INTUITIONISTIC ANALYSIS AT THE END OF TIME

JOAN RAND MOSCHOVAKIS

Abstract. Kripke recently suggested viewing the intuitionistic continuum as an expansion *in time* of a definite classical continuum. We prove the classical consistency of a three-sorted intuitionistic formal system **IC**, simultaneously extending Kleene's intuitionistic analysis **I** and a negative copy $\mathbf{C}^\circ$ of the classically correct part of **I**, with an "end of time" axiom ET asserting that no choice sequence can be guaranteed not to be pointwise equal to a definite (classical or lawlike) sequence. "Not every sequence is pointwise equal to a definite sequence" is independent of **IC**. The proofs are by $^{\mathcal{C}}$realizability interpretations based on classical ω-models $\mathcal{M} = (\omega, \mathcal{C})$ of $\mathbf{C}^\circ$.

1. Introduction

L. E. J. Brouwer agreed with Kant that the intuition of time is *a priori*, but unlike Kant he considered it the basis of all mathematical reasoning. The intuitionistic continuum is composed of *point cores* or equivalence classes of convergent sequences of rational segments or rational numbers. The *reduced continuum* consists of *definite*, "lawlike" fundamental sequences, all of whose values are determined in advance. The *full continuum* also includes point cores determined by *indefinite, unfinished* convergent sequences whose rational values are generated by successive, more or less free, choices.

Brouwer abstracted from the full continuum to the "universal spread," his intuitionistic version of Baire space. An arbitrary *choice sequence* α of natural numbers is *potentially infinite*; at any given time, only a finite initial segment of α may have been determined. This intuition justifies Brouwer's controversial continuity principles.

In contrast, as Troelstra observed in [7], lawlike sequences may allow classical logic. Now Kripke has proposed considering the intuitionistic full continuum as an expansion *in time* of the classical continuum, depending on the actions of a creating subject.

Kleene's formal system **I** of intuitionistic analysis, including countable choice, bar induction and a classically false continuity principle, is consistent relative to its neutral subsystem **B** by [4] and consistent with "there are no non-recursive sequences" by [6]. "Every sequence is recursive" is inconsistent with **B** by Lemma 9.8 of [4].

Classical analysis with countable choice **C** ($\equiv \mathbf{B} + \neg\neg\mathrm{A} \rightarrow \mathrm{A}$) is classically equivalent to its negative translation, which is consistent with **I**. Negative formulas (no $\exists$ or $\vee$) are stable under double negation even with intuitionistic logic. Let $\mathcal{M} = (\omega, \mathcal{C})$ be an ω-model of a negative version $\mathbf{C}^\circ$ of **C**. We define $^{\mathcal{C}}$realizability to prove classically that a three-sorted extension **IC** of **I** and $\mathbf{C}^\circ$ asserting "there are no indefinite sequences" is consistent, and does not decide "not every sequence is definite" provided $\mathcal{C}$ may $\neq \omega^\omega$.[1]

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[1] **C** and $\mathbf{C}^\circ$ have the same classical ω-models. $\mathcal{C}$ can be thought of informally as representing either classical Baire space or Brouwer's species of lawlike sequences. See also the last section of this paper.

2. Just the basics

For Brouwer a statement A was in general stronger than its double negation $\neg\neg$A, since intuitionistic negation expresses inconsistency. Thus (A $\rightarrow \neg\neg$A) holds in general, as does ($\neg\neg\neg$A $\rightarrow \neg$A), but not always ($\neg\neg$A $\rightarrow$ A). Even $\neg\neg$(A $\vee$ B) $\rightarrow \neg\neg$A $\vee \neg\neg$B fails under the constructive interpretation of disjunction; and while $\exists$xA(x) asserts that a witness can be designated, $\neg\neg\exists xA(x)$ says only that $\forall$x$\neg$A(x) is inconsistent.

Classical logic, on the other hand, can be formulated in a negative language with only &, $\neg, \rightarrow$ and $\forall$, since A $\vee$ B and $\exists$xA(x) are classically equivalent to $\neg(\neg$A & $\neg$B) and $\neg\forall$x$\neg$A(x) respectively. The language $\mathcal{L}(\mathbf{C}^\circ)$ of classical analysis $\mathbf{C}^\circ$ has two sorts of variables: $\mathrm{i, j, \ldots, p, q, w, x, y, z, i_1, \ldots}$ intended to range over natural numbers, and $\mathrm{a, b, c, d, e, a_1, b_1, c_1, \ldots}$ intended to range over sequences of natural numbers; constants for primitive recursive functions; Church's λ; parentheses, used both to denote function application and also to indicate the scopes of &, $\neg, \rightarrow$, $\forall$x and $\forall$b in formulas; and equality = between number terms. For ease of reading we sometimes abbreviate

$$\neg(\neg\mathrm{A} \,\&\, \neg\mathrm{B}) \text{ by } \mathrm{A} \overset{\circ}{\vee} \mathrm{B}, \quad \neg\forall\mathrm{x}\neg\mathrm{A(x)} \text{ by } \exists^\circ\mathrm{xA(x)}, \quad \text{and } \neg\forall\mathrm{b}\neg\mathrm{A(b)} \text{ by } \exists^\circ\mathrm{bA(b)}.$$

The Peano axioms are negative in form when the schema of mathematical induction is restricted to formulas of the negative language. The equality axiom x = y $\rightarrow$ b(x) = b(y) is negative. Primitive recursive functions have negative definitions. The axiom of countable choice is represented by its negative translation. *Even with intuitionistic logic* the classical law of double negation $\neg\neg$E $\rightarrow$ E holds for formulas E of this language.

The three-sorted axiomatic system **IC** combines Kleene and Vesley's intuitionistic formal system **I**, which has variables $\alpha, \beta, \gamma, \ldots$ ranging over arbitrary choice sequences, with the formal system resulting from $\mathbf{C}^\circ$ by extending its language and logic to include $\vee$, $\exists$x, $\exists$b and their intuitionistic postulates. The only new axiom explicitly connecting the two sorts of sequence variables is $\forall\alpha\neg\forall\mathrm{b}\neg\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$, or equivalently

$$\forall\alpha\neg\neg\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}.$$

The idea is that when mathematical activity has ended and all values of an arbitrary choice sequence α have been specified, *it will turn out* that α coincides with some definite (classical or "lawlike") sequence.[2] This correlation may not be made in advance; **IC** proves neither $\neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$ nor $\neg\neg\forall\alpha\exists\mathrm{b}\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}$. However, **IC** proves

$$\forall\mathrm{b}\exists\alpha\forall\mathrm{x}\ \alpha(\mathrm{x}) = \mathrm{b(x)}.$$

Thus every definite sequence is extensionally equal to a choice sequence, and "at the end of time" intuitionistic and classical Baire space will be indistinguishable.

In order to establish the consistency of **IC** we assume a classical ω-model $\mathcal{M} = (\omega, \mathcal{C})$ of $\mathbf{C}^\circ$ exists and use it to define a modified ${}^{\mathcal{C}}$realizability interpretation. The *potential* ${}^{\mathcal{C}}$realizers belong to ω^ω and the *actual* ${}^{\mathcal{C}}$realizers belong to the recursively closed set $\mathcal{C}$. All theorems of **IC** are ${}^{\mathcal{C}}$realizable but $0 = 1$ is not, so **IC** is consistent.

Kleene observed (Lemma 8.4a of [4]) that true negative sentences of the language of **I** have primitive recursive realizers. All sentences of the language of $\mathbf{C}^\circ$ which are true in $\mathcal{M}$ are ${}^{\mathcal{C}}$realized by primitive recursive functions, and thus are consistent with **IC**.

[2]This addresses an objection, from a member of the audience after Kripke's talk in Amsterdam, to the effect that the classical continuum is already complete.

3. The formal systems $\mathbf{C}^\circ$, $\mathbf{B}$, $\mathbf{I}$ and $\mathbf{IC}$

3.1. **A negative formal system $\mathbf{C}^\circ$ for classical analysis with countable choice.** The two-sorted language $\mathcal{L}(\mathbf{C}^\circ)$ was described briefly in the preceding section. Now we adopt Kleene's finite list $\mathrm{f}_0, \ldots, \mathrm{f}_p$ of constants representing selected primitive recursive functions, with $\mathrm{f}_0 = 0, \mathrm{f}_1 = {}', \mathrm{f}_2 = +, \mathrm{f}_3 = \cdot$ and $\mathrm{f}_4(\mathrm{x}, \mathrm{y}) = \mathrm{x}^\mathrm{y}$. The list, including bounded sum and bounded product, may be expanded by definition as needed.

C°-*terms* (type-0 terms) and C°-*functors* (type-1 terms) are defined simultaneously inductively. The number variables and the constant 0 are C°-*terms.* The lawlike sequence variables, the successor symbol $'$ and constants representing primitive recursive functions of one type-0 argument are C°-*functors.* If $\mathrm{f_i}$ is a constant representing a primitive recursive function of k_i type-0 and m_i type-1 variables, and if $\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}$ are C°-*terms* and $\mathrm{u}_1, \ldots, \mathrm{u}_{m_i}$ are C°-*functors*, then $\mathrm{f_i}(\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}, \mathrm{u}_1, \ldots, \mathrm{u}_{m_i})$ is a C°-*term.* If u is a C°-*functor* and t is a C°-*term* then (u)(t) (sometimes written u(t)) is a C°-*term.* If x is a number variable and s is a C°-*term* then λx(s) (sometimes written λx.s) is a C°-*functor.* This completes the definition.

The *prime formulas* are the expressions of the form $\mathrm{s} = \mathrm{t}$ where s, t are C°-*terms.* Equality at type 1 is defined extensionally, with $\mathrm{a} = \mathrm{b}$ abbreviating $\forall \mathrm{x}(\mathrm{a}(\mathrm{x}) = \mathrm{b}(\mathrm{x}))$. *Compound formulas* are built from prime formulas and both sorts of variables using $\&, \neg, \rightarrow, \forall$ and parentheses as usual. $(\mathrm{A} \leftrightarrow \mathrm{B})$ abbreviates $(\mathrm{A} \rightarrow \mathrm{B}) \,\&\, (\mathrm{B} \rightarrow \mathrm{A})$. All formulas of $\mathcal{L}(\mathbf{C}^\circ)$ are *negative* (they contain neither $\vee$ nor $\exists$).

The logical axioms and rules are Kleene's ([2], [4]) adapted to $\mathcal{L}(\mathbf{C}^\circ)$, so $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{A}(\mathrm{x})$ and $\mathrm{A}(\mathrm{b})$ are negative formulas. We retain Kleene's numbers for comparison.

1a. $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{A})$.
1b. $(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow ((\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{C})) \rightarrow (\mathrm{A} \rightarrow \mathrm{C}))$.
2. (Modus Ponens) $\mathrm{A},\ \mathrm{A} \rightarrow \mathrm{B}\ /\ \mathrm{B}$.
3. $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{A} \,\&\, \mathrm{B})$.
4a. $\mathrm{A} \,\&\, \mathrm{B} \rightarrow \mathrm{A}$.
4b. $\mathrm{A} \,\&\, \mathrm{B} \rightarrow \mathrm{B}$.
7. $(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow ((\mathrm{A} \rightarrow \neg \mathrm{B}) \rightarrow \neg \mathrm{A})$.
8^I. $\neg \mathrm{A} \rightarrow (\mathrm{A} \rightarrow \mathrm{B})$.
9N. $\mathrm{B} \rightarrow \mathrm{A}(\mathrm{x})\ /\ \mathrm{B} \rightarrow \forall \mathrm{x} \mathrm{A}(\mathrm{x})$, where x is not free in B.
10N. $\forall \mathrm{x} \mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{t})$, where t is a C°-term free for x in A(x).
$9\mathrm{C}^\circ$. $\mathrm{B} \rightarrow \mathrm{A}(\mathrm{b})\ /\ \mathrm{B} \rightarrow \forall \mathrm{b} \mathrm{A}(\mathrm{b})$, where b is not free in B.
$10\mathrm{C}^\circ$. $\forall \mathrm{b} \mathrm{A}(\mathrm{b}) \rightarrow \mathrm{A}(\mathrm{u})$, where u is a C°-functor free for b in A(b).

Mathematical axioms assert that $=$ is an equivalence relation, 0 is not a successor, $'$ is one-to-one, and $\mathrm{x} = \mathrm{y} \rightarrow \mathrm{a}(\mathrm{x}) = \mathrm{a}(\mathrm{y})$. The primitive recursive defining equations for $+, \cdot$ and $\mathrm{f}_4, \ldots, \mathrm{f}_p$ (Postulate Group D of [4], [3]) are axioms, as is the mathematical induction schema $\mathrm{A}(0) \,\&\, \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{x}')) \rightarrow \mathrm{A}(\mathrm{x})$ for formulas A(x) of $\mathcal{L}(\mathbf{C}^\circ)$. For C°-terms $\mathrm{r}(\mathrm{x}), \mathrm{t}$ the λ-reduction schema is

$$(\lambda \mathrm{x}.\mathrm{r}(\mathrm{x}))(\mathrm{t}) = \mathrm{r}(\mathrm{t}),$$

where r(t) results by substituting t for all free occurrences of x in r(x). The axiom schema of countable choice, for formulas $\mathrm{A}(\mathrm{x}, \mathrm{b})$ of $\mathcal{L}(\mathbf{C}^\circ)$ with a, x free for b, is

$$\mathrm{AC}_{01}^{\mathrm{C}^\circ}. \quad \forall \mathrm{x} \neg \forall \mathrm{b} \neg \mathrm{A}(\mathrm{x}, \mathrm{b}) \rightarrow \neg \forall \mathrm{a} \neg \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\mathrm{a}(2^\mathrm{x} \cdot 3^\mathrm{y})).$$

3.2. **Properties of $\mathbf{C}^\circ$.** To avoid unnecessary formal reasoning, first observe that the Deduction Theorem (Theorem 1 on p. 97 of [2]) holds for $\mathbf{C}^\circ$ (using the same arguments for the relevant cases), so the Hilbert-style logical axioms and rules can be replaced by natural deduction rules for $\rightarrow$, $\&$, $\neg$ and $\forall$ (as in Theorem 2 on pp. 98-99 of [2]).

3.2.1. *Lemma.* For all C°-terms s, t and all formulas A, B of $\mathcal{L}(\mathbf{C}^\circ)$, $\mathbf{C}^\circ$ proves

(a) $\neg\neg \mathrm{s} = \mathrm{t} \rightarrow \mathrm{s} = \mathrm{t}$.
(b) $\mathrm{A} \rightarrow \mathrm{A}$.
(c) $\mathrm{A} \rightarrow \neg\neg\mathrm{A}$.
(d) $(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow (\neg\mathrm{B} \rightarrow \neg\mathrm{A})$.
(e) $\neg\neg\mathrm{A} \rightarrow \mathrm{A}$.

Proofs. (a) follows (by $\forall$-introduction and then $\forall$-elimination) from $\neg\neg \mathrm{x} = \mathrm{y} \rightarrow \mathrm{x} = \mathrm{y}$ which is provable by double mathematical induction. (b) - (d) are exercises in negative propositional logic. (e) is by formula induction from the axioms and (a), (c) and (d). □

Note that (e) is Kleene's classical negation-elimination axiom schema 8°, restricted in this case to negative formulas. All the logical postulates which were omitted because they contain $\vee$ or $\exists$ have negative versions provable in $\mathbf{C}^\circ$.

3.2.2. *Lemma.* For all formulas $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{A}(\mathrm{x}), \mathrm{A}(\mathrm{b})$ of $\mathcal{L}(\mathbf{C}^\circ)$, $\mathbf{C}^\circ$ proves

5a°. $\mathrm{A} \rightarrow \mathrm{A} \overset{\circ}{\vee} \mathrm{B}$.
5b°. $\mathrm{B} \rightarrow \mathrm{A} \overset{\circ}{\vee} \mathrm{B}$.
6°. $(\mathrm{A} \rightarrow \mathrm{C}) \rightarrow ((\mathrm{B} \rightarrow \mathrm{C}) \rightarrow (\mathrm{A} \overset{\circ}{\vee} \mathrm{B} \rightarrow \mathrm{C}))$.
11N°. $\mathrm{A}(\mathrm{t}) \rightarrow \exists^\circ \mathrm{x}\mathrm{A}(\mathrm{x})$ if t is a C°-term free for x in $\mathrm{A}(\mathrm{x})$.
11C°. $\mathrm{A}(\mathrm{u}) \rightarrow \exists^\circ \mathrm{b}\mathrm{A}(\mathrm{b})$ if u is a C°-functor free for b in $\mathrm{A}(\mathrm{b})$.

Moreover, for all formulas $\mathrm{A}(\mathrm{x}), \mathrm{A}(\mathrm{b}), \mathrm{B}$ of $\mathcal{L}(\mathbf{C}^\circ)$, $\mathbf{C}^\circ$ obeys the rules

12N°. $\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{B} \;/\; \exists^\circ \mathrm{x}\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{B}$, where x is not free in B and x is held constant in the deduction of $\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{B}$.
12C°. $\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{B} \;/\; \exists^\circ \mathrm{b}\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{B}$, where b is not free in B and b is held constant in the deduction of $\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{B}$.

Proofs. 5a° follows from an instance $\neg\mathrm{A} \,\&\, \neg\mathrm{B} \rightarrow \neg\mathrm{A}$ of axiom 4a by Lemma 3.2.1(c,d) and 5b° follows from an instance of axiom 4b. For 6°, assume $\mathrm{A} \rightarrow \mathrm{C}$ and $\mathrm{B} \rightarrow \mathrm{C}$; then $\neg\mathrm{C} \rightarrow \neg\mathrm{A}$ and $\neg\mathrm{C} \rightarrow \neg\mathrm{B}$, so $\neg\mathrm{C} \rightarrow \neg\mathrm{A} \,\&\, \neg\mathrm{B}$ using axiom 3, so $\neg(\neg\mathrm{A} \,\&\, \neg\mathrm{B}) \rightarrow \mathrm{C}$ by Lemma 3.2.1(d,e). Similarly, 12N° follows from 9N, and 12C° follows from 9C°. □

3.3. **Kleene's intuitionistic formal systems B and I.** The neutral basic system **B** has axioms for two-sorted intuitionistic logic and arithmetic, countable choice and bar induction. Intuitionistic analysis **I** is **B** together with Brouwer's classically false principle of continuous choice, which is consistent relative to **B** by function-realizability.

The language resembles a richer version of $\mathcal{L}(\mathbf{C}^\circ)$. Instead of variables $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{a}_1, \ldots$ over classical sequences, $\mathcal{L}(\mathbf{B})$ ($\equiv \mathcal{L}(\mathbf{I})$) has variables $\alpha, \beta, \gamma, \delta, \alpha_1, \ldots$ intended to range over arbitrary choice sequences. In addition to $=, \lambda$, parentheses and the logical symbols $\&, \neg, \rightarrow$ and universal quantifiers $\forall \mathrm{x}, \forall \alpha$, $\mathcal{L}(\mathbf{B})$ has disjunction $\vee$ and existential quantifiers $\exists \mathrm{x}, \exists \alpha$ of both sorts. With the same constants $\mathrm{f}_0, \ldots, \mathrm{f}_p$ representing the same primitive recursive functions, the simultaneous inductive definition of *term* and *functor* is like that of C°-term and C°-functor but with $\alpha, \beta, \ldots$ in place of $\mathrm{a}, \mathrm{b}, \ldots$.

Prime formulas are expressions of the form $s = t$ where s, t are terms. Compound formulas are built from prime formulas and both sorts of variables using $\&, \neg, \rightarrow, \vee, \forall, \exists$ and parentheses as needed. $\alpha = \beta$ abbreviates the negative formula $\forall x(\alpha(x) = \beta(x))$.

The logical rules and axioms include 1a - 8^I and 9N - 12N, as for $\mathbf{C}^\circ$ except that now A, B, C and $A(x)$ may be any formulas of $\mathcal{L}(\mathbf{B})$; t is a term free for x in $A(x)$; $\vee$ and $\exists$ replace $\overset{\circ}{\vee}$ and $\exists^\circ$ respectively; and 5a, 5b, 6, 11N and 12N are postulates rather than theorems. In the following replacements for $9C^\circ$ - $12C^\circ$, $A(\beta)$ and B may be any formulas of $\mathcal{L}(\mathbf{B})$:

9F. $B \rightarrow A(\beta)$ / $B \rightarrow \forall\beta A(\beta)$ if β is not free in B.
10F. $\forall\beta A(\beta) \rightarrow A(u)$ if u is a functor free for β in $A(\beta)$.
11F. $A(u) \rightarrow \exists\beta A(\beta)$ if u is a functor free for β in $A(\beta)$.
12F. $A(\beta) \rightarrow B$ / $\exists\beta A(\beta) \rightarrow B$ if β is not free in B.

The mathematical axioms of $\mathbf{B}$ include those of $\mathbf{C}^\circ$, but with $\alpha, \beta, \ldots$ instead of $a, b, \ldots$ and with the following adaptations. For the mathematical induction schema, $A(x)$ may be any formula of $\mathcal{L}(\mathbf{B})$. For the λ-reduction schema $(\lambda x.r(x))(t) = r(t)$ both $r(x)$ and t are terms of $\mathcal{L}(\mathbf{B})$. The axiom schema of countable choice for $\mathbf{B}$ is

$$AC_{01}.\quad \forall x\exists\alpha A(x,\alpha) \rightarrow \exists\beta\forall x A(x, \lambda y.\beta(2^x \cdot 3^y))$$

where $A(x, \alpha)$ is any formula of $\mathcal{L}(\mathbf{B})$ with β, x free for α.

Brouwer's most important contributions to the foundations of intuitionistic mathematics were his "bar theorem," which is classically valid, and his continuity principle, which is not. An axiom schema of bar induction completes Kleene's neutral system $\mathbf{B}$, and the full intuitionistic system $\mathbf{I}$ comes from $\mathbf{B}$ by adding a principle of continuous choice. These are more complicated to state.

Finite sequences of natural numbers are coded formally using the function constants of $\mathcal{L}(\mathbf{B})$. In [4] $f_{19}(i) = p_i$ denotes the ith prime, with $p_0 = 2$; $f_{20}(y, i) = (y)_i$ denotes the exponent of p_i in the prime factorization of y; and $\langle x_0, \ldots, x_k\rangle$ abbreviates $\Pi_{i<k} p_i^{x_i}$. Let $Seq(y)$ abbreviate $\forall i < lh(y)\ (y)_i > 0$, where $lh(y)$ is a term denoting the number of nonzero exponents in the prime factorization of y. Then $\langle\,\rangle = 1$ codes the empty sequence; $\langle x_0 + 1, \ldots, x_k + 1\rangle$ codes the sequence $(x_0, \ldots, x_k)$; the concatenation of the finite sequences coded by w and z (assuming $Seq(w) \,\&\, Seq(z)$) is coded by $w * z$; and $w * \alpha$ codes the sequence defined by prefixing the finite sequence coded by w to α.

Let $\overline{\alpha}(n)$ abbreviate the code $\Pi_{i<n} p_i^{\alpha(i)+1}$ of the initial segment of α of length n (so $\overline{\alpha}(0) = 1$). The last axiom schema of $\mathbf{B}$ is the principle of bar induction (with a thin bar, essentially ^{x}26.3c on p. 55 of [4]), where $\exists!xR(\overline{\alpha}(x))$ is an abbreviation for $\exists x(R(\overline{\alpha}(x)) \,\&\, \forall y(R(\overline{\alpha}(y)) \rightarrow y = x))$:

$$BI!.\quad \forall\alpha\exists!xR(\overline{\alpha}(x)) \,\&\, \forall w[Seq(w) \,\&\, (R(w) \vee \forall nA(w * \langle n+1\rangle)) \rightarrow A(w)] \rightarrow A(\langle\,\rangle).$$

This description of Kleene's neutral basic system $\mathbf{B}$ of intuitionistic analysis summarizes Postulate Groups A-D, §§1-6 of [4].

The full intuitionistic system $\mathbf{I}$ comes from $\mathbf{B}$ by adding a principle of continuous choice ("Brouwer's principle for a function," cf. ^{x}27.1 on p. 73 of [4]):

$$CC_{11}.\quad \forall\alpha\exists\beta A(\alpha,\beta) \rightarrow \exists\sigma\forall\alpha(\forall x\exists!y\sigma(\langle x+1\rangle * \overline{\alpha}(y)) > 0 \,\&\, \forall\beta(\forall x\exists y\sigma(\langle x+1\rangle * \overline{\alpha}(y)) = \beta(x)+1 \rightarrow A(\alpha,\beta))).$$

3.3.1. *The classical version* **C** *of* **B**. A formal system **C** of classical analysis, with the axiom of countable choice, results from **B** by omitting BI! and replacing 8^{I} (*ex falso sequitur quodlibet*) by 8°. $\neg\neg\mathrm{E} \rightarrow \mathrm{E}$ for all formulas E of $\mathcal{L}(\mathbf{B})$. Because BI! follows from AC_{01} by classical logic (*26.1° on p. 53 of [4]), **B** is a subsystem of **C**.

Clearly **C** is inconsistent with **I**. The negative translation of AC_{01}, which is consistent with **I** by Kleene's function-realizability, is not a theorem schema of **B** so **C** cannot be interpreted negatively in its subsystem **B**.

3.3.2. *Proposition.* There is a faithful negative translation $\mathrm{A} \mapsto \mathrm{A}^{\mathrm{tr}}$ of **C** to $\mathbf{C}^\circ$.

Proof. In each formula A of $\mathcal{L}(\mathbf{C})$ ($\equiv \mathcal{L}(\mathbf{B})$) first replace $\vee, \exists\mathrm{x}, \exists\beta$ by $\overset{\circ}{\vee}, \exists^\circ\mathrm{x}, \exists^\circ\beta$ to obtain A', then replace $\alpha, \beta, \ldots$ by $\mathrm{a}, \mathrm{b}, \ldots$ to obtain A^{tr}. Clearly **C** proves $\mathrm{A}' \leftrightarrow \mathrm{A}$.

The translation E^{tr} of each axiom E of **C** is an axiom of $\mathbf{C}^\circ$, or a theorem of $\mathbf{C}^\circ$ by Lemma 3.2.1(e) or Lemma 3.2.2. Since the translation of every logical rule of **C** is an admissible rule of $\mathbf{C}^\circ$, deductions in **C** can be replaced by corresponding deductions in $\mathbf{C}^\circ$. It follows that A^{tr} is a theorem of $\mathbf{C}^\circ$ if and only if A is a theorem of **C**. □

3.4. **The formal system IC.** In order to compare the intuitionistic continuum with a definite (either classical or "reduced") continuum, two sorts of sequence variables are needed. One sort of number variables suffices, but in **IC** even arithmetical formulas will not always be provably equivalent to their negative translations.

The three-sorted language $\mathcal{L}(\mathbf{IC})$ extending both $\mathcal{L}(\mathbf{C}^\circ)$ and $\mathcal{L}(\mathbf{I})$ has three sorts of variables with or without subscripts, also used as metavariables:

$\mathrm{i}, \mathrm{j}, \mathrm{k}, \ldots, \mathrm{p}, \mathrm{q}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z}$ over natural numbers,
$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}$ over definite (classical or "lawlike") sequences,
$\alpha, \beta, \gamma, \ldots$ over arbitrary choice sequences;

finitely many constants $\mathrm{f}_0 = 0$, $\mathrm{f}_1 = {}'$ (successor), $\mathrm{f}_2 = +$, $\mathrm{f}_3 = \cdot$, $\mathrm{f}_4 = \exp$, $\mathrm{f}_5, \ldots, \mathrm{f}_p$ for primitive recursive functions and functionals; the binary predicate constant =; Church's λ denoting function abstraction; parentheses (,) denoting function application; and the logical symbols $\&, \vee, \rightarrow, \neg$ and quantifiers $\forall$ and $\exists$ over each sort of variable.

Terms and *functors* are defined simultaneously inductively as for **B**, except that now all definite sequence variables and all arbitrary choice sequence variables are functors. *Prime formulas* are of the form $\mathrm{s} = \mathrm{t}$ where s, t are terms. If u, v are functors then $\mathrm{u} = \mathrm{v}$ abbreviates $\forall\mathrm{x}\,\mathrm{u}(\mathrm{x}) = \mathrm{v}(\mathrm{x})$. Composite formulas are formed as usual.

Terms, functors and formulas with no occurrences of arbitrary choice sequence variables are C-*terms*, C-*functors* and C-*formulas*, respectively.

The logical axioms and rules of **I** carry over to **IC**, where the $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{A}(\mathrm{x}), \mathrm{A}(\beta)$ may now be any formulas of $\mathcal{L}(\mathbf{IC})$. In addition, **IC** has logical rules 9C and 12C, and axiom schemas 10C and 11C, for all formulas B, A(b) of $\mathcal{L}(\mathbf{IC})$:

9C. $\mathrm{B} \rightarrow \mathrm{A}(\mathrm{b})\ /\ \mathrm{B} \rightarrow \forall\mathrm{b}\mathrm{A}(\mathrm{b})$ where b is not free in B.
10C. $\forall\mathrm{b}\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{A}(\mathrm{u})$ where u is a C-functor free for b in A(b).
11C. $\mathrm{A}(\mathrm{u}) \rightarrow \exists\mathrm{b}\mathrm{A}(\mathrm{b})$ where u is a C-functor free for b in A(b).
12C. $\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{B}\ /\ \exists\mathrm{b}\mathrm{A}(\mathrm{b}) \rightarrow \mathrm{B}$ where b is not free in B.

The mathematical axioms of **I**, and $\mathrm{AC}_{01}^{\mathbf{C}^\circ}$ for negative C-formulas $\mathrm{A}(\mathrm{x}, \mathrm{b})$, become axioms of **IC**. For the λ-reduction schema, r(x) and t may be any terms of $\mathcal{L}(\mathbf{IC})$. A(x) in the mathematical induction schema, $\mathrm{A}(\mathrm{x}, \alpha)$ in AC_{01}, R(w) and A(w) in BI!, and $\mathrm{A}(\alpha, \beta)$ in CC_{11} may be any formulas of $\mathcal{L}(\mathbf{IC})$ satisfying the conditions of the schemas.

3.4.1. *Lemma.* For terms s, t and formulas A, B of $\mathcal{L}(\mathbf{IC})$, parts (a)-(d) of Lemma 3.2.1 also hold for **IC**. In addition, **IC** proves

(e) $\neg\neg A \rightarrow A$ if A is negative (no $\exists$ or $\vee$).
(f) $A \vee \neg A$ if A is quantifier-free.

3.4.2. *Lemma.* **IC** proves $\forall b \exists \alpha \forall x [b(x) = \alpha(x)]$.

Proof. An easy consequence of AC_{01} is

$$AC_{00}. \quad \forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)).$$

From $b(x) = b(x)$ conclude $\exists y[b(x) = y]$, then use $\forall$x-introduction, AC_{00}, Modus Ponens and $\forall$b-introduction. □

Brouwer's arbitrary choice sequences included his lawlike sequences. In **IC** each definite sequence, all of whose values are fixed in advance, can be imitated by a choice sequence under construction.

3.4.3. *The end of time axiom.* The only completely new axiom of **IC** is

$$ET. \quad \forall \alpha \neg \forall b \neg \forall x [\alpha(x) = b(x)],$$

which is equivalent in **IC** to $\forall \alpha \neg\neg \exists b \forall x \; \alpha(x) = b(x)$. The intuitionistic double negation expresses *persistent consistency*; as the values of an arbitrary choice sequence α are chosen one by one, the possibility that α may coincide with a definite (classical or lawlike) sequence can never be excluded.

If ET were strengthened to $\forall \alpha \exists b \forall x \; \alpha(x) = b(x)$ and $\neg\neg E \rightarrow E$ was assumed for all C-formulas E, the result would be inconsistent by the following result.

3.4.4. *Proposition.* **IC** proves

(a) $\forall b \neg\neg(\forall x b(x) = 0 \vee \neg \forall x b(x) = 0)$.
(b) $\neg \forall \alpha (\forall x \alpha(x) = 0 \vee \neg \forall x \alpha(x) = 0)$.

Proofs. (a) holds because $\neg\neg(A \vee \neg A)$ is a theorem of intuitionistic logic. (b) holds because because **IC** proves $A \vee \neg A \rightarrow \exists y (y = 0 \leftrightarrow A)$ and "Brouwer's principle for a number" (*27.2 of [4]):

$$CC_{10}. \; \forall \alpha \exists x A(\alpha, x) \rightarrow \exists \sigma \forall \alpha (\exists ! y \sigma(\overline{\alpha}(y)) \neq 0 \,\&\, \forall x \forall z (\exists y \sigma(\overline{\alpha}(y)) = z + 1 \rightarrow A(\alpha, z))).$$

4. $^{\mathcal{C}}$REALIZABILITY AND THE CONSISTENCY OF IC

4.1. **From now on, assume that $\mathcal{M} = (\omega, \mathcal{C})$ is a classical ω-model of $\mathbf{C}^{\circ}$.** Then $\mathcal{M}$ is also an ω-model of **B** and **C** (cf. §3.3.1 above) under the classical interpretation of $\vee$ and $\exists$. Observe that $\mathcal{C}$ is closed under "recursive in," i.e. if γ is recursive in finitely many elements of $\mathcal{C}$ then $\gamma \in \mathcal{C}$, so C-functors represent elements of $\mathcal{C}$.

For the proof that **IC** is consistent it is not necessary to assume $\mathcal{C}$ is countable, or even that $\mathcal{C} \neq \omega^\omega$. The proof that **IC** is consistent with $\neg \forall \alpha \exists b \forall x \; \alpha(x) = b(x)$, on the other hand, will depend on the additional assumption $\mathcal{C} \neq \omega^\omega$.

Kleene's curly bracket and Λ notations are described in Section 8 of [4]. Briefly, $\{\alpha\}[\beta](x)$ is *defined and equal to* y ($\{\alpha\}[\beta](x) \simeq y$) if for some z: $\alpha(\langle x+1 \rangle * \overline{\beta}(z)) = y+1$ and $\alpha(\langle x + 1 \rangle * \overline{\beta}(j)) = 0$ for all $j < z$. Thus $\{\alpha\}[\beta]$ is a recursive partial functional of α and β. In general, $\{\alpha\}[x]$ will abbreviate $\{\alpha\}[\lambda t.x]$, and $\{\alpha\}$ will abbreviate $\{\alpha\}[0]$.

If $\Phi[\alpha,\beta,x,y]$ is a partial functional of the indicated variables which is recursive in functions Δ, by Kleene's enumeration theorem there are index functions $\Lambda\alpha\,\Phi[\alpha,\beta,x,y]$, $\Lambda x\,\Phi[\alpha,\beta,x,y]$, $\Lambda\,\Phi[\alpha,\beta,x,y]$ primitive recursive in Δ such that for all α,β,x,y,z:

$$(\{\Lambda\alpha\,\Phi[\alpha,\beta,x,y]\}[\alpha])(z) \simeq (\Phi[\alpha,\beta,x,y])(z) \simeq (\{\Lambda x\,\Phi[\alpha,\beta,x,y]\}[x])(z)$$

and $\{\Lambda\,\Phi[\alpha,\beta,x,y]\} \simeq \Phi[\alpha,\beta,x,y]$. Similarly for $\Phi[\alpha_1,\ldots,\alpha_j,x_1,\ldots,x_k,y_1,\ldots,y_m]$.

The fundamental difference between modified and plain realizability was described elegantly by van Oosten: modified realizability requires *two* sets of realizers, the *potential* realizers and the *actual* realizers. As in [6] we avoid explicitly assigning types to our potential realizers via a notion of "agreement" which makes the types implicit.

4.2. **Definition.** By induction on the logical form of a formula E of $\mathcal{L}(\mathbf{IC})$ we define when $\varepsilon \in \omega^\omega$ *agrees with* E, as follows, where $(\varepsilon)_i$ abbreviates $\lambda y.(\varepsilon(y))_i$.

(1) ε *agrees with* a prime formula s = t, for each ε.
(2) ε *agrees with* A & B, if $(\varepsilon)_0$ *agrees with* A and $(\varepsilon)_1$ *agrees with* B.
(3) ε *agrees with* A $\vee$ B, if $(\varepsilon(0))_0 = 0$ implies that $(\varepsilon)_1$ *agrees with* A, while $(\varepsilon(0))_0 \neq 0$ implies that $(\varepsilon)_1$ *agrees with* B.
(4) ε *agrees with* A $\to$ B, if, whenever α *agrees with* A, $\{\varepsilon\}[\alpha]$ is completely defined and *agrees with* B.
(5) ε *agrees with* $\neg$A, if ε *agrees with* A $\to$ 1 = 0 by the preceding clause.
(6) ε *agrees with* $\exists$xA(x), if $(\varepsilon)_1$ *agrees with* A(x).
(7) ε *agrees with* $\forall$xA(x), if, for each x, $\{\varepsilon\}[x]$ is completely defined and *agrees with* A(x).
(8) ε *agrees with* $\exists$bA(b), if $\{(\varepsilon)_0\}$ is completely defined and belongs to $\mathcal{C}$, and $(\varepsilon)_1$ *agrees with* A(b).
(9) ε *agrees with* $\forall$bA(b), if, for each $\beta \in \mathcal{C}$, $\{\varepsilon\}[\beta]$ is completely defined and *agrees with* A(b).
(10) ε *agrees with* $\exists\alpha$A(α), if $\{(\varepsilon)_0\}$ is completely defined and $(\varepsilon)_1$ *agrees with* A(α).
(11) ε *agrees with* $\forall\alpha$A(α), if, for each sequence $\alpha \in \omega^\omega$, $\{\varepsilon\}[\alpha]$ is completely defined and *agrees with* A(α).

4.2.1. *Lemma.*

(a) If s is a term free for y in A(y), then ε agrees with A(y) if and only if ε agrees with A(s). Similarly if v is a functor free for β in A(β), or u is a C-functor free for b in A(b).
(b) ε agrees with E if and only if ε agrees with the result of replacing each part of E of the form $\neg$A by (A $\to$ 1 = 0).
(c) For each formula E of $\mathcal{L}(\mathbf{IC})$ there is a primitive recursive function ε^{E} which agrees with E.

Proofs. By induction on the logical form of E. Only (c) is nontrivial. If E is prime then ε^{E} is $\lambda t.0$. Given ε^{A} and ε^{B} agreeing with A and B respectively, let $\varepsilon^{\mathrm{A\&B}} = \langle \varepsilon^{\mathrm{A}}, \varepsilon^{\mathrm{B}} \rangle$, $\varepsilon^{\mathrm{A\vee B}} = \langle \lambda t.0, \varepsilon^{\mathrm{A}} \rangle$, $\varepsilon^{\mathrm{A\to B}} = \Lambda\alpha\,\varepsilon^{\mathrm{B}}$ and $\varepsilon^{\neg\mathrm{A}} = \Lambda\pi\lambda t.0$. Given $\varepsilon^{\mathrm{A(x)}}$ agreeing with A(x), let $\varepsilon^{\exists\mathrm{xA(x)}} = \langle \lambda t.0, \varepsilon^{\mathrm{A(x)}} \rangle$ and $\varepsilon^{\forall\mathrm{xA(x)}} = \Lambda x\,\varepsilon^{\mathrm{A(x)}}$. Given $\varepsilon^{\mathrm{A(b)}}$, let $\varepsilon^{\exists\mathrm{bA(b)}} = \langle \Lambda\,\lambda t.0, \varepsilon^{\mathrm{A(b)}} \rangle$ and $\varepsilon^{\forall\mathrm{bA(b)}} = \Lambda\beta\,\varepsilon^{\mathrm{A(b)}}$. Given $\varepsilon^{\mathrm{A}(\alpha)}$, let $\varepsilon^{\exists\alpha\mathrm{A}(\alpha)} = \langle \Lambda\,\lambda t.0, \varepsilon^{\mathrm{A}(\alpha)} \rangle$ and $\varepsilon^{\forall\alpha\mathrm{A}(\alpha)} = \Lambda\alpha\,\varepsilon^{\mathrm{A}(\alpha)}$. $\square$

4.3. **Definition.** By induction on the logical form of a formula E of $\mathcal{L}(\mathbf{IC})$ containing free at most the distinct variables Ψ we define when a sequence ε, *belonging to* $\mathcal{C}$, ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ E, where $\mathit{\Psi}$ are elements of ω, $\mathcal{C}$ and $\mathcal{C}$ corresponding respectively to the number, lawlike sequence, and choice sequence variables in the list Ψ, as follows.

(1) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ a prime formula P, if P is true-$\mathit{\Psi}$ in $\mathcal{M}$.
(2) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ A & B, if $(\varepsilon)_0$ ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ A and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ B.
(3) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ A $\vee$ B, if either $(\varepsilon(0))_0 = 0$ and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ A, or $(\varepsilon(0))_0 \neq 0$ and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ B.
(4) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ A $\to$ B, if ε agrees with A $\to$ B and, whenever $\alpha \in \mathcal{C}$ and α ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ A, $\{\varepsilon\}[\alpha]$ is completely defined and ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ B.
(5) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ $\neg$A, if ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ A $\to 1 = 0$ by the preceding clause.
(6) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ $\exists$xA(x), if $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}, (\varepsilon(0))_0$ A(x).
(7) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ $\forall$xA(x), if, for each natural number n, $\{\varepsilon\}[n]$ is completely defined and ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}, n$ A(x).
(8) ε ${}^{\mathcal{C}}$*realizes* $\exists$bA(b), if $\{(\varepsilon)_0\}$ is completely defined (hence belongs to $\mathcal{C}$) and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}, \{(\varepsilon)_0\}$ A(b).
(9) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ $\forall$bA(b), if, for each sequence $\beta \in \mathcal{C}$, $\{\varepsilon\}[\beta]$ is completely defined and ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}, \beta$ A(b).
(10) ε ${}^{\mathcal{C}}$*realizes* $\exists\alpha$A(α), if $\{(\varepsilon)_0\} \in \mathcal{C}$ and $(\varepsilon)_1$ ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}, \{(\varepsilon)_0\}$ A(α).
(11) ε ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}$ $\forall\alpha$A(α), if ε agrees with $\forall\alpha$A(α) and, for each $\beta \in \mathcal{C}$, $\{\varepsilon\}[\beta]$ (is completely defined and) ${}^{\mathcal{C}}$*realizes*-$\mathit{\Psi}, \beta$ A(α).

A sentence E of $\mathcal{L}(\mathbf{IC})$ is ${}^{\mathcal{C}}$*realizable* if and only if E is ${}^{\mathcal{C}}$realized by some *general recursive* sequence ε, and a formula is ${}^{\mathcal{C}}$*realizable* if and only if its universal closure is ${}^{\mathcal{C}}$realizable.

4.3.1. *Lemma.* Let $\mathit{\Psi}$ be a list of numbers and elements of $\mathcal{C}$.

(a) If ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ a formula E of $\mathcal{L}(\mathbf{IC})$, then ε agrees with E and $\varepsilon \in \mathcal{C}$.
(b) ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ a formula E of $\mathcal{L}(\mathbf{IC})$ if and only if ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ the result of replacing each part of E of the form $\neg$A by (A $\to 1 = 0$).
(c) For no formula E of $\mathcal{L}(\mathbf{IC})$ and no sequences $\varepsilon_1, \varepsilon_2 \in \mathcal{C}$ is it the case that ε_1 ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ E and ε_2 ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ $\neg$E.

4.3.2. *Lemma.*

(a) Let A(y) be a formula of $\mathcal{L}(\mathbf{IC})$ containing free at most the distinct variables Ψ, y, let s be a term containing free at most Ψ, y and free for y in A(y), and let $s(\mathit{\Psi}, y)$ be the number expressed by s when Ψ, y take the values $\mathit{\Psi}, y$ in $\mathcal{C}$ and ω. Let $\varepsilon \in \mathcal{C}$. Then ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}, y$ A(s) if and only if ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}, s(\mathit{\Psi}, y)$ A(y).
(b) Let A(β) be a formula of $\mathcal{L}(\mathbf{IC})$ containing free at most the distinct variables Ψ, β, let u be a functor containing free at most Ψ, β and free for β in A(β), and let $u[\mathit{\Psi}, \beta]$ be the element of $\mathcal{C}$ expressed by u when Ψ, β take the values $\mathit{\Psi}, \beta$ in $\mathcal{C}$ and ω. Let $\varepsilon \in \mathcal{C}$. Then ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}, \beta$ A(u) if and only if ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}, u[\mathit{\Psi}, \beta]$ A(β). Similarly for A(b) where u is a C-functor free for b.

4.3.3. *Lemma.* Let E $\equiv$ E($\alpha_1, \ldots, \alpha_{\mathrm{j}}, \mathrm{b}_1, \ldots, \mathrm{b}_{\mathrm{k}}, \mathrm{y}_1, \ldots, \mathrm{y}_{\mathrm{m}}$) be a formula of $\mathcal{L}(\mathbf{IC})$ with only the indicated distinct variables free. Then E is ${}^{\mathcal{C}}$realizable if and only if there is a recursive partial functional $\mathit{\Phi}[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$ such that, for all $\alpha_1, \ldots, \alpha_j \in \omega^\omega$, all $\beta_1, \ldots, \beta_k \in \mathcal{C}$ and all $y_1, \ldots, y_m \in \omega$:

(a) $\Phi[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$ is defined (so belongs to ω^ω) and agrees with $\mathrm{E}(\alpha_1, \ldots, \alpha_\mathrm{j}, \mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m})$.
(b) If also $\alpha_1, \ldots, \alpha_j \in \mathcal{C}$ then $\Phi[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$ belongs to $\mathcal{C}$ and ${}^{\mathcal{C}}$realizes-$\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m$ E.

4.3.4. *Lemma.* For every negative C-formula E of $\mathcal{L}(\mathbf{IC})$ (so for every formula of $\mathcal{L}(\mathbf{C}^\circ)$) with only the distinct variables Ψ free there is a primitive recursive function τ_E such that τ_E agrees with E, and for each interpretation $\mathit{\Psi}$ of Ψ by elements of $\mathcal{C}$ and ω:

(a) If E is ${}^{\mathcal{C}}$realized-$\mathit{\Psi}$ by some $\varepsilon \in \mathcal{C}$ then E is true-$\mathit{\Psi}$ in $\mathcal{M}$.
(b) If E is true-$\mathit{\Psi}$ in $\mathcal{M}$ then τ_E ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ E.

Proof. For each negative C-formula E let τ_E be the primitive recursive function ε^E defined in proving Lemma 4.2.1(c). τ_E agrees with E by the lemma, and satisfies (a) and (b) by formula induction. We give the case for $\mathrm{E} \equiv \neg\mathrm{A}$. Assume τ_A satisfies (a) and (b) for A. If ($\varepsilon \in \mathcal{C}$ and) ε ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ ¬A, then (since $0 = 1$ is false in $\mathcal{M}$) no $\delta \in \mathcal{C}$ can ${}^{\mathcal{C}}$realize-$\mathit{\Psi}$ A, so A is false-$\mathit{\Psi}$ in $\mathcal{M}$ by (b) for τ_A, so ¬A is true-$\mathit{\Psi}$ in $\mathcal{M}$, so (a) holds for ¬A. If ¬A is true-$\mathit{\Psi}$ in $\mathcal{M}$ then A is false-$\mathit{\Psi}$ in $\mathcal{M}$, so no $\varepsilon \in \mathcal{C}$ can ${}^{\mathcal{C}}$realize-$\mathit{\Psi}$ A by (a) for τ_A, so $\tau_{\neg\mathrm{A}} = \Lambda\pi\, \lambda t.0$ ${}^{\mathcal{C}}$realizes-$\mathit{\Psi}$ ¬A, so (b) holds for ¬A. □

4.4. **Theorem.** If $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}, \mathrm{E}$ are formulas of $\mathcal{L}(\mathbf{IC})$ such that $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n} \vdash_{\mathbf{IC}} \mathrm{E}$ and $\mathrm{F}_1, \ldots, \mathrm{F}_\mathrm{n}$ are all ${}^{\mathcal{C}}$ realizable, then E is ${}^{\mathcal{C}}$realizable. Therefore, **IC** is consistent.

Proof. For each axiom or axiom schema of **IC** containing free at most the distinct variables in the list $\Psi = \alpha_1, \ldots, \alpha_\mathrm{j}, \mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m}$ we give a ${}^{\mathcal{C}}$realizing functional $\Phi[\mathit{\Psi}] = \Phi[\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, y_1, \ldots, y_m]$, as in Lemma 4.3.3; and assuming that such a $\Phi'[\mathit{\Psi}']$ exists for each premise of a rule of inference, we provide a $\Phi[\mathit{\Psi}]$ for the conclusion.

Logical Axioms 1a, 1b, 3-7, 10N, 11N, 10F, 11F (exactly as in [4]) and 10C, 11C:

1a. $\mathrm{A} \to (\mathrm{B} \to \mathrm{A})$. $\Lambda\alpha\Lambda\beta\,\alpha$.
1b. $(\mathrm{A} \to \mathrm{B}) \to ((\mathrm{A} \to (\mathrm{B} \to \mathrm{C})) \to (\mathrm{A} \to \mathrm{C}))$. $\Lambda\pi\Lambda\rho\Lambda\alpha\{\{\rho\}[\alpha]\}[\{\pi\}[\alpha]]$.
3. $\mathrm{A} \to (\mathrm{B} \to \mathrm{A}\ \&\ \mathrm{B})$. $\Lambda\alpha\Lambda\beta\,\langle\alpha, \beta\rangle$.
4a. $\mathrm{A}\ \&\ \mathrm{B} \to \mathrm{A}$. $\Lambda\alpha(\alpha)_0$.
4b. $\mathrm{A}\ \&\ \mathrm{B} \to \mathrm{B}$. $\Lambda\alpha(\alpha)_1$.
5a. $\mathrm{A} \to \mathrm{A} \vee \mathrm{B}$. $\Lambda\alpha\,\langle\lambda t.0, \alpha\rangle$.
5b. $\mathrm{B} \to \mathrm{A} \vee \mathrm{B}$. $\Lambda\alpha\,\langle\lambda t.1, \alpha\rangle$.
6. $(\mathrm{A} \to \mathrm{C}) \to ((\mathrm{B} \to \mathrm{C}) \to (\mathrm{A} \vee \mathrm{B} \to \mathrm{C}))$.
$\Lambda\pi\Lambda\rho\Lambda\delta\,\lambda t.(1 \dot{-} (\delta(0))_0)\{\pi\}[(\delta)_1](t) + (\delta(0)_0)\{\rho\}[(\delta)_1](t)$.
7. $(\mathrm{A} \to \mathrm{B}) \to ((\mathrm{A} \to \neg\mathrm{B}) \to \neg\mathrm{A})$. Same as for 1b.
8. $\neg\mathrm{A} \to (\mathrm{A} \to \mathrm{B})$. $\Lambda\delta\Lambda\alpha\,\varepsilon^\mathrm{B}$.
10N. $\forall\mathrm{x}\mathrm{A}(\mathrm{x}) \to \mathrm{A}(\mathrm{t})$ where $\mathrm{t}(\Psi)$ is a term free for x in A(x). $\Lambda\delta\,\{\delta\}[t(\mathit{\Psi})]$.
11N. $\mathrm{A}(\mathrm{t}) \to \exists\mathrm{x}\mathrm{A}(\mathrm{x})$ where $\mathrm{t}(\Psi)$ is a term free for x in A(x). $\Lambda\delta\,\langle\lambda y.t(\mathit{\Psi}), \delta\rangle$.
10C. $\forall\mathrm{b}\mathrm{A}(\mathrm{b}) \to \mathrm{A}(\mathrm{u})$ where $\mathrm{u}[\Psi] = \mathrm{u}[\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m}]$ is a C-functor free for b in A(b). $\Lambda\delta\,\{\delta\}[u(\beta_1, \ldots, \beta_k, y_1, \ldots, y_m)]$.
11C. $\mathrm{A}(\mathrm{u}) \to \exists\mathrm{b}\mathrm{A}(\mathrm{b})$ where $\mathrm{u}[\Psi] = \mathrm{u}[\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{k}, \mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{m}]$ is a C-functor free for b in A(b). $\Lambda\delta\langle\Lambda\,u[\beta_1, \ldots, \beta_k, y_1, \ldots, y_m], \delta\rangle$.
10F. $\forall\alpha\mathrm{A}(\alpha) \to \mathrm{A}(\mathrm{u})$ where $\mathrm{u}[\Psi]$ is a functor free for α in $\mathrm{A}(\alpha)$. $\Lambda\delta\,\{\delta\}[u[\mathit{\Psi}]]$.
11F. $\mathrm{A}(\mathrm{u}) \to \exists\alpha\mathrm{A}(\alpha)$ where $\mathrm{u}[\Psi]$ is a functor free for α in $\mathrm{A}(\alpha)$. $\Lambda\delta\,\langle\Lambda\,u[\mathit{\Psi}], \delta\rangle$.

Axioms for 3-sorted intuitionistic number theory: As in [4], $\lambda t.0$, $\Lambda\pi\lambda t.0$ and $\Lambda\pi\Lambda\sigma\lambda t.0$ take care of the prime axioms, including $(\lambda \mathrm{x.r(x)})(\mathrm{t}) = \mathrm{r(t)}$; $\mathrm{x} = \mathrm{y} \rightarrow \alpha(\mathrm{x}) = \alpha(\mathrm{y})$ and axioms 14, 15, 17 from [2]; and axiom 16 from [2], respectively.

The mathematical induction schema (13 in [2]) is $\mathrm{A}(0)\ \&\ \forall \mathrm{x}(\mathrm{A(x)} \rightarrow \mathrm{A(x')}) \rightarrow \mathrm{A(x)}$. A ${}^{\mathcal{C}}$realizing functional is $\Lambda\pi\rho[x,\pi]$ where ρ is defined by the functional recursion $\rho[0,\pi] = (\pi)_0$ and $\rho[x',\pi] = \{\{(\pi)_1\}[x]\}[\rho[x,\pi]]$ (cf. [4], page 106).

Axiom of countable choice: AC_{01}. $\forall \mathrm{x}\exists\alpha \mathrm{A}(\mathrm{x},\alpha) \rightarrow \exists\beta\forall \mathrm{x}\mathrm{A}(\mathrm{x}, \lambda \mathrm{y}.\beta(2^{\mathrm{x}}\cdot 3^{\mathrm{y}}))$. Exactly as in [4], $\Lambda\pi\langle \Lambda\,\lambda z.\{(\{\pi\}[(z)_0])_0\}((z)_1), \Lambda x\,(\{\pi\}[x])_1\rangle$. Agreement is obvious. Assume $\pi \in \mathcal{C}$ and π ${}^{\mathcal{C}}$realizes-Ψ $\forall \mathrm{x}\exists\alpha\mathrm{A}(\mathrm{x},\alpha)$. Then $\langle \Lambda\,\lambda z.\{(\{\pi\}[(z)_0])_0\}((z)_1), \Lambda x\,(\{\pi\}[x])_1\rangle$ is in $\mathcal{C}$ and ${}^{\mathcal{C}}$realizes-Ψ $\exists\beta\forall\mathrm{x}\mathrm{A}(\mathrm{x},\lambda\mathrm{y}.\beta(2^{\mathrm{x}}\cdot 3^{\mathrm{y}}))$.

Countable choice for $\mathbf{C}^\circ$: $\mathrm{AC}_{01}^{\mathrm{C}^\circ}$. $\forall\mathrm{x}\neg\forall\mathrm{b}\neg\mathrm{A}(\mathrm{x},\mathrm{b}) \rightarrow \neg\forall\mathrm{a}\neg\forall\mathrm{x}\mathrm{A}(\mathrm{x},\lambda\mathrm{y}.\mathrm{a}(2^{\mathrm{x}}\cdot 3^{\mathrm{y}}))$ where $\mathrm{A}(\mathrm{x},\mathrm{b})$ is a (negative) formula of $\mathcal{L}(\mathbf{C}^\circ)$. Use Lemma 4.3.4.

End of time axiom: ET. $\forall\alpha\neg\forall\mathrm{b}\neg\forall\mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$. $\varepsilon^{\forall\alpha\neg\forall\mathrm{b}\neg\forall\mathrm{x}[\alpha(\mathrm{x})=\mathrm{b}(\mathrm{x})]} = \Lambda\alpha\Lambda\pi\,\lambda t.0$ agrees with the axiom by Lemma 4.2.1(b,c) and ${}^{\mathcal{C}}$realizes the axiom because if $\alpha \in \mathcal{C}$ then $\Lambda\pi\,\lambda t.0$ is in $\mathcal{C}$ and ${}^{\mathcal{C}}$realizes-α $\neg\forall\mathrm{b}\neg\forall\mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$, since no $\pi \in \mathcal{C}$ ${}^{\mathcal{C}}$realizes-α,α $\neg\forall\mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$ so no $\pi\in\mathcal{C}$ ${}^{\mathcal{C}}$realizes-α $\forall\mathrm{b}\neg\forall\mathrm{x}[\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})]$.

Bar induction BI!: $\Lambda\pi\zeta[\pi,1]$ where $\zeta[\pi,w]$ is a recursive partial function defined using the recursion theorem. Let $G(\pi,w)$ abbreviate "$w = \overline{(w*\lambda t.0)}((\{(\pi)_0\}[w*\lambda t.0](0))_0)$," $H(\pi,w)$ abbreviate "$Seq(w)\,\&\,lh(w) \geq (\{(\pi)_0\}[w*\lambda t.0](0))_0$," and $J(\pi,w)$ abbreviate "$G(\pi,w)\,\&\,\forall u,v < w(u*v = w \rightarrow \neg G(\pi,u))$." If π ${}^{\mathcal{C}}$realizes-Ψ the hypothesis of BI! then $\forall\alpha\exists! x_\alpha J(\pi,\overline{\alpha}(x_\alpha))$ (because $y_\alpha \equiv (\{(\pi)_0\}[\alpha](0))_0$ is uniquely determined by an initial segment of α, and $((\{(\pi)_0\}[\alpha])_1)_0$ ${}^{\mathcal{C}}$realizes-$\Psi,\overline{\alpha}(y_\alpha)$ R(w), so $G(\pi,\overline{\alpha}(y_\alpha))$). Let

$$\zeta[\pi,w] = \begin{cases} \varepsilon^{\mathrm{A}(\langle\rangle)} & \text{if } H(\pi,w)\,\&\,\neg G(\pi,w) \\ \{\{(\pi)_1\}[w]\}[\langle\lambda t.0, \langle\lambda t.0, ((\{(\pi)_0\}[w*\lambda t.0])_1)_0\rangle\rangle] & \text{if } G(\pi,w) \\ \{\{(\pi)_1\}[w]\}[\langle\lambda t.0, \langle\lambda t.1, \Lambda n\zeta[\pi, w*\langle n+1\rangle]\rangle\rangle] & \text{otherwise, if } Seq(w) \end{cases}$$

If π ${}^{\mathcal{C}}$realizes-Ψ the hypothesis of BI! then $\zeta[\pi,1]$ ${}^{\mathcal{C}}$realizes-Ψ $\mathrm{A}(\langle\,\rangle)$ by an informal bar induction with $J(\pi,w)$ determining the thin bar and $K(\pi,w)$ (abbreviating "($\zeta[\pi,w]$ ${}^{\mathcal{C}}$realizes-Ψ,w A(w))$\,\&\,\forall u,v < w(u*v = w \rightarrow \neg G(\pi,u))$") as the inductive predicate.

Continuous choice CC_{11}: As in [4], [6]: $\Lambda\pi\langle\Lambda\,\sigma, \Lambda\alpha\langle\rho,\tau\rangle\rangle$ where $\sigma = \Lambda\alpha\{(\{\pi\}[\alpha])_0\}$, $\rho = \Lambda x\langle\mu y(\sigma(2^{x+1}*\overline{\alpha}(y)) > 0), \langle\lambda t.0, \Lambda z\Lambda\delta\lambda t.0\rangle\rangle$ and $\tau = \Lambda\beta\Lambda\delta(\{\pi\}[\alpha])_1$.

Rules of inference: Modus ponens and 9N, 12N, 9F, 12F (as in [4]) and 9C, 12C:

2. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for A, $\Phi''[\Psi]$ is a ${}^{\mathcal{C}}$realizing functional for A $\rightarrow$ B and $\Psi' \subseteq \Psi$, then $\Phi[\Psi] = \{\Phi''[\Psi]\}[\Phi'[\Psi']]$ is a ${}^{\mathcal{C}}$realizing functional for B.

9N. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for B $\rightarrow$ A(x), where $\Psi' = \Psi, x$ and x is not free in B, then $\Phi[\Psi] = \Lambda\delta\Lambda x\,\{\Phi'[\Psi,x]\}[\delta]$ is a ${}^{\mathcal{C}}$realizing functional for B $\rightarrow \forall$xA(x).

12N. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for A(x) $\rightarrow$ B, where $\Psi' = \Psi, x$ and x is not free in B, then $\Phi[\Psi] = \Lambda\pi\,\{\Phi'[\Psi,(\pi(0))_0]\}[(\pi)_1]$ is a ${}^{\mathcal{C}}$realizing functional for $\exists x A(x) \rightarrow B$.

9C. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for B $\rightarrow$ A(b), where $\Psi' = \Psi,\beta$ and b is not free in B, then $\Phi[\Psi] = \Lambda\delta\Lambda\beta\,\{\Phi'[\Psi,\beta]\}[\delta]$ is a ${}^{\mathcal{C}}$realizing functional for B $\rightarrow \forall$bA(b).

12C. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for A(b) $\rightarrow$ B, where $\Psi' = \Psi,\beta$ and b is not free in B, then $\Lambda\pi\{\Phi'[\Psi,\{(\pi)_0\}]\}[(\pi)_1]$ is a ${}^{\mathcal{C}}$realizing functional for $\exists$bA(b) $\rightarrow$ B.

9F. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for B $\rightarrow$ A(α), where $\Psi' = \Psi,\alpha$ and α is not free in B, then $\Phi[\Psi] = \Lambda\delta\Lambda\alpha\,\{\Phi'[\Psi,\alpha]\}[\delta]$ is a ${}^{\mathcal{C}}$realizing functional for B $\rightarrow \forall\alpha$A($\alpha$).

12F. If $\Phi'[\Psi']$ is a ${}^{\mathcal{C}}$realizing functional for $\mathrm{A}(\alpha) \to \mathrm{B}$, where $\Psi' = \Psi, \alpha$ and α is not free in B, then $\Lambda\pi\,\{\Phi'[\Psi, \{(\pi)_0\}]\}[(\pi)_1]$ is a ${}^{\mathcal{C}}$realizing functional for $\exists\alpha A(\alpha) \to B$.

This completes the proof that every theorem of **IC** is ${}^{\mathcal{C}}$realizable. By assumption $\mathcal{M}$ is a classical model of $\mathbf{C}^{\circ}$ so $0 = 1$ is not true in $\mathcal{M}$, and therefore not ${}^{\mathcal{C}}$realizable. □

4.4.1. *Corollary.* **IC** is consistent with all sentences of $\mathcal{L}(\mathbf{C}^{\circ})$ which are true in $\mathcal{M}$.

Proof. By the theorem with Lemma 4.3.4, every formula of $\mathcal{L}(\mathbf{IC})$ which is provable in **IC** from sentences of $\mathcal{L}(\mathbf{C}^{\circ})$ true in $\mathcal{M}$ is ${}^{\mathcal{C}}$realizable. □

4.4.2. *Corollary.* If $\mathcal{C} = \omega^{\omega}$ then $\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is ${}^{\mathcal{C}}$realizable, and if $\mathcal{C} \neq \omega^{\omega}$ then $\neg\neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is not ${}^{\mathcal{C}}$realizable. Hence if both $(\omega, \omega^{\omega})$ and some $(\omega, \mathcal{C})$ with $\mathcal{C} \neq \omega^{\omega}$ are classical models of $\mathbf{C}^{\circ}$ then $\neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is independent of **IC**.

Proof. If $\mathcal{C} = \omega^{\omega}$ then ε agrees with $\forall\alpha \mathrm{B}(\alpha)$ if and only if ε agrees with $\forall \mathrm{aB(a)}$, so $\Lambda\alpha\langle\Lambda\,\alpha, \Lambda x\lambda t.0\rangle$ ${}^{\mathcal{C}}$realizes $\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$.

If $\mathcal{C} \neq \omega^{\omega}$ then $\neg\neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is not ${}^{\mathcal{C}}$realizable. Since $\mathcal{C}$ is dense in ω^{ω}, if $\{\varepsilon\}[\alpha]$ is defined for all $\alpha \in \omega^{\omega}$ and $\{\varepsilon\}[\beta] = \beta$ for all $\beta \in \mathcal{C}$ then $\{\varepsilon\}[\alpha] = \alpha$ for all $\alpha \in \omega^{\omega}$, so no $\varepsilon \in \mathcal{C}$ can ${}^{\mathcal{C}}$realize $\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, so $\Lambda\pi\,\lambda t.0$ ${}^{\mathcal{C}}$realizes $\neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\,\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$, so $\neg\neg\forall\alpha\exists \mathrm{b}\forall \mathrm{x}\alpha(\mathrm{x}) = \mathrm{b}(\mathrm{x})$ is not ${}^{\mathcal{C}}$realizable. □

4.4.3. *Definition.* A formula E of $\mathcal{L}(\mathbf{IC})$ is ${}^{\mathcal{C}}$*realizable*/$\mathcal{C}$ if and only if its universal closure is ${}^{\mathcal{C}}$realized by some element of $\mathcal{C}$.

4.4.4. *Lemma.* If the truth function κ for classical arithmetic is an element of $\mathcal{C}$, then to each arithmetical formula E of $\mathcal{L}(\mathbf{IC})$ with at most the distinct variables $\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{k}$ free there is a partial functional $\Gamma_{\mathrm{E}}[y_1, \ldots, y_k]$ recursive in κ which agrees with E and satisfies, for all $y_1, \ldots, y_k \in \omega$ and corresponding numerals $\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k}$:

(a) If $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ then $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ is true in $\mathcal{M}$.

(b) If $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ is true in $\mathcal{M}$ then $\Gamma_{\mathrm{E}}[y_1, \ldots, y_k]$ ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$.

Proof. Since $\mathcal{M}$ is a classical ω-model of $\mathbf{C}^{\circ}$ and hence of **C**, an arithmetical sentence A is classically true if and only if it is true in $\mathcal{M}$, if and only if $\kappa(\ulcorner \mathrm{A} \urcorner) = 1$. By Lemma 4.3.2(a), ε ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{y}_1, \ldots, \mathbf{y}_\mathrm{k})$ if and only if ε ${}^{\mathcal{C}}$realizes-$y_1, \ldots, y_k$ E.

We use induction on the logical form of E. Prime formulas, &, $\vee, \to, \neg$ and $\forall x$ follow the proof of Lemma 4.3.4; e.g. if $\mathrm{E(y)} \equiv \forall \mathrm{xA(x, y)}$ has only y free and (a), (b) hold for A with $\Gamma_{\mathrm{A}}[x, y]$ recursive in κ, then (a) and (b) hold for E with $\Gamma_{\mathrm{E}}[y] = \Lambda x \Gamma_{\mathrm{A}}[x, y]$.

Suppose $\mathrm{E(x, y)} \equiv \mathrm{A(x, y)} \vee \mathrm{B(x, y)}$ where (a), (b) hold for A, B with Γ_{A}, Γ_{B} recursive in κ. Let $\Gamma_{\mathrm{E}}[x, y] = \langle 0, \Gamma_{\mathrm{A}}[x, y]\rangle$ if $\kappa(\ulcorner \mathrm{A}(\mathbf{x}, \mathbf{y}) \urcorner) = 1$, otherwise $\Gamma_{\mathrm{E}}[x, y] = \langle 1, \Gamma_{\mathrm{B}}[x, y]\rangle$. If $\mathrm{E}(\mathbf{x}, \mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$, then $\mathrm{A}(\mathbf{x}, \mathbf{y})$ or $\mathrm{B}(\mathbf{x}, \mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ so true in $\mathcal{M}$. If $\mathrm{E}(\mathbf{x}, \mathbf{y})$ is true in $\mathcal{M}$ then $\mathrm{A}(\mathbf{x}, \mathbf{y})$ or $\mathrm{B}(\mathbf{x}, \mathbf{y})$ is true in $\mathcal{M}$ so $\Gamma_{\mathrm{E}}[x, y]$ ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{x}, \mathbf{y})$.

Suppose E(y) is $\exists \mathrm{xA(x, y)}$ where (a) and (b) hold for A(x, y) with Γ_{A} recursive in κ. Let $\Gamma_{\mathrm{E}}[y] = \langle \lambda t.z, \Gamma_{\mathrm{A}}[z, y]\rangle$ where $z \simeq \mu x \kappa(\ulcorner \mathrm{A}(\mathbf{x}, \mathbf{y}) \urcorner) = 1$. If $\mathrm{E}(\mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ then $\mathrm{A}(\mathbf{n}, \mathbf{y})$ is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ and so true in $\mathcal{M}$ for some $n \in \omega$, so $\mathrm{E}(\mathbf{y})$ is true in $\mathcal{M}$. If $\mathrm{E}(\mathbf{y})$ is true in $\mathcal{M}$ then $\mathrm{A}(\mathbf{n}, \mathbf{y})$ is true in $\mathcal{M}$ for some least $\mathbf{n}$, so $\Gamma_{\mathrm{E}}[y]$ ${}^{\mathcal{C}}$realizes $\mathrm{E}(\mathbf{n})$. □

4.4.5. *Corollary.* **IC** is consistent with all classically true arithmetical sentences.

Proof. Lemma 4.3.3 and Theorem 4.4 relativize to ${}^{\mathcal{C}}$realizability/$\mathcal{C}$. All ${}^{\mathcal{C}}$realizable formulas and all consequences in **IC** of ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ formulas are ${}^{\mathcal{C}}$realizable/$\mathcal{C}$. By Lemma 4.4.4, classically true arithmetical sentences are ${}^{\mathcal{C}}$realizable/$\mathcal{C}$; $0 = 1$ is not. □

4.5. **Markov's Principle MP_1, Weak Kripke's Schema, WWKS, VS and $IP_\neg$.** Late in his life Brouwer introduced "creating subject" arguments to refute e.g.

$$\mathrm{MP}_1. \quad \neg\neg\exists x\alpha(x) = 0 \to \exists x\alpha(x) = 0$$

("Markov's Principle"), which is consistent with **I** but not ${}^{\mathcal{C}}$realizable or ${}^{\mathcal{C}}$realizable/$\mathcal{C}$. Efforts to formalize Brouwer's creating subject arguments led to Weak Kripke's Schema

$$\mathrm{WKS}. \quad \exists\beta[(\exists x\beta(x) \neq 0 \to A) \,\&\, (\forall x\beta(x) = 0 \to \neg A)]$$

where β is not free in A. If choice sequence variables are allowed to occur free in A then WKS (with $\forall x\alpha(x) = 0$ as the A) conflicts with CC_{11}, so **I** + WKS is inconsistent. *A fortiori*, so is **IC** + WKS. For arbitrary A without β free, a classically (but not intuitionistically) equivalent version of WKS is Weaker Weak Kripke's Schema:

$$\mathrm{WWKS}. \quad \exists\beta[\forall x\beta(x) = 0 \leftrightarrow \neg A]$$

which conflicts with CC_{11} by the same argument.

4.5.1. *Proposition.* If A(y) is a negative $\mathcal{C}$-formula with at most y free, and if the truth function $\lambda y.\kappa(y)$ for A(y) over $\mathcal{M} = (\omega, \mathcal{C})$ is an element of $\mathcal{C}$, then the instance of WKS for A(y) is ${}^{\mathcal{C}}$realizable/$\mathcal{C}$ and hence consistent with **IC**.

Proof. Use Lemma 4.3.4. If $D(\beta)$ is $[(\exists x\beta(x) \neq 0 \to A(y)) \,\&\, (\forall x\beta(x) = 0 \to \neg A(y))]$, then $\delta = \Lambda y\langle \Lambda\,\lambda t.\kappa(y), \langle \Lambda\sigma\, \tau_{A(y)}, \Lambda\sigma\, \tau_{\neg A(y)}\rangle\rangle$ ${}^{\mathcal{C}}$realizes/$\mathcal{C}$ $\forall y\exists\beta D(\beta)$. □

4.5.2. *Proposition.* For all sentences A of $\mathcal{L}(\mathbf{IC})$, WWKS is classically ${}^{\mathcal{C}}$realizable and therefore consistent with **IC**.

Proof. If $E(\beta)$ is $[(\forall x\beta(x) = 0 \to \neg A) \,\&\, (\neg A \to \forall x\beta(x) = 0)]$ where A is a sentence of $\mathcal{L}(\mathbf{IC})$, then $\varepsilon = \langle \Lambda\rho\Lambda\tau\lambda t.0, \Lambda\tau\Lambda x\lambda t.0\rangle$ ${}^{\mathcal{C}}$realizes-$\lambda t.0$ E if $\neg A$ is ${}^{\mathcal{C}}$realizable; otherwise ε ${}^{\mathcal{C}}$realizes-$\lambda t.1$ E. □

Thezs relative consistency proofs for restricted WKS and WWKS evidently have nothing to do with the stage-by-stage activity of a creating subject. Another approach to Brouwer's counterexamples was suggested by Richard Vesley.

4.5.3. *Vesley's Schema.* Richard Vesley [9] proved that the axiom schema

$$\mathrm{VS}. \quad \forall w[\mathrm{Seq}(w) \to \exists\alpha(\overline{\alpha}(\mathrm{lh}(w)) = w \,\&\, \neg A(\alpha))] \,\&\, \forall\alpha[\neg A(\alpha) \to \exists\beta B(\alpha,\beta)] \to \forall\alpha\exists\beta[\neg A(\alpha) \to B(\alpha,\beta)]$$

(with β not free in $A(\alpha)$) is consistent with **I** and suffices to refute the universal closure of MP_1 and for other "creating subject" counterexamples. He also proved that **I** is consistent with a stronger "independence of premise" schema (with β not free in A):

$$\mathrm{IP}_\neg. \quad (\neg A \to \exists\beta B(\beta)) \to \exists\beta(\neg A \to B(\beta)).$$

4.5.4. *Proposition.* $IP_\neg$ (and therefore VS) is ${}^{\mathcal{C}}$realizable.

Proof. $\Lambda\sigma\langle(\{\sigma\}[\Lambda\pi\,\lambda t.0])_0, \Lambda\delta\,(\{\sigma\}[\Lambda\pi\,\lambda t.0])_1\rangle$ is a ${}^{\mathcal{C}}$realizing functional for IP. □

4.5.5. *Corollary.* **IC** + $IP_\neg$ is consistent.

5. Epilogue

5.1. **Choice sequences revisited.** At each nth stage in the generation of a choice sequence α, when $\overline{\alpha}(n)$ has already been determined and $\alpha(n)$ is to be chosen, Brouwer allowed (but did not require) restrictions to be placed on all future choices, consistent with any restrictions inherited from previous stages. Many variations of his original notion appear in the literature.

Troelstra's "hesitant sequences" α ([8] §4.6.2) are related to lawlike sequences b by $\neg\neg\exists b\forall x\alpha(x) = b(x)$ as the choice sequences of the present model are related to elements of $\mathcal{C}$, but with an additional restriction. A hesitant sequence α proceeds freely until and unless at some finite stage "in time" a particular lawlike b is deliberately correlated to α (which then becomes lawlike).

Kreisel's "lawless sequences" α ([8] §12.2), for which $\neg\exists b\forall x\alpha(x) = b(x)$ holds, are beyond the scope of our interpretation, as are projections of lawless sequences.[3]

5.2. **Understanding $\mathcal{M}$.** Kleene proved that while Brouwer's fan theorem is classically true for the arithmetical sequences, even all the hyperarithmetical sequences do not suffice to form a classical ω-model of **C**. Brouwer would have had no need for choice sequences if his reduced continuum was complete. An intuitionist might reject the idea of a definite classical continuum, *a fortiori* the idea of a classical ω-model of $\mathbf{C}^\circ$ or **C**.

However, an intuitionist might understand $\mathcal{M} = (\omega, \mathcal{C})$ by taking $\mathcal{C}$ to be the species of Brouwer's lawlike sequences and assuming that $\mathcal{M}$ satisfies lawlike versions of all the axioms of **B** except AC_{01} and BI!, plus "unique choice" $\mathrm{AC}_{00}!$ (like AC_{00} but with the stronger hypothesis $\forall \mathrm{x}\exists ! \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y})$) and the negative interpretation of AC_{01}. By [3] these axioms could only prove the existence of definite, in fact recursive, sequences. From this point of view, ET simply asserts that at the nth stage in the generation of a choice sequence α, the possibility that α will turn out to be pointwise equal to a lawlike sequence cannot be excluded. This seems reasonable provided that at each stage only lawlike restrictions on future values are allowed (for example, restriction to a spread with a lawlike spread-law).

Evidently **IC** gives no further insight into the stage-by-stage activity of a creating subject. All we can claim is that from the perspective (unattainable by the creating subject) of the end of time, Kripke's idea is classically feasible.

5.3. **Copyright information.**

[3] Apart from [6], two related earlier investigations were pointed out to me after this work was done. V. Lifschitz [5] introduced a distinction between the constructive or "calculable" numbers and the classical natural numbers, with a formal theory (expressing both classical and recursive arithmetic) which proves that not every classical number is calculable but there is no non-calculable classical number. Birkedal and van Oosten [1] abstractly described toposes corresponding to ${}^{\mathcal{C}}$realizability, with elementary sub-partial combinatory algebras of ω^ω playing the role of $\mathcal{C}$.

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