf{B}$, $\mathbf{I}$ and $\mathbf{IC}$

3.1. **A negative formal system $\mathbf{C}^\circ$ for classical analysis with countable choice.** The two-sorted language $\mathcal{L}(\mathbf{C}^\circ)$ was described briefly in the preceding section. Now we adopt Kleene's finite list $\mathrm{f}_0, \ldots, \mathrm{f}_p$ of constants representing selected primitive recursive functions, with $\mathrm{f}_0 = 0, \mathrm{f}_1 = {}', \mathrm{f}_2 = +, \mathrm{f}_3 = \cdot$ and $\mathrm{f}_4(\mathrm{x}, \mathrm{y}) = \mathrm{x}^{\mathrm{y}}$. The list, including bounded sum and bounded product, may be expanded by definition as needed.

C°-*terms* (type-0 terms) and C°-*functors* (type-1 terms) are defined inductively. The number variables and the constant 0 are C°-*terms*. The lawlike sequence variables, the successor symbol $'$ and constants representing primitive recursive functions of one type-0 argument are C°-*functors*. If f_i is a constant representing a primitive recursive function of k_i type-0 and m_i type-1 variables, and if $\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}$ are C°-*terms* and $\mathrm{u}_1, \ldots, \mathrm{u}_{m_i}$ are C°-*functors*, then $\mathrm{f}_\mathrm{i}(\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}, \mathrm{u}_1, \ldots, \mathrm{u}_{m_i})$ is a C°-*term*. If u is a C°-*functor* and t is a C°-*term* then (u)(t) (sometimes written u(t)) is a C°-*term*. If x is a number variable and s is a C°-*term* then λx(s) (sometimes written λx.s) is a C°-*functor*. This completes the definition.

The *prime formulas* are the expressions of the form $\mathrm{s} = \mathrm{t}$ where s, t are C°-*terms*. Equality at type 1 is defined extensionally, with $\mathrm{a} = \mathrm{b}$ abbreviating $\forall\mathrm{x}(\mathrm{a}(\mathrm{x}) = \mathrm{b}(\mathrm{x}))$. *Compound formulas* are built from prime formulas and both sorts of variables using $\&, \neg, \rightarrow, \forall$ and parentheses as usual. $(\mathrm{A} \leftrightarrow \mathrm{B})$ abbreviates $(\mathrm{A} \rightarrow \mathrm{B}) \,\&\, (\mathrm{B} \rightarrow \mathrm{A})$. All formulas of $\mathcal{L}(\mathbf{C}^\circ)$ are *negative* (they contain neither $\vee$ nor $\exists$).

The logical axioms and rules are Kleene's ([1], [3]) adapted to $\mathcal{L}(\mathbf{C}^\circ)$, so A, B, C, A(x) and A(b) are negative formulas. We retain Kleene's numbers for comparison.

1a. $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{A})$.
1b. $(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow ((\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{C})) \rightarrow (\mathrm{A} \rightarrow \mathrm{C}))$.
2. (Modus Ponens) $\mathrm{A},\ \mathrm{A} \rightarrow \mathrm{B}\ /\ \mathrm{B}$.
3. $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{A} \,\&\, \mathrm{B})$.
4a. $\mathrm{A} \,\&\, \mathrm{B} \rightarrow \mathrm{A}$.
4b. $\mathrm{A} \,\&\, \mathrm{B} \rightarrow \mathrm{B}$.
7. $(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow ((\mathrm{A} \rightarrow \neg\mathrm{B}) \rightarrow \neg\mathrm{A})$.
8^{I}. $\neg\mathrm{A} \rightarrow (\mathrm{A} \rightarrow \mathrm{B})$.
9N. $\mathrm{B} \rightarrow \mathrm{A}(\mathrm{x})\ /\ \mathrm{B} \rightarrow \forall\mathrm{x}\mathrm{A}(\mathrm{x})$, where x is not free in B.
10N. $\forall\mathrm{x}\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{t})$, where t is a C°-term free for x in A(x).
$9\mathrm{C}^\circ$. $\mathrm{B} \rightarrow \mathrm{A}(\mathrm{b})\ /\ \mathrm{B} \rightarrow \forall\mathrm{b}\mathrm{A}(\mathrm{b})$, where b is not free in B.
$10\mathrm{C}^\circ$. $\forall\mathrm{b}\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{A}(\mathrm{u})$, where u is a C°-functor free for b in A(b).

Mathematical axioms assert that $=$ is an equivalence relation, 0 is not a successor, $'$ is one-to-one, and $\mathrm{x} = \mathrm{y} \rightarrow \mathrm{a}(\mathrm{x}) = \mathrm{a}(\mathrm{y})$. The primitive recursive defining equations for $+, \cdot$ and $\mathrm{f}_4, \ldots, \mathrm{f}_p$ (Postulate Group D of [3], [2]) are axioms, as is the mathematical induction schema $\mathrm{A}(0) \,\&\, \forall\mathrm{x}(\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{x}')) \rightarrow \mathrm{A}(\mathrm{x})$. for formulas A(x) of $\mathcal{L}(\mathbf{C}^\circ)$. For C°-terms r(x), t the λ-reduction schema is

$$(\lambda\mathrm{x}.\mathrm{r}(\mathrm{x}))(\mathrm{t}) = \mathrm{r}(\mathrm{t}),$$

where r(t) results by substituting t for all free occurrences of x in r(x). The axiom schema of countable choice, for formulas A(x, b) of $\mathcal{L}(\mathbf{C}^\circ)$ with a, x free for b, is

$$\mathrm{AC}_{01}^{\mathrm{C}^\circ}. \quad \forall\mathrm{x}\neg\forall\mathrm{b}\neg\mathrm{A}(\mathrm{x}, \mathrm{b}) \rightarrow \neg\forall\mathrm{a}\neg\forall\mathrm{x}\mathrm{A}(\mathrm{x}, \lambda\mathrm{y}.\mathrm{a}(2^{\mathrm{x}} \cdot 3^{\mathrm{y}})).$$

3.2. **Properties of $\mathbf{C}^\circ$.** To avoid unnecessary formal reasoning, first observe that the Deduction Theorem (Theorem 1 on p. 97 of [1]) holds for $\mathbf{C}^\circ$ (using the same arguments for the relevant cases), so the Hilbert-style logical axioms and rules can be replaced by natural deduction rules for $\to$, $\&$, $\neg$ and $\forall$ (as in Theorem 2 on pp. 98-99 of [1]).

3.2.1. *Lemma.* For all C°-terms s, t and all formulas A, B of $\mathcal{L}(\mathbf{C}^\circ)$, $\mathbf{C}^\circ$ proves

(a) $\neg\neg \mathrm{s} = \mathrm{t} \to \mathrm{s} = \mathrm{t}$.
(b) $\mathrm{A} \to \mathrm{A}$.
(c) $\mathrm{A} \to \neg\neg\mathrm{A}$.
(d) $(\mathrm{A} \to \mathrm{B}) \to (\neg\mathrm{B} \to \neg\mathrm{A})$.
(e) $\neg\neg\mathrm{A} \to \mathrm{A}$.

Proofs. (a) follows (by $\forall$-introduction and then $\forall$-elimination) from $\neg\neg \mathrm{x} = \mathrm{y} \to \mathrm{x} = \mathrm{y}$ which is provable by double mathematical induction. (b) - (d) are exercises in negative propositional logic. (e) is by formula induction from the axioms and (a), (c) and (d).

Note that (e) is Kleene's classical negation-elimination axiom schema 8°, restricted in this case to negative formulas. All the logical postulates which were omitted because they contain $\vee$ or $\exists$ have negative versions provable in $\mathbf{C}^\circ$.

3.2.2. *Lemma.* For all formulas $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{A}(\mathrm{x}), \mathrm{A}(\mathrm{b})$ of $\mathcal{L}(\mathbf{C}^\circ)$, $\mathbf{C}^\circ$ proves

5a°. $\mathrm{A} \to \mathrm{A} \overset{\circ}{\vee} \mathrm{B}$.
5b°. $\mathrm{B} \to \mathrm{A} \overset{\circ}{\vee} \mathrm{B}$.
6°. $(\mathrm{A} \to \mathrm{C}) \to ((\mathrm{B} \to \mathrm{C}) \to (\mathrm{A} \overset{\circ}{\vee} \mathrm{B} \to \mathrm{C}))$.
11N°. $\mathrm{A}(\mathrm{t}) \to \exists^\circ \mathrm{x}\mathrm{A}(\mathrm{x})$ if t is a C°-term free for x in $\mathrm{A}(\mathrm{x})$.
11C°. $\mathrm{A}(\mathrm{u}) \to \exists^\circ \mathrm{b}\mathrm{A}(\mathrm{b})$ if u is a C°-functor free for b in $\mathrm{A}(\mathrm{b})$.

Moreover, for all formulas $\mathrm{A}(\mathrm{x}), \mathrm{A}(\mathrm{b}), \mathrm{B}$ of $\mathcal{L}(\mathbf{C}^\circ)$, $\mathbf{C}^\circ$ obeys the rules

12N°. $\mathrm{A}(\mathrm{x}) \to \mathrm{B}\ /\ \exists^\circ \mathrm{x}\mathrm{A}(\mathrm{x}) \to \mathrm{B}$, where x is not free in B and x is held constant in the deduction of $\mathrm{A}(\mathrm{x}) \to \mathrm{B}$.
12C°. $\mathrm{A}(\mathrm{b}) \to \mathrm{B}\ /\ \exists^\circ \mathrm{b}\mathrm{A}(\mathrm{b}) \to \mathrm{B}$, where b is not free in B and b is held constant in the deduction of $\mathrm{A}(\mathrm{b}) \to \mathrm{B}$.

Proofs. 5a° follows from an instance $\neg\mathrm{A} \,\&\, \neg\mathrm{B} \to \neg\mathrm{A}$ of axiom 4a by Lemma 3.2.1(c,d) and 5b° follows from an instance of axiom 4b. For 6°, assume $\mathrm{A} \to \mathrm{C}$ and $\mathrm{B} \to \mathrm{C}$; then $\neg\mathrm{C} \to \neg\mathrm{A}$ and $\neg\mathrm{C} \to \neg\mathrm{B}$, so $\neg\mathrm{C} \to \neg\mathrm{A} \,\&\, \neg\mathrm{B}$ using axiom 3, so $\neg(\neg\mathrm{A} \,\&\, \neg\mathrm{B}) \to \mathrm{C}$ by Lemma 3.2.1(d,e). Similarly, 12N° follows from 9N, and 12C° follows from 9C°.

3.3. **Kleene's intuitionistic formal systems B and I.** The neutral basic system $\mathbf{B}$ has axioms for two-sorted intuitionistic logic and arithmetic, countable choice and bar induction. Intuitionistic analysis $\mathbf{I}$ is $\mathbf{B}$ together with Brouwer's classically false principle of continuous choice, which is consistent relative to $\mathbf{B}$ by function-realizability.

The language resembles a richer version of $\mathcal{L}(\mathbf{C}^\circ)$. Instead of variables $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{a}_1, \ldots$ over classical sequences, $\mathcal{L}(\mathbf{B})$ ($\equiv \mathcal{L}(\mathbf{I})$) has variables $\alpha, \beta, \gamma, \delta, \alpha_1, \ldots$ intended to range over arbitrary choice sequences. In addition to $=, \lambda$, parentheses and the logical symbols $\&, \neg, \to$ and universal quantifiers $\forall \mathrm{x}, \forall\alpha$, $\mathcal{L}(\mathbf{B})$ has disjunction $\vee$ and existential quantifiers $\exists \mathrm{x}, \exists\alpha$ of both sorts. With the same constants $\mathrm{f}_0, \ldots, \mathrm{f}_p$ representing the same primitive recursive functions, the inductive definition of *term* and *functor* is like that of C°-term and C°-functor but with $\alpha, \beta, \ldots$ in place of $a, b, \ldots$.

Prime formulas are expressions of the form $s = t$ where s, t are terms. Compound formulas are built from prime formulas and both sorts of variables using $\&, \neg, \rightarrow, \vee, \forall, \exists$ and parentheses as needed. $\alpha = \beta$ abbreviates the negative formula $\forall x(\alpha(x) = \beta(x))$.

The logical rules and axioms include 1a - 8^I and 9N - 12N, as for $\mathbf{C}^\circ$ except that now A, B, C and $A(x)$ may be any formulas of $\mathcal{L}(\mathbf{B})$; t is a term free for x in $A(x)$; $\vee$ and $\exists$ replace $\overset{\circ}{\vee}$ and $\exists^\circ$ respectively; and 5a, 5b, 6, 11N and 12N are postulates rather than theorems. In the following replacements for $9C^\circ$ - $12C^\circ$, $A(\beta)$ and B may be any formulas of $\mathcal{L}(\mathbf{B})$:

9F. $B \rightarrow A(\beta)$ / $B \rightarrow \forall\beta A(\beta)$ if β is not free in B.
10F. $\forall\beta A(\beta) \rightarrow A(u)$ if u is a functor free for β in $A(\beta)$.
11F. $A(u) \rightarrow \exists\beta A(\beta)$ if u is a functor free for β in $A(\beta)$.
12F. $A(\beta) \rightarrow B$ / $\exists\beta A(\beta) \rightarrow B$ if β is not free in B.

The mathematical axioms of $\mathbf{B}$ include those of $\mathbf{C}^\circ$, but with $\alpha, \beta, \ldots$ instead of $a, b, \ldots$ and with the following adaptations. For the mathematical induction schema, $A(x)$ may be any formula of $\mathcal{L}(\mathbf{B})$. For the λ-reduction schema $(\lambda x.r(x))(t) = r(t)$ both $r(x)$ and t are terms of $\mathcal{L}(\mathbf{B})$. The axiom schema of countable choice for $\mathbf{B}$ is

$$AC_{01}. \quad \forall x\exists\alpha A(x, \alpha) \rightarrow \exists\beta\forall x A(x, \lambda y.\beta(2^x \cdot 3^y))$$

where $A(x, \alpha)$ is any formula of $\mathcal{L}(\mathbf{B})$ with β, x free for α.

Brouwer's most important contributions to the foundations of intuitionistic mathematics were his "bar theorem," which is classically valid, and his continuity principle, which is not. An axiom schema of bar induction completes Kleene's neutral system $\mathbf{B}$, and the full intuitionistic system $\mathbf{I}$ comes from $\mathbf{B}$ by adding a principle of continuous choice. These are more complicated to state.

Finite sequences of natural numbers are coded formally using the function constants of $\mathcal{L}(\mathbf{B})$. In [3] $f_{19}(i) = p_i$ denotes the ith prime, with $p_0 = 2$; $f_{20}(y, i) = (y)_i$ denotes the exponent of p_i in the prime factorization of y; and $\langle x_0, \ldots, x_k\rangle$ abbreviates $\Pi_{i<k} p_i^{x_i}$. Let $Seq(y)$ abbreviate $\forall i < lh(y)\ (y)_i > 0$, where $lh(y)$ is a term denoting the number of nonzero exponents in the prime factorization of y. Then $\langle\,\rangle = 1$ codes the empty sequence; $\langle x_0 + 1, \ldots, x_k + 1\rangle$ codes the sequence $(x_0, \ldots, x_k)$; the concatenation of the finite sequences coded by w and z (assuming $Seq(w) \,\&\, Seq(z)$) is coded by $w * z$; and $w * \alpha$ codes the sequence defined by prefixing the finite sequence coded by w to α.

Let $\overline{\alpha}(n)$ abbreviate the code $\Pi_{i<n} p_i^{\alpha(i)+1}$ of the initial segment of α of length n (so $\overline{\alpha}(0) = 1$). The last axiom schema of $\mathbf{B}$ is the principle of bar induction (with a thin bar, essentially ^{x}26.3c on p. 55 of [3]), where ! expresses uniqueness:

$$BI!. \quad \forall\alpha\exists!xR(\overline{\alpha}(x)) \,\&\, \forall w[Seq(w) \,\&\, (R(w) \vee \forall nA(w * \langle n+1\rangle)) \rightarrow A(w)] \rightarrow A(\langle\,\rangle).$$

This description of Kleene's neutral basic system $\mathbf{B}$ of intuitionistic analysis summarizes Postulate Groups A-D, §§1-6 of [3].

The full intuitionistic system $\mathbf{I}$ comes from $\mathbf{B}$ by adding a principle of continuous choice ("Brouwer's principle for a function," cf. ^{x}27.1 on p. 73 of [3]):

$$CC_{11}. \quad \forall\alpha\exists\beta A(\alpha, \beta) \rightarrow \exists\sigma\forall\alpha(\forall x\exists!y\sigma(\langle x+1\rangle * \overline{\alpha}(y)) > 0 \,\& \\ \forall\beta(\forall x\exists y\sigma(\langle x+1\rangle * \overline{\alpha}(y)) = \beta(x) + 1 \rightarrow A(\alpha, \beta))).$$

3.3.1. *The classical version* **C** *of* **B**. A formal system **C** of classical analysis, with the axiom of countable choice, results from **B** by omitting BI! and replacing 8^{I} (*ex falso sequitur quodlibet*) by 8°. $\neg\neg\mathrm{E} \to \mathrm{E}$ for all formulas E of $\mathcal{L}(\mathbf{B})$. Because BI! follows from AC_{01} by classical logic (*26.1° on p. 53 of [3]), **B** is a subsystem of **C**.

Clearly **C** is inconsistent with **I**. The negative translation of AC_{01} is not a theorem schema of **B** so **C** cannot be interpreted negatively in its subsystem **B**.

3.3.2. *Proposition.* There is a faithful negative translation $\mathrm{A} \mapsto \mathrm{A}^{\mathrm{tr}}$ of **C** to **C**°.

Proof. In each formula A of $\mathcal{L}(\mathbf{C})$ ($\equiv \mathcal{L}(\mathbf{B})$) first replace $\vee, \exists\mathrm{x}, \exists\beta$ by $\overset{\circ}{\vee}, \exists^{\circ}\mathrm{x}, \exists^{\circ}\beta$ to obtain A', then replace $\alpha, \beta, \ldots$ by $\mathrm{a}, \mathrm{b}, \ldots$ to obtain A^{tr}. Clearly **C** proves $\mathrm{A}' \leftrightarrow \mathrm{A}$.

The translation E^{tr} of each axiom E of **C** is an axiom of **C**°, or a theorem of **C**° by Lemma 3.2.1(e) or Lemma 3.2.2. Since the translation of every logical rule of **C** is an admissible rule of **C**°, deductions in **C** can be replaced by corresponding deductions in **C**°. It follows that A^{tr} is a theorem of **C**° if and only if A is a theorem of **C**.

3.4. **The formal system IC.** The common extension **IC** of **C**° and **I** has both lawlike and choice sequence variables, since **I** essentially involves constructive existence and **C**° does not. One sort of number variables suffices, but in **IC** arithmetical formulas will not always be provably equivalent to their negative translations.[2]

The three-sorted language $\mathcal{L}(\mathbf{IC})$ extending both $\mathcal{L}(\mathbf{C}^{\circ})$ and $\mathcal{L}(\mathbf{I})$ has three sorts of variables with or without subscripts, also used as metavariables:

$\mathrm{i}, \mathrm{j}, \mathrm{k}, \ldots, \mathrm{p}, \mathrm{q}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z}$ over natural numbers,
$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}$ over classical lawlike sequences,
$\alpha, \beta, \gamma, \ldots$ over arbitrary choice sequences;

finitely many constants $\mathrm{f}_0 = 0$, $\mathrm{f}_1 = {}'$ (successor), $\mathrm{f}_2 = +$, $\mathrm{f}_3 = \cdot$, $\mathrm{f}_4 = \exp$, $\mathrm{f}_5, \ldots, \mathrm{f}_p$ for primitive recursive functions and functionals; the binary predicate constant =; Church's λ denoting function abstraction; parentheses (,) denoting function application; and the logical symbols & , $\vee, \to, \neg$ and quantifiers $\forall$ and $\exists$ over each sort of variable.

Terms and *functors* are defined inductively just as for **B** except that now all classical lawlike sequence variables and all arbitrary choice sequence variables are functors. *Prime formulas* are of the form $\mathrm{s} = \mathrm{t}$ where s, t are terms. If u, v are functors then $\mathrm{u} = \mathrm{v}$ abbreviates $\forall\mathrm{x}\, \mathrm{u}(\mathrm{x}) = \mathrm{v}(\mathrm{x})$. Composite formulas are formed as usual.

Terms, functors and formulas with no occurrences of arbitrary choice sequence variables are C-*terms*, C-*functors* and C-*formulas*, respectively.

The logical axioms and rules of **I** carry over to **IC**, where the $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{A}(\mathrm{x}), \mathrm{A}(\beta)$ may now be any formulas of $\mathcal{L}(\mathbf{IC})$. In addition, **IC** has logical rules 9C and 12C, and axiom schemas 10C and 11C, for all formulas $\mathrm{B}, \mathrm{A}(\mathrm{b})$ of $\mathcal{L}(\mathbf{IC})$:

9C. $\mathrm{B} \to \mathrm{A}(\mathrm{b})\ /\ \mathrm{B} \to \forall\mathrm{b}\mathrm{A}(\mathrm{b})$ where b is not free in B.
10C. $\forall\mathrm{b}\mathrm{A}(\mathrm{b}) \to \mathrm{A}(\mathrm{u})$ where u is a C-functor free for b in A(b).
11C. $\mathrm{A}(\mathrm{u}) \to \exists\mathrm{b}\mathrm{A}(\mathrm{b})$ where u is a C-functor free for b in A(b).
12C. $\mathrm{A}(\mathrm{b}) \to \mathrm{B}\ /\ \exists\mathrm{b}\mathrm{A}(\mathrm{b}) \to \mathrm{B}$ where b is not free in B.

[2]Under the assumption that a classical ω-model of **C** exists, the modified realizability interpretation introduced in the next section can show that **IC** is consistent with $\neg\neg\mathrm{E} \to \mathrm{E}$ for *all* formulas E of the language $\mathcal{L}(\mathbf{IC})$ containing no lawlike or choice sequence variables, but there is no need to add full first-order classical arithmetic to the formal system as we already have its negative translation.

The mathematical axioms of **I**, and $\mathrm{AC}_{01}^{\mathrm{C}^\circ}$ for negative C-formulas $\mathrm{A(x,b)}$, become axioms of **IC**. For the λ-reduction schema, $\mathrm{r(x)}$ and t may be any terms of $\mathcal{L}(\mathbf{IC})$. $\mathrm{A(x)}$ in the mathematical induction schema, $\mathrm{A}(\mathrm{x},\alpha)$ in AC_{01}, $\mathrm{R(w)}$ and $\mathrm{A(w)}$ in BI!, and $\mathrm{A}(\alpha,\beta)$ in CC_{11} may be any formulas of $\mathcal{L}(\mathbf{IC})$ satisfying the conditions of the schemas.

3.4.1. *Lemma.* For terms s, t and formulas A, B of $\mathcal{L}(\mathbf{IC})$, parts (a)-(d) of Lemma 3.2.1 also hold for **IC**. In addition, **IC** proves
(e) $\neg\neg\mathrm{A} \rightarrow \mathrm{A}$ if A is negative (no $\exists$ or $\vee$).
(f) $\mathrm{A} \vee \neg\mathrm{A}$ if A is quantifier-free.

3.4.2. *Lemma.* **IC** proves $\forall\mathrm{b}\exists\alpha\forall\mathrm{x}[\mathrm{b(x)} = \alpha(\mathrm{x})]$.

Proof. An easy consequence of AC_{01} is

$$\mathrm{AC}_{00}. \quad \forall\mathrm{x}\exists\mathrm{y}\mathrm{A(x,y)} \rightarrow \exists\alpha\forall\mathrm{x}\mathrm{A}(\mathrm{x},\alpha(\mathrm{x})).$$

From $\mathrm{b(x)} = \mathrm{b(x)}$ conclude $\exists\mathrm{y}[\mathrm{b(x)} = \mathrm{y}]$, then use $\forall$x-introduction, AC_{00}, Modus Ponens and $\forall$b-introduction.

Brouwer's arbitrary choice sequences included his lawlike sequences. In **IC** each classical sequence, all of whose values are fixed in advance, can be imitated by a choice sequence under construction.

3.4.3. *The end of time axiom.* The only completely new axiom of **IC** is

$$\mathrm{ET}. \quad \forall\alpha\neg\forall\mathrm{b}\neg\forall\mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b(x)}],$$

which is equivalent in **IC** to $\forall\alpha\neg\neg\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$. The intuitionistic double negation expresses *persistent consistency*; as the values of an arbitrary choice sequence α are chosen one by one, the possibility that α may coincide with a lawlike sequence can never be excluded.

If ET were strengthened to $\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$ and $\neg\neg\mathrm{E} \rightarrow \mathrm{E}$ was assumed for all C-formulas E, the result would be inconsistent by the following result.

3.4.4. *Proposition.* **IC** proves
(a) $\forall\mathrm{b}\neg\neg(\forall\mathrm{x}\mathrm{b(x)} = 0 \vee \neg\forall\mathrm{x}\mathrm{b(x)} = 0)$.
(b) $\neg\forall\alpha(\forall\mathrm{x}\alpha(\mathrm{x}) = 0 \vee \neg\forall\mathrm{x}\alpha(\mathrm{x}) = 0)$.

Proofs. (a) holds because $\neg\neg(\mathrm{A} \vee \neg\mathrm{A})$ is a theorem of intuitionistic logic. (b) holds because because **IC** proves $\mathrm{A} \vee \neg\mathrm{A} \rightarrow \exists\mathrm{y}(\mathrm{y} = 0 \leftrightarrow \mathrm{A})$ and "Brouwer's principle for a number" (*27.2 of [3]):

$$\mathrm{CC}_{10}. \forall\alpha\exists\mathrm{x}\mathrm{A}(\alpha,\mathrm{x}) \rightarrow \exists\sigma\forall\alpha(\exists!\mathrm{y}\sigma(\overline{\alpha}(\mathrm{y})) \neq 0\ \&\ \forall\mathrm{x}\forall\mathrm{z}(\exists\mathrm{y}\sigma(\overline{\alpha}(\mathrm{y})) = \mathrm{z}+1 \rightarrow \mathrm{A}(\alpha,\mathrm{z}))).$$

4. $^{\mathcal{C}}$REALIZABILITY AND THE CONSISTENCY OF IC

4.1. **From now on, assume that $\mathcal{M} = (\omega,\mathcal{C})$ is a classical ω-model of $\mathbf{C}^\circ$.** Then $\mathcal{M}$ is also an ω-model of **B** and **C** (cf. §3.3.1 above) under the classical interpretation of $\vee$ and $\exists$. Observe that $\mathcal{C}$ is closed under "recursive in," i.e. if γ is recursive in finitely many elements of $\mathcal{C}$ then $\gamma \in \mathcal{C}$, so C-functors represent elements of $\mathcal{C}$.

For the proof that **IC** is consistent it is not necessary to assume $\mathcal{C}$ is countable, or even that $\mathcal{C} \neq \omega^\omega$. The proof that **IC** is consistent with $\neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$, on the other hand, will depend on the additional assumption $\mathcal{C} \neq \omega^\omega$.

Kleene's curly bracket and Λ notations are described in Section 8 of [3]. Briefly, $\{\alpha\}[\beta](x)$ is *defined and equal to* y ($\{\alpha\}[\beta](x) \simeq y$) if for some z: $\alpha(\langle x+1\rangle * \overline{\beta}(z)) = y+1$ and $\alpha(\langle x+1\rangle * \overline{\beta}(j)) = 0$ for all $j < z$. Thus $\{\alpha\}[\beta]$ is a recursive partial functional of α and β. In general, $\{\alpha\}[x]$ will abbreviate $\{\alpha\}[\lambda t.x]$, and $\{\alpha\}$ will abbreviate $\{\alpha\}[0]$.

If $\Phi[\alpha, \beta, x, y]$ is a partial functional of the indicated variables which is recursive in functions Δ, by Kleene's enumeration theorem there are index functions $\Lambda\alpha\, \Phi[\alpha, \beta, x, y]$, $\Lambda x\, \Phi[\alpha, \beta, x, y]$, $\Lambda\, \Phi[\alpha, \beta, x, y]$ primitive recursive in Δ such that for all α, β, x, y, z:

$$(\{\Lambda\alpha\, \Phi[\alpha, \beta, x, y]\}[\alpha])(z) \simeq (\Phi[\alpha, \beta, x, y])(z) \simeq (\{\Lambda x\, \Phi[\alpha, \beta, x, y]\}[x])(z)$$

and $\{\Lambda\, \Phi[\alpha, \beta, x, y]\} \simeq \Phi[\alpha, \beta, x, y]$. Similarly for $\Phi[\alpha_1, \ldots, \alpha_j, x_1, \ldots, x_k, y_1, \ldots, y_m]$.

4.2. **Definition.** By induction on the logical form of a formula E of $\mathcal{L}(\mathbf{IC})$ we define when $\varepsilon \in \omega^\omega$ *agrees with* E, as follows, where $(\varepsilon)_i$ abbreviates $\lambda y.(\varepsilon(y))_i$.

(1) ε *agrees with* a prime formula s = t, for each ε.
(2) ε *agrees with* A & B, if $(\varepsilon)_0$ *agrees with* A and $(\varepsilon)_1$ *agrees with* B.
(3) ε *agrees with* A $\vee$ B, if $(\varepsilon(0))_0 = 0$ implies that $(\varepsilon)_1$ *agrees with* A, while $(\varepsilon(0))_0 \neq 0$ implies that $(\varepsilon)_1$ *agrees with* B.
(4) ε *agrees with* A $\to$ B, if, whenever α *agrees with* A, $\{\varepsilon\}[\alpha]$ is completely defined and *agrees with* B.
(5) ε *agrees with* $\neg$A, if ε *agrees with* A $\to$ 1 = 0 by the preceding clause.
(6) ε *agrees with* $\exists$xA(x), if $(\varepsilon)_1$ *agrees with* A(x).
(7) ε *agrees with* $\forall$xA(x), if, for each x, $\{\varepsilon\}[x]$ is completely defined and *agrees with* A(x).
(8) ε *agrees with* $\exists$bA(b), if $\{(\varepsilon)_0\}$ is completely defined and belongs to $\mathcal{C}$, and $(\varepsilon)_1$ *agrees with* A(b).
(9) ε *agrees with* $\forall$bA(b), if, for each $\beta \in \mathcal{C}$, $\{\varepsilon\}[\beta]$ is completely defined and *agrees with* A(b).
(10) ε *agrees with* $\exists\alpha$A(α), if $\{(\varepsilon)_0\}$ is completely defined and $(\varepsilon)_1$ *agrees with* A(α).
(11) ε *agrees with* $\forall\alpha$A(α), if, for each sequence α, $\{\varepsilon\}[\alpha]$ is completely defined and *agrees with* A(α).

4.2.1. *Lemma.*

(a) If s is a term free for y in A(y), then ε agrees with A(y) if and only if ε agrees with A(s). Similarly if v is a functor free for β in A(β), or u is a C-functor free for b in A(b).
(b) ε agrees with E if and only if ε agrees with the result of replacing each part of E of the form $\neg$A by (A $\to$ 1 = 0).
(c) For each formula E of $\mathcal{L}(\mathbf{IC})$ there is a primitive recursive function ε^{E} which agrees with E.

Proofs. By induction on the logical form of E. Only (c) is nontrivial. If E is prime then ε^{E} is $\lambda t.0$. Given ε^{A} and ε^{B} agreeing with A and B respectively, let $\varepsilon^{\mathrm{A\&B}} = \langle \varepsilon^{\mathrm{A}}, \varepsilon^{\mathrm{B}} \rangle$, $\varepsilon^{\mathrm{A\vee B}} = \langle \lambda t.0, \varepsilon^{\mathrm{A}} \rangle$, $\varepsilon^{\mathrm{A\to B}} = \Lambda\alpha\; \varepsilon^{\mathrm{B}}$ and $\varepsilon^{\neg\mathrm{A}} = \Lambda\pi\lambda t.0$. Given $\varepsilon^{\mathrm{A(x)}}$ agreeing with A(x), let $\varepsilon^{\exists\mathrm{xA(x)}} = \langle \lambda t.0, \varepsilon^{\mathrm{A(x)}} \rangle$ and $\varepsilon^{\forall\mathrm{xA(x)}} = \Lambda x\; \varepsilon^{\mathrm{A(x)}}$. Given $\varepsilon^{\mathrm{A(b)}}$, let $\varepsilon^{\exists\mathrm{bA(b)}} = \langle \Lambda\; \lambda t.0, \varepsilon^{\mathrm{A(b)}} \rangle$ and $\varepsilon^{\forall\mathrm{bA(b)}} = \Lambda\beta\; \varepsilon^{\mathrm{A(b)}}$. Given $\varepsilon^{\mathrm{A}(\alpha)}$, let $\varepsilon^{\exists\alpha\mathrm{A}(\alpha)} = \langle \Lambda\; \lambda t.0, \varepsilon^{\mathrm{A}(\alpha)} \rangle$ and $\varepsilon^{\forall\alpha\mathrm{A}(\alpha)} = \Lambda\alpha\; \varepsilon^{\mathrm{A}(\alpha)}$.

4.3. **Definition.** By induction on the logical form of a formula E of $\mathcal{L}(\mathbf{IC})$ containing free at most the distinct variables Ψ we define when a sequence ε, *belonging to* $\mathcal{C}$, ${}^{\mathcal{C}}$*realizes-*$\varPsi$ E, where $\varPsi$ are elements of ω, $\mathcal{C}$ and $\mathcal{C}$ corresponding respectively to the number, lawlike sequence, and choice sequence variables in the list Ψ, as follows.

(1) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ a prime formula P, if P is true-$\varPsi$ in $\mathcal{M}$.
(2) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ A & B, if $(\varepsilon)_0$ ${}^{\mathcal{C}}$*realizes-*$\varPsi$ A and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes-*$\varPsi$ B.
(3) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ A $\vee$ B, if either $(\varepsilon(0))_0 = 0$ and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes-*$\varPsi$ A, or $(\varepsilon(0))_0 \neq 0$ and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes-*$\varPsi$ B.
(4) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ A $\rightarrow$ B, if ε agrees with A $\rightarrow$ B and, whenever $\alpha \in \mathcal{C}$ and α ${}^{\mathcal{C}}$*realizes-*$\varPsi$ A, $\{\varepsilon\}[\alpha]$ is completely defined and ${}^{\mathcal{C}}$*realizes-*$\varPsi$ B.
(5) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ $\neg$A, if ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ A $\rightarrow 1 = 0$ by the preceding clause.
(6) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ $\exists$xA(x), if $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes-*$\varPsi, (\varepsilon(0))_0$ A(x).
(7) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ $\forall$xA(x), if, for each natural number n, $\{\varepsilon\}[n]$ is completely defined and ${}^{\mathcal{C}}$*realizes-*$\varPsi, n$ A(x).
(8) ε ${}^{\mathcal{C}}$*realizes* $\exists$bA(b), if $\{(\varepsilon)_0\}$ is completely defined (hence belongs to $\mathcal{C}$) and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes-*$\varPsi, \{(\varepsilon)_0\}$ A(b).
(9) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ $\forall$bA(b), if, for each sequence $\beta \in \mathcal{C}$, $\{\varepsilon\}[\beta]$ is completely defined and ${}^{\mathcal{C}}$*realizes-*$\varPsi, \beta$ A(b).
(10) ε ${}^{\mathcal{C}}$*realizes* $\exists\alpha$A(α), if $\{(\varepsilon)_0\} \in \mathcal{C}$ and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes-*$\varPsi, \{(\varepsilon)_0\}$ A(α).
(11) ε ${}^{\mathcal{C}}$*realizes-*$\varPsi$ $\forall\alpha$A(α), if ε agrees with $\forall\alpha$A(α) and, for each $\beta \in \mathcal{C}$, $\{\varepsilon\}[\beta]$ (is completely defined and) ${}^{\mathcal{C}}$*realizes-*$\varPsi, \beta$ A(α).

A sentence E of $\mathcal{L}(\mathbf{IC})$ is ${}^{\mathcal{C}}$*realizable* if and only if E is ${}^{\mathcal{C}}$realized by some *general recursive* sequence ε, and a formula is ${}^{\mathcal{C}}$*realizable* if and only if its universal closure is ${}^{\mathcal{C}}$realizable.

4.3.1. *Lemma.* Let $\varPsi$ be a list of numbers and elements of $\mathcal{C}$.

(a) If ε ${}^{\mathcal{C}}$realizes-$\varPsi$ a formula E of $\mathcal{L}(\mathbf{IC})$, then ε agrees with E and $\varepsilon \in \mathcal{C}$.
(b) ε ${}^{\mathcal{C}}$realizes-$\varPsi$ a formula E of $\mathcal{L}(\mathbf{IC})$ if and only if ε ${}^{\mathcal{C}}$realizes-$\varPsi$ the result of replacing each part of E of the form $\neg$A by (A $\rightarrow 1 = 0$).
(c) For no formula E of $\mathcal{L}(\mathbf{IC})$ and no sequences $\varepsilon_1, \varepsilon_2 \in \mathcal{C}$ is it the case that ε_1 ${}^{\mathcal{C}}$realizes-$\varPsi$ E and ε_2 ${}^{\mathcal{C}}$realizes-$\varPsi$ $\neg$E.

4.3.2. *Lemma.*

(a) Let A(y) be a formula of $\mathcal{L}(\mathbf{IC})$ containing free at most the distinct variables Ψ, y, let s be a term containing free at most Ψ, y and free for y in A(y), and let $s(\varPsi, y)$ be the number expressed by s when Ψ, y take the values $\varPsi, y$ in $\mathcal{C}$ and ω. Let $\varepsilon \in \mathcal{C}$. Then ε ${}^{\mathcal{C}}$realizes-$\varPsi, y$ A(s) if and only if ε ${}^{\mathcal{C}}$realizes-$\varPsi, s(\varPsi, y)$ A(y).
(b) Let A(β) be a formula of $\mathcal{L}(\mathbf{IC})$ containing free at most the distinct variables Ψ, β, let u be a functor containing free at most Ψ, β and free for β in A(β), and let $u[\varPsi, \beta]$ be the element of $\mathcal{C}$ expressed by u when Ψ, β take the values $\varPsi, \beta$ in $\mathcal{C}$ and ω. Let $\varepsilon \in \mathcal{C}$. Then ε ${}^{\mathcal{C}}$realizes-$\varPsi, \beta$ A(u) if and only if ε ${}^{\mathcal{C}}$realizes-$\varPsi, u[\varPsi, \beta]$ A(β). Similarly for A(b) where u is a C-functor free for b.

4.3.3. *Lemma.* Let $\mathrm{E} \equiv \mathrm{E}(\alpha_1, \ldots, \alpha_j, \mathrm{b}_1, \ldots, \mathrm{b}_k, \mathrm{y}_1, \ldots, \mathrm{y}_m)$ be a formula of $\mathcal{L}(\mathbf{IC})$ with only the indicated distinct variables free. Then E is ${}^{\mathcal{C}}$realizable if and only if there is a recursive partial functional $\varPhi[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$ such that, for all $\alpha_1, \ldots, \alpha_j \in \omega^\omega$, all $\beta_1, \ldots, \beta_k \in \mathcal{C}$ and all $y_1, \ldots, y_m \in \omega$:

(a) $\Phi[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$ is defined (so belongs to ω^ω) and agrees with $\mathrm{E}(\alpha_1, \ldots, \alpha_\mathrm{j}, \mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m})$.

(b) If also $\alpha_1, \ldots, \alpha_j \in \mathcal{C}$ then $\Phi[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$ belongs to $\mathcal{C}$ and ${}^{\mathcal{C}}$realizes-$\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m$ E.

4.3.4. *Lemma.* For every negative C-formula E of $\mathcal{L}(\mathbf{IC})$ (so for every formula of $\mathcal{L}(\mathbf{C}^\circ)$) with only the distinct variables Ψ free there is a primitive recursive function τ_E such that τ_E agrees with E, and for each interpretation $\mathit{\Psi}$ of Ψ by elements of $\mathcal{C}$ and ω:

(a) If E is ${}^{\mathcal{C}}$realized-$\mathit{\Psi}$ by some $\varepsilon \in \mathcal{C}$ then E is true-$\mathit{\Psi}$ in $\mathcal{M}$.

(b) If E is true-$\mathit{\Psi}$ in $\mathcal{M}$ then τ_E ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ E.

Proof. For each negative C-formula E let τ_E be the primitive recursive function ε^E defined in proving Lemma 4.2.1(c). τ_E agrees with E by the lemma, and satisfies (a) and (b) by formula induction. We give the case for $\mathrm{E} \equiv \neg\mathrm{A}$. Assume τ_A satisfies (a) and (b) for A. If ($\varepsilon \in \mathcal{C}$ and) ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ $\neg$A, then (since $0 = 1$ is false in $\mathcal{M}$) no $\delta \in \mathcal{C}$ can ${}^{\mathcal{C}}$realize-$\mathit{\Psi}$ A, so A is false-$\mathit{\Psi}$ in $\mathcal{M}$ by (b) for τ_A, so $\neg$A is true-$\mathit{\Psi}$ in $\mathcal{M}$, so (a) holds for $\neg$A. If $\neg$A is true-$\mathit{\Psi}$ in $\mathcal{M}$ then A is false-$\mathit{\Psi}$ in $\mathcal{M}$, so no $\varepsilon \in \mathcal{C}$ can ${}^{\mathcal{C}}$realize-$\mathit{\Psi}$ A by (a) for τ_A, so $\tau_{\neg\mathrm{A}} = \Lambda\pi\, \lambda t.0$ ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ $\neg$A, so (b) holds for $\neg$A.

4.4. **Theorem.** If $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}, \mathrm{E}$ are formulas of $\mathcal{L}(\mathbf{IC})$ such that $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n} \vdash_{\mathbf{IC}} \mathrm{E}$ and $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}$ are all ${}^{\mathcal{C}}$ realizable, then E is ${}^{\mathcal{C}}$realizable. Therefore, **IC** is consistent.

Proof. For each axiom or axiom schema of **IC** containing free at most the distinct variables in the list $\Psi = \alpha_1, \ldots, \alpha_\mathrm{j}, \mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m}$ we give a ${}^{\mathcal{C}}$realizing functional $\Phi[\mathit{\Psi}] = \Phi[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$, as in Lemma 4.3.3; and assuming that such a $\Phi'[\mathit{\Psi}']$ exists for each premise of a rule of inference, we provide a $\Phi[\mathit{\Psi}]$ for the conclusion.

Logical Axioms 1a, 1b, 3-7, 10N, 11N, 10F, 11F (exactly as in [3]) and 10C, 11C:

1a. $\mathrm{A} \to (\mathrm{B} \to \mathrm{A})$. $\Lambda\alpha\Lambda\beta\, \alpha$.

1b. $(\mathrm{A} \to \mathrm{B}) \to ((\mathrm{A} \to (\mathrm{B} \to \mathrm{C})) \to (\mathrm{A} \to \mathrm{C}))$. $\Lambda\pi\Lambda\rho\Lambda\alpha\{\{\rho\}[\alpha]\}[\{\pi\}[\alpha]]$.

3. $\mathrm{A} \to (\mathrm{B} \to \mathrm{A}\ \&\ \mathrm{B})$. $\Lambda\alpha\Lambda\beta\, \langle\alpha, \beta\rangle$.

4a. $\mathrm{A}\ \&\ \mathrm{B} \to \mathrm{A}$. $\Lambda\alpha(\alpha)_0$.

4b. $\mathrm{A}\ \&\ \mathrm{B} \to \mathrm{B}$. $\Lambda\alpha(\alpha)_1$.

5a. $\mathrm{A} \to \mathrm{A} \vee \mathrm{B}$. $\Lambda\alpha\, \langle\lambda t.0, \alpha\rangle$.

5b. $\mathrm{B} \to \mathrm{A} \vee \mathrm{B}$. $\Lambda\alpha\, \langle\lambda t.1, \alpha\rangle$.

6. $(\mathrm{A} \to \mathrm{C}) \to ((\mathrm{B} \to \mathrm{C}) \to (\mathrm{A} \vee \mathrm{B} \to \mathrm{C}))$.
$\Lambda\pi\Lambda\rho\Lambda\delta\, \lambda t.(1 \dot{-} (\delta(0))_0)\{\pi\}[(\delta)_1](t) + (\delta(0)_0)\{\rho\}[(\delta)_1](t)$.

7. $(\mathrm{A} \to \mathrm{B}) \to ((\mathrm{A} \to \neg\mathrm{B}) \to \neg\mathrm{A})$. Same as for 1b.

8. $\neg\mathrm{A} \to (\mathrm{A} \to \mathrm{B})$. $\Lambda\delta\Lambda\alpha\, \varepsilon^\mathrm{B}$.

10N. $\forall\mathrm{xA}(\mathrm{x}) \to \mathrm{A}(\mathrm{t})$ where $\mathrm{t}(\Psi)$ is a term free for x in A(x). $\Lambda\delta\, \{\delta\}[t(\mathit{\Psi})]$.

11N. $\mathrm{A}(\mathrm{t}) \to \exists\mathrm{xA}(\mathrm{x})$ where $\mathrm{t}(\Psi)$ is a term free for x in A(x). $\Lambda\delta\, \langle\lambda y.t(\mathit{\Psi}), \delta\rangle$.

10C. $\forall\mathrm{bA}(\mathrm{b}) \to \mathrm{A}(\mathrm{u})$ where $\mathrm{u}[\Psi] = \mathrm{u}[\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m}]$ is a C-functor free for b in A(b). $\Lambda\delta\, \{\delta\}[u(\beta_1, \ldots, \beta_k, y_1, \ldots, y_m)]$.

11C. $\mathrm{A}(\mathrm{u}) \to \exists\mathrm{bA}(\mathrm{b})$ where $\mathrm{u}[\Psi] = \mathrm{u}[\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m}]$ is a C-functor free for b in A(b). $\Lambda\delta\langle\Lambda\, u[\beta_1, \ldots, \beta_k, y_1, \ldots, y_m], \delta\rangle$.

10F. $\forall\alpha\mathrm{A}(\alpha) \to \mathrm{A}(\mathrm{u})$ where $\mathrm{u}[\Psi]$ is a functor free for α in $\mathrm{A}(\alpha)$. $\Lambda\delta\, \{\delta\}[u[\mathit{\Psi}]]$.

11F. $\mathrm{A}(\mathrm{u}) \to \exists\alpha\mathrm{A}(\alpha)$ where $\mathrm{u}[\Psi]$ is a functor free for α in $\mathrm{A}(\alpha)$. $\Lambda\delta\, \langle\Lambda\, u[\mathit{\Psi}], \delta\rangle$.

Axioms for 3-sorted intuitionistic number theory: As in [3], $\lambda t.0$, $\Lambda\pi\lambda t.0$ and $\Lambda\pi\Lambda\sigma\lambda t.0$ take care of the prime axioms, including $(\lambda \mathrm{x}.\mathrm{r}(\mathrm{x}))(\mathrm{t}) = \mathrm{r}(\mathrm{t})$); $\mathrm{x} = \mathrm{y} \rightarrow \alpha(\mathrm{x}) = \alpha(\mathrm{y})$ and axioms 14, 15, 17 from [1]; and axiom 16 from [1], respectively.

The mathematical induction schema (13 in [1]) is $\mathrm{A}(0)\ \&\ \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{x}')) \rightarrow \mathrm{A}(\mathrm{x})$. A ${}^{\mathcal{C}}$realizing functional is $\Lambda\pi\rho[x,\pi]$ where ρ is defined by the functional recursion $\rho[0,\pi] = (\pi)_0$ and $\rho[x',\pi] = \{\{(\pi)_1\}[x]\}[\rho[x,\pi]]$ (cf. [3], page 106).

Axiom of countable choice: AC_{01}. $\forall \mathrm{x}\exists\alpha \mathrm{A}(\mathrm{x},\alpha) \rightarrow \exists\beta\forall \mathrm{x}\mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\beta(2^{\mathrm{x}} \cdot 3^{\mathrm{y}}))$. Exactly as in [3], $\Lambda\pi\langle\Lambda\,\lambda z.\{(\{\pi\}[(z)_0])_0\}((z)_1), \Lambda x\,(\{\pi\}[x])_1\rangle$. Agreement is obvious. Assume $\pi \in \mathcal{C}$ and π ${}^{\mathcal{C}}$realizes-Ψ $\forall \mathrm{x}\exists\alpha \mathrm{A}(\mathrm{x},\alpha)$. Then $\langle\Lambda\,\lambda z.\{(\{\pi\}[(z)_0])_0\}((z)_1), \Lambda x\,(\{\pi\}[x])_1\rangle$ is in $\mathcal{C}$ and ${}^{\mathcal{C}}$realizes-Ψ $\exists\beta\forall \mathrm{x}\mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\beta(2^{\mathrm{x}} \cdot 3^{\mathrm{y}}))$.

Countable choice for $\mathbf{C}^\circ$: $\mathrm{AC}_{01}^{\mathrm{C}^\circ}$. $\forall \mathrm{x}\neg\forall \mathrm{b}\neg \mathrm{A}(\mathrm{x},\mathrm{b}) \rightarrow \neg\forall \mathrm{a}\neg\forall \mathrm{x}\mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\mathrm{a}(2^{\mathrm{x}} \cdot 3^{\mathrm{y}}))$ where $\mathrm{A}(\mathrm{x},\mathrm{b})$ is a (negative) formula of $\mathcal{L}(\mathbf{C}^\circ)$. Use Lemma 4.3.4.

End of time axiom: ET. $\forall\alpha\neg\forall \mathrm{b}\neg\forall \mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$. $\varepsilon^{\forall\alpha\neg\forall \mathrm{b}\neg\forall \mathrm{x}[\alpha(\mathrm{x})=\mathrm{b}(\mathrm{x})]} = \Lambda\alpha\Lambda\pi\,\lambda t.0$ agrees with the axiom by Lemma 4.2.1(b,c) and ${}^{\mathcal{C}}$realizes the axiom because if $\alpha \in \mathcal{C}$ then $\Lambda\pi\,\lambda t.0$ is in $\mathcal{C}$ and ${}^{\mathcal{C}}$realizes-α $\neg\forall \mathrm{b}\neg\forall \mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$, since no $\pi \in \mathcal{C}$ ${}^{\mathcal{C}}$realizes-α, α $\neg\forall \mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$ so no $\pi \in \mathcal{C}$ ${}^{\mathcal{C}}$realizes-α $\forall \mathrm{b}\neg\forall \mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$.

Bar induction BI!: $\Lambda\pi\zeta[\pi,1]$ where $\zeta[\pi,w]$ is a recursive partial function defined using the recursion theorem. Let $G(\pi,w)$ abbreviate "$w = \overline{(w * \lambda t.0)}((\{(\pi)_0\}[w * \lambda t.0](0))_0)$," $H(\pi,w)$ abbreviate "$Seq(w)\,\&\,lh(w) \geq (\{(\pi)_0\}[w * \lambda t.0](0))_0$," and $J(\pi,w)$ abbreviate "$G(\pi,w)\,\&\,\forall u,v < w(u * v = w \rightarrow \neg G(\pi,u))$." If π ${}^{\mathcal{C}}$realizes-Ψ the hypothesis of BI! then $\forall\alpha\exists!x_\alpha J(\pi,\overline{\alpha}(x_\alpha))$ (because $y_\alpha \equiv (\{(\pi)_0\}[\alpha](0))_0$ is determined by a finite initial segment of α); and $(\{(\pi)_0\}[\alpha])_1$ ${}^{\mathcal{C}}$realizes-$\Psi, \overline{\alpha}(x_\alpha)$ R(w) (since $\mathcal{C}$ is dense in ω^ω). Let

$$\zeta[\pi,w] = \begin{cases} \varepsilon^{\mathrm{A}(\langle\,\rangle)} & \text{if } H(\pi,w)\,\&\,\neg G(\pi,w) \\ \{\{(\pi)_1\}[w]\}[\langle\lambda t.0, \langle\lambda t.0, (\{(\pi)_0\}[w * \lambda t.0])_1\rangle\rangle] & \text{if } G(\pi,w) \\ \{\{(\pi)_1\}[w]\}[\langle\lambda t.0, \langle\lambda t.1, \Lambda n\zeta[\pi, w * \langle n+1\rangle]\rangle\rangle] & \text{otherwise, if } Seq(w) \end{cases}$$

If π ${}^{\mathcal{C}}$realizes-Ψ the hypothesis of BI! then $\zeta[\pi,1]$ ${}^{\mathcal{C}}$realizes-Ψ $\mathrm{A}(\langle\,\rangle)$ by an informal bar induction with $J(\pi,w)$ determining the thin bar and $K(\pi,w)$ (abbreviating "($\zeta[\pi,w]$ ${}^{\mathcal{C}}$realizes-Ψ, w A(w)) $\&\,\forall u,v < w(u * v = w \rightarrow \neg G(\pi,u))$") as the inductive predicate.

Continuous choice CC_{11}: As in [3], [4]: $\Lambda\pi\langle\Lambda\,\sigma, \Lambda\alpha\langle\rho,\tau\rangle\rangle$ where $\sigma = \Lambda\alpha\{(\{\pi\}[\alpha])_0\}$, $\rho = \Lambda x\langle\mu y(\sigma(2^{x+1} * \overline{\alpha}(y)) > 0), \langle\lambda t.0, \Lambda z\Lambda\delta\lambda t.0\rangle\rangle$ and $\tau = \Lambda\beta\Lambda\delta(\{\pi\}[\alpha])_1$.

Rules of inference: Modus ponens and 9N, 12N, 9F, 12F (as in [3]) and 9C, 12C:

2. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for A, $\Phi''[\Psi]$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{A} \rightarrow \mathrm{B}$ and $\Psi' \subseteq \Psi$, then $\Phi[\Psi] = \{\Phi''[\Psi]\}[\Phi'[\Psi']]$ is a ${}^{\mathcal{C}}$realizing functional for B.

9N. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{B} \rightarrow \mathrm{A}(\mathrm{x})$, where $\Psi' = \Psi, x$ and x is not free in B, then $\Phi[\Psi] = \Lambda\delta\Lambda x\,\{\Phi'[\Psi,x]\}[\delta]$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{B} \rightarrow \forall \mathrm{x}\mathrm{A}(\mathrm{x})$.

12N. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{B}$, where $\Psi' = \Psi, x$ and x is not free in B, then $\Phi[\Psi] = \Lambda\pi\,\{\Phi'[\Psi,(\pi(0))_0]\}[(\pi)_1]$ is a ${}^{\mathcal{C}}$realizing functional for $\exists x A(x) \rightarrow B$.

9C. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{B} \rightarrow \mathrm{A}(\mathrm{b})$, where $\Psi' = \Psi, \beta$ and b is not free in B, then $\Phi[\Psi] = \Lambda\delta\Lambda\beta\,\{\Phi'[\Psi,\beta]\}[\delta]$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{B} \rightarrow \forall \mathrm{b}\mathrm{A}(\mathrm{b})$.

12C. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{B}$, where $\Psi' = \Psi, \beta$ and b is not free in B, then $\Lambda\pi\{\Phi'[\Psi,\{(\pi)_0\}]\}[(\pi)_1]$ is a ${}^{\mathcal{C}}$realizing functional for $\exists \mathrm{b}\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{B}$.

9F. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{B} \rightarrow \mathrm{A}(\alpha)$, where $\Psi' = \Psi, \alpha$ and α is not free in B, then $\Phi[\Psi] = \Lambda\delta\Lambda\alpha\,\{\Phi'[\Psi,\alpha]\}[\delta]$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{B} \rightarrow \forall\alpha\mathrm{A}(\alpha)$.

12F. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{A}(\alpha) \to \mathrm{B}$, where $\Psi' = \Psi, \alpha$ and α is not free in B, then $\Lambda\pi\,\{\Phi'[\Psi, \{(\pi)_0\}]\}[(\pi)_1]$ is a ${}^{\mathcal{C}}$realizing functional for $\exists\alpha A(\alpha) \to B$.

This completes the proof that every theorem of **IC** is ${}^{\mathcal{C}}$realizable. By assumption $\mathcal{M}$ is a classical model of $\mathbf{C}^{\circ}$ so $0 = 1$ is not true in $\mathcal{M}$, and therefore not ${}^{\mathcal{C}}$realizable.

4.4.1. *Corollary.* **IC** is consistent with all sentences of $\mathcal{L}(\mathbf{C}^{\circ})$ which are true in $\mathcal{M}$.

Proof. By the theorem with Lemma 4.3.4, every formula of $\mathcal{L}(\mathbf{IC})$ which is provable in **IC** from sentences of $\mathcal{L}(\mathbf{C}^{\circ})$ true in $\mathcal{M}$ is ${}^{\mathcal{C}}$realizable.

4.4.2. *Corollary.* If $\mathcal{C} \neq \omega^{\omega}$ then $\mathbf{IC} + \neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is consistent (and consistent with all sentences of $\mathcal{L}(\mathbf{C}^{\circ})$ which are true in $\mathcal{M}$).

Proof. It will be enough to show that if $\mathcal{C} \neq \omega^{\omega}$ then $\neg\neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is not ${}^{\mathcal{C}}$realizable. Since $\mathcal{C}$ is dense in ω^{ω}, if $\{\varepsilon\}[\alpha]$ is defined for all $\alpha \in \omega^{\omega}$ and $\{\varepsilon\}[\beta] = \beta$ for all $\beta \in \mathcal{C}$ then $\{\varepsilon\}[\alpha] = \alpha$ for all $\alpha \in \omega^{\omega}$, so no $\varepsilon \in \mathcal{C}$ can ${}^{\mathcal{C}}$realize $\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, so $\Lambda\pi\,\lambda t.0$ ${}^{\mathcal{C}}$realizes $\neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, so $\neg\neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is not ${}^{\mathcal{C}}$realizable.

4.4.3. *Definition.* A formula E of $\mathcal{L}(\mathbf{IC})$ is ${}^{\mathcal{C}}$*realizable*/$\mathcal{C}$ if and only if its universal closure is ${}^{\mathcal{C}}$realized by some element of $\mathcal{C}$.

4.4.4. *Lemma.* If the truth function κ for classical arithmetic is an element of $\mathcal{C}$, then to each arithmetical formula E of $\mathcal{L}(\mathbf{IC})$ with at most the distinct variables $\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{k}$ free there is a partial functional $\Gamma_{\mathrm{E}}[y_1, \ldots, y_k]$ recursive in κ which agrees with E and satisfies, for all $y_1, \ldots, y_k \in \omega$ and corresponding numerals $\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k}$:

(a) If $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ then $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ is true in $\mathcal{M}$.
(b) If $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ is true in $\mathcal{M}$ then $\Gamma_{\mathrm{E}}[y_1, \ldots, y_k]$ ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$.

Proof. Since $\mathcal{M}$ is a classical ω-model of $\mathbf{C}^{\circ}$ and hence of $\mathbf{C}$, an arithmetical sentence A is classically true if and only if it is true in $\mathcal{M}$, if and only if $\kappa(\ulcorner\mathrm{A}\urcorner) = 1$. By Lemma 4.3.2(a), ε ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ if and only if ε ${}^{\mathcal{C}}$realizes-$y_1, \ldots, y_k$ E.

We use induction on the logical form of E. Prime formulas, &, $\vee, \to, \neg$ and $\forall x$ follow the proof of Lemma 4.3.4; e.g. if $\mathrm{E}(\mathrm{y}) \equiv \forall\mathrm{x}\mathrm{A}(\mathrm{x}, \mathrm{y})$ has only y free and (a), (b) hold for A with $\Gamma_{\mathrm{A}}[x, y]$ recursive in κ, then (a) and (b) hold for E with $\Gamma_{\mathrm{E}}[y] = \Lambda x \Gamma_{\mathrm{A}}[x, y]$.

Suppose $\mathrm{E}(\mathrm{x}, \mathrm{y}) \equiv \mathrm{A}(\mathrm{x}, \mathrm{y}) \vee \mathrm{B}(\mathrm{x}, \mathrm{y})$ where (a), (b) hold for A, B with Γ_{A}, Γ_{B} recursive in κ. Let $\Gamma_{\mathrm{E}}[x, y] = \langle 0, \Gamma_{\mathrm{A}}[x, y]\rangle$ if $\kappa(\ulcorner\mathrm{A}(\mathbf{x}, \mathbf{y})\urcorner) = 1$, otherwise $\Gamma_{\mathrm{E}}[x, y] = \langle 1, \Gamma_{\mathrm{B}}[x, y]\rangle$. If $\mathrm{E}(\mathbf{x}, \mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$, then $\mathrm{A}(\mathbf{x}, \mathbf{y})$ or $\mathrm{B}(\mathbf{x}, \mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ so true in $\mathcal{M}$. If $\mathrm{E}(\mathbf{x}, \mathbf{y})$ is true in $\mathcal{M}$ then $\mathrm{A}(\mathbf{x}, \mathbf{y})$ or $\mathrm{B}(\mathbf{x}, \mathbf{y})$ is true in $\mathcal{M}$ so $\Gamma_{\mathrm{E}}[x, y]$ ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{x}, \mathbf{y})$.

Suppose $\mathrm{E}(\mathrm{y})$ is $\exists\mathrm{x}\mathrm{A}(\mathrm{x}, \mathrm{y})$ where (a) and (b) hold for $\mathrm{A}(\mathrm{x}, \mathrm{y})$ with Γ_{A} recursive in κ. Let $\Gamma_{\mathrm{E}}[y] = \langle z, \Gamma_{\mathrm{A}}[z, y]\rangle$ where $z \simeq \mu x \kappa(\ulcorner\mathrm{A}(\mathbf{x}, \mathbf{y})\urcorner) = 1$. If $\mathrm{E}(\mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ then $\mathrm{A}(\mathbf{n}, \mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ and so true in $\mathcal{M}$ for some $n \in \omega$, so $\mathrm{E}(\mathbf{y})$ is true in $\mathcal{M}$. If $\mathrm{E}(\mathbf{y})$ is true in $\mathcal{M}$ then $\mathrm{A}(\mathbf{n}, \mathbf{y})$ is true in $\mathcal{M}$ for some least $\mathbf{n}$, so $\Gamma_{\mathrm{E}}[n, y]$ ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{n})$.

4.4.5. *Corollary.* **IC** is consistent with all classically true arithmetical sentences.

Proof. Lemma 4.3.3 and Theorem 4.4 relativize to ${}^{\mathcal{C}}$realizability/$\mathcal{C}$. Every ${}^{\mathcal{C}}$realizable formula is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$. Consequences in **IC** of ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ formulas are ${}^{\mathcal{C}}$realizable/$\mathcal{C}$. Classically true arithmetical sentences are ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ by Lemma 4.4.4; $0 = 1$ is not.

4.5. **Markov's Principle, Weak Kripke's Schema and a weaker alternative.** Late in his life Brouwer introduced "creating subject" arguments to refute e.g.

$$\text{MP}_1. \quad \neg\neg\exists \text{x}\alpha(\text{x}) = 0 \rightarrow \exists \text{x}\alpha(\text{x}) = 0$$

("Markov's Principle"), which is consistent with **I** but not ${}^{\mathcal{C}}$realizable or ${}^{\mathcal{C}}$realizable/$\mathcal{C}$. Efforts to formalize Brouwer's creating subject arguments led to Weak Kripke's Schema

$$\text{WKS.} \quad \exists\beta[(\exists \text{x}\beta(\text{x}) \neq 0 \rightarrow \text{A}) \ \& \ (\forall \text{x}\beta(\text{x}) = 0 \rightarrow \neg\text{A})]$$

where β is not free in A. If choice sequence variables are allowed to occur free in A then WKS (with $\forall \text{x}\alpha(\text{x}) = 0$ as the A) conflicts with CC_{11}, so **I** + WKS is inconsistent. *A fortiori*, so is **IC** + WKS. For arbitrary A without β free, a classically (but not intuitionistically) equivalent version of WKS is Weaker Weak Kripke's Schema:

$$\text{WWKS.} \quad \exists\beta[\forall \text{x}\beta(\text{x}) = 0 \leftrightarrow \neg\text{A}]$$

which conflicts with CC_{11} by the same argument.

4.5.1. *Proposition.* $\neg\neg$WWKS is ${}^{\mathcal{C}}$realizable for formulas A of $\mathcal{L}(\mathbf{IC})$ with no free β.

Proof. If A is a formula of $\mathcal{L}(\mathbf{IC})$ with only the distinct variables $\alpha, \text{c}, \text{z}$ free, let $\text{E}(\beta)$ be $[(\forall \text{x}\beta(\text{x}) = 0 \rightarrow \neg\text{A}) \ \& \ (\neg\text{A} \rightarrow \forall \text{x}\beta(\text{x}) = 0)]$. Then $\Lambda\alpha\Lambda\gamma\Lambda z\Lambda\pi\, \lambda t.0$ ${}^{\mathcal{C}}$realizes $\forall\alpha\forall \text{c}\forall \text{z}\neg\neg\exists\beta\text{E}(\beta)$. Agreement is obvious. If $\alpha, \gamma, \pi \in \mathcal{C}$, $z \in \omega$, and π ${}^{\mathcal{C}}$realizes-α, γ, z $\neg\exists\beta\text{E}(\beta)$, then no $\rho \in \mathcal{C}$ ${}^{\mathcal{C}}$realizes-α, γ, z $\text{E}(\lambda\text{t}.0)$; so no $\sigma \in \mathcal{C}$ ${}^{\mathcal{C}}$realizes-α, γ, z $\neg$A; so $\langle \Lambda\rho\Lambda\tau\lambda t.0, \Lambda\tau\Lambda x\lambda t.0\rangle$ ${}^{\mathcal{C}}$realizes-α, γ, z $\text{E}(\lambda\text{t}.1)$, contradicting the hypothesis on π.

4.5.2. *Corollary.* For all sentences A of $\mathcal{L}(\mathbf{IC})$, WWKS is classically ${}^{\mathcal{C}}$realizable.

4.5.3. *Vesley's Schema.* Richard Vesley (cf. [4]) proved that the axiom schema

$$\text{VS.} \quad \forall \text{w}[\text{Seq}(\text{w}) \rightarrow \exists\alpha(\overline{\alpha}(\text{lh}(\text{w})) = \text{w} \ \& \ \neg\text{A}(\alpha))] \ \& \ \forall\alpha[\neg\text{A}(\alpha) \rightarrow \exists\beta\text{B}(\alpha,\beta)] \rightarrow \forall\alpha\exists\beta[\neg\text{A}(\alpha) \rightarrow \text{B}(\alpha,\beta)]$$

(with β not free in $\text{A}(\alpha)$) is consistent with **I** and suffices for the refutation of MP_1 and other "creating subject" counterexamples. VS is evidently derivable by intuitionistic logic from the "independence of premise" schema (with β not free in A)

$$\text{IP.} \quad (\neg\text{A} \rightarrow \exists\beta\text{B}(\beta)) \rightarrow \exists\beta(\neg\text{A} \rightarrow \text{B}(\beta)).$$

4.5.4. *Proposition.* IP (and therefore VS) is ${}^{\mathcal{C}}$realizable.

Proof. $\Lambda\sigma\langle(\{\sigma\}[\Lambda\pi\, \lambda t.0])_0, \Lambda\delta\, (\{\sigma\}[\Lambda\pi\, \lambda t.0])_1\rangle$ is a ${}^{\mathcal{C}}$realizing functional for IP.

4.5.5. *Corollary.* **IC** + $\neg\neg$WWKS + IP is consistent.

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