# Minimum Classical Extensions of Constructive Theories

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Constructive and classical mathematics differ in two ways:

1. Logic and logical language: either intuitionistic or classical.
   - In *intuitionistic* logic $\vee$, $\exists$, &, $\neg$, $\forall$, $\rightarrow$ are *independent*.
   - *Classically*, $\vee$ and $\exists$ may be *omitted* because (classically) $(\mathrm{A} \vee \mathrm{B}) \leftrightarrow \neg(\neg \mathrm{A} \ \& \ \neg \mathrm{B})$ and $\exists \mathrm{x} \mathrm{A}(\mathrm{x}) \leftrightarrow \neg \forall \mathrm{x} \neg \mathrm{A}(\mathrm{x})$. This is the basis of *Gentzen's negative interpretation*.

2. Mathematical axioms describe properties of *intended objects*.
   - Constructive and classical natural numbers are standard, and constructive and classical primitive recursive functions agree.
   - Existence criteria for constructive infinite sequences, sets or functions are typically stronger than for classical counterparts.

According to Ishihara, constructive reverse mathematics aims "to classify . . . theorems in intuitionistic, constructive and recursive mathematics by logical principles, function existence axioms and their combinations" over a weak base with intuitionistic logic.

Weak constructive base theories include

- intuitionistic two-sorted arithmetic $\mathsf{IA}_1$,
- primitive recursive arithmetic of finite types $\mathsf{HA}^\omega$,
- Troelstra's EL $\equiv \mathsf{IA}_1 + \text{QF-AC}_{00}$ and Veldman's BIM.

$\mathsf{IA}_1$ and $\mathsf{HA}^\omega$ contain their negative interpretations, but a classical logical principle ($\Sigma^0_1$-double negation shift) must be added to EL or BIM to negatively interpret $\text{QF-AC}_{00}$ or recursive comprehension.

So EL is weaker than its negative interpretation, which is also the negative interpretation of EL $+ (\neg\neg \mathrm{A} \to \mathrm{A})$.

In this talk, "S $\vdash \mathrm{E}$" always means "by intuitionistic logic."
Classical logic is indicated by "S $\vdash^\circ$ E", following Kleene.

Every formal system S based on intuitionistic logic has a **classical twin**:

$$\mathsf{S}^\circ \equiv_{Def} \mathsf{S} + (\neg\neg \mathrm{A} \to \mathrm{A})$$

with *the same mathematical axioms*. Logic is the only difference.

**Our Question:** Exactly what classical logical axioms and function existence principles need to be added to a constructive system S based on intuitionistic logic, in order to prove the Gentzen negative interpretation of S (a faithful copy of the classical version $\mathsf{S}^\circ$ of S)?

In other words, what would be the precise constructive cost of accepting the classical interpretation of our mathematical axioms?

The *language of arithmetic* $\mathcal{L}$(Ar) is any first-order language with constants $=$, 0, $'$, $+$, $\cdot$, variables m, n, . . . , x, y, z over numbers.

The *language of analysis* $\mathcal{L}$(An) adds variables $\alpha, \beta, \gamma, \ldots$ over *infinite sequences*, and primitive recursive function(al) constants.

The **Gentzen negative interpretation** $\mathrm{E}^{g}$ of a formula $\mathrm{E}$ in $\mathcal{L}$(Ar) or $\mathcal{L}$(An) is defined inductively:

- Prime formulas are unchanged: $(\mathrm{s} = \mathrm{t})^{g} \equiv (\mathrm{s} = \mathrm{t})$.
- Negative operations pass through: $(\neg \mathrm{A})^{g} \equiv \neg(\mathrm{A}^{g})$
  $(\mathrm{A}\ \&\ \mathrm{B})^{g} \equiv (\mathrm{A}^{g}\ \&\ \mathrm{B}^{g})$ $\quad (\mathrm{A} \to \mathrm{B})^{g} \equiv (\mathrm{A}^{g} \to \mathrm{B}^{g})$
  $(\forall \mathrm{x} \mathrm{A}(\mathrm{x}))^{g} \equiv \forall \mathrm{x}(\mathrm{A}(\mathrm{x}))^{g}$ $\quad (\forall \alpha \mathrm{A}(\alpha))^{g} \equiv \forall \alpha(\mathrm{A}(\alpha))^{g}$.
- $\vee$ and $\exists$ are interpreted classically: $(\mathrm{A} \vee \mathrm{B})^{g} \equiv \neg(\neg \mathrm{A}^{g} \& \neg \mathrm{B}^{g})$
  $(\exists \mathrm{x} \mathrm{A}(\mathrm{x}))^{\mathrm{g}} \equiv \neg \forall \mathrm{x} \neg(\mathrm{A}(\mathrm{x}))^{g}$ $\quad (\exists \alpha \mathrm{A}(\mathrm{x}))^{\mathrm{g}} \equiv \neg \forall \alpha \neg(\mathrm{A}(\alpha))^{g}$

# Classical Soundness and Classical Content

**Definitions.** We identify the **classical content** of a formula $\mathrm{E}$ with its Gentzen negative translation $\mathrm{E}^g$, where $\vdash^\circ (\mathrm{E} \leftrightarrow \mathrm{E}^g)$.

The **classical content** $\Gamma^g$ of a set $\Gamma$ of formulas is the closure under intuitionistic logic of $\{\mathrm{E}^g : \mathrm{E} \in \Gamma\}$.

A formal system S in $\mathcal{L}$(Ar) or $\mathcal{L}$(An) is **classically sound** if and only if S has a *classical $\omega$-model* (a model with standard integers).

The **classical content** of a classically sound formal system S is

$$\mathsf{S}^g \equiv_{Def} \{\mathrm{E}^g : \mathrm{S} \vdash \mathrm{E}\}.$$

**Lemma.** 1. If S is classically sound then $\mathsf{S}^\circ$ is consistent.

2. If $\Gamma \vdash^\circ \mathrm{E}$ then $\Gamma^g \vdash \mathrm{E}^g$, and $(\mathrm{E}^g)^g = \mathrm{E}^g$ for every formula $\mathrm{E}$.

3. If S and T differ only by *classical logical* axioms then $\mathsf{S}^g = \mathsf{T}^g$.

*Some* constructive systems S *contain their classical content*, e.g.:

**First-Order Arithmetic:** In $\mathcal{L}$(Ar), intuitionistic arithmetic HA has full mathematical induction. PA = $\text{HA}^\circ$ and $\text{HA}^g \subseteq$ HA.

**Two-Sorted Intuitionistic Arithmetic:** In $\mathcal{L}$(An) with constants for primitive recursive function(al)s, $\lambda$-abstraction and $\lambda$-reduction, $\text{IA}_1$ extends HA (so $\text{IA}_1^\circ$ extends PA). $(\text{IA}_1)^g \subseteq \text{IA}_1$.

- In $\text{IA}_1$ as in HA, equality at type 0 is primitive and decidable.
- $\alpha = \beta \equiv_{Def} \forall x \alpha(x) = \beta(x)$ and $\text{IA}_1 \vdash x = y \rightarrow \alpha(x) = \alpha(y)$.
- $\vdash \neg\neg(\alpha = \beta) \rightarrow \alpha = \beta$ but $\text{IA}_1 \nvdash \alpha = \beta \vee \neg(\alpha = \beta)$.

**Arithmetic of Finite Types:** $\text{HA}^\omega$ extends HA to include primitive recursive functions of all finite types. $(\text{HA}^\omega)^g \subseteq \text{HA}^\omega$.

$\omega$-models of $\text{IA}_1$, $\text{HA}^\omega$ require only primitive recursive functions.

If S is classically sound and includes an axiom or axiom schema of countable choice or comprehension, then $S^g \not\subseteq S$. Examples:

**$IA_1$ + Recursive Comprehension:** IRA or Troelstra's EL.

- $IRA \equiv IA_1 + \forall x \exists y \rho(\langle x, y\rangle) = 0 \to \exists \alpha \forall x \rho(\langle x, \alpha(x)\rangle) = 0$.
- $EL \equiv IA_1 + QF\text{-}AC_{00}$ (*quantifier-free countable choice*).

**Kleene's Neutral Basic Analysis:** B extends $IA_1$ (and IRA).

- Intended objects: numbers; infinitely proceeding sequences.
- Axioms include *countable choice for sequences*

$$AC_{01} : \forall x \exists \alpha A(x, \alpha) \to \exists \beta \forall x A(x, \lambda y.\beta(\langle x, y\rangle)).$$

  and *bar induction* $BI_d$ or $BI_1$.
- $B^\circ = (IA_1 + AC_{01})^\circ$ so $B^g = (IA_1 + AC_{01})^g$.

**Subsystems** of B weaken $AC_{01}$ to $AC_{00}$ or *unique choice* $AC_{00}!$, and/or omit bar induction or replace it by fan induction.

Intuitionistic analysis is *consistent but not classically sound*.

**Kleene's Intuitionistic Analysis:** $\mathsf{I} \equiv_{Def} \mathsf{B} + \mathsf{CC}_{11}$.

- $\mathsf{CC}_{11}$ is a strong *continuous choice* principle.
- $\mathsf{I} \vdash \neg\forall\alpha(\forall x\alpha(x) = 0 \vee \neg\forall x\alpha(x) = 0)$.

**Vesley's Intuitionistic Analysis:** $\mathsf{I} + \mathsf{VS}$ refutes classical logical principles for whose refutation Brouwer used a "creative subject."

- $\mathsf{I} + \mathsf{VS} \vdash \neg\forall\alpha(\neg\forall x\alpha(x) = 0 \rightarrow \exists x\alpha(x) \neq 0)$.

**Troelstra's Realizable Intuitionistic Analysis:** $\mathsf{B} + \mathsf{GC}$.

- Troelstra's generalized continuous choice principle GC extends $\mathsf{CC}_{11}$ to relations whose domain is *almost negative*.
- B + GC characterizes Kleene's function-realizability.

**van Oosten's Lifshitz Realizable Analysis** weakens GC to $\mathsf{GC}_L$.

Constructive recursive mathematics studies the properties of numbers and general recursive functions.

**Constructive Recursive Mathematics** is axiomatized in $\mathcal{L}(\text{Ar})$ by Troelstra and van Dalen as CRM $\equiv$ HA + MP + $\text{ECT}_0$, where

- MP is Markov's Principle for decidable relations, and
- $\text{ECT}_0$ is *Extended Church's Thesis* for $\mathrm{A}(\mathrm{x})$ *almost negative*:

$$\forall \mathrm{x}[\mathrm{A}(\mathrm{x}) \to \exists \mathrm{y} \mathrm{B}(\mathrm{x}, \mathrm{y})] \to \exists \mathrm{e} \forall \mathrm{x}[\mathrm{A}(\mathrm{x}) \to \{\mathrm{e}\}(\mathrm{x})\downarrow \,\&\, \mathrm{B}(\mathrm{x}, \{\mathrm{e}\}(\mathrm{x}))]$$

CRM is consistent but not classically sound, and $\text{CRM}^g \not\subseteq$ CRM.

In $\mathcal{L}(\text{An})$ one might be interested in MRA $\equiv$ IRA + $\text{MP}_1$ + $\text{CT}_1$.

- $\text{MP}_1$ is $\forall \alpha(\neg \forall \mathrm{x} \alpha(\mathrm{x}) = 0 \to \exists \mathrm{x} \alpha(\mathrm{x}) \neq 0)$.
- $\text{CT}_1$ can be abbreviated by $\forall \alpha \exists \mathrm{e} \forall \mathrm{x}(\alpha(\mathrm{x}) = \{\mathrm{e}\}(\mathrm{x}))$.

MRA is classically sound and $\text{MRA}^g \subseteq$ MRA, but MRA $\vdash^\circ \neg \text{BI}_1$.

# Minimum Classical Extension of S

**Main Definition:** The **minimum classical extension** $\mathsf{S}^{+g}$ of a classically sound formal system S, based on intuitionisti logic in $\mathcal{L}(\mathsf{Ar})$ or $\mathcal{L}(\mathsf{An})$, is the closure under intuitionistic logic of $\mathsf{S} \cup \mathsf{S}^g$.

**Challenges:**

1. Given such a formal system S, find a characterization of $\mathsf{S}^{+g}$ which clarifies the constructive cost of expanding S to include the negative interpretation of its classical twin.
2. What if S is consistent but not classically sound? Is there a preferred way to define $\mathsf{S}^{+g}$ in that case?

# Double Negation Shift Principles

In $\mathcal{L}(\mathsf{Ar})$ or $\mathcal{L}(An)$, **double negation shift for integers** is

$$\mathsf{DNS}_0: \quad \forall x \neg\neg A(x) \rightarrow \neg\neg\forall x A(x)$$

where $A(x)$ may contain additional free variables.

**Proposition 1.** If S proves a version of the countable axiom of choice, then S + $\mathsf{DNS}_0$ proves its negative interpretation. E.g.

$$\Sigma^0_1\text{-}\mathsf{DNS}_0: \quad \forall\rho(\forall x\neg\neg\exists y\, \rho(\langle x, y\rangle) = 0 \rightarrow \neg\neg\forall x\exists y\, \rho(\langle x, y\rangle) = 0)$$

characterizes the minimum classical extension of EL or IRA.

1. $\mathsf{EL}^{+g} = \mathsf{EL} + \Sigma^0_1\text{-}\mathsf{DNS}_0$ where $\mathsf{EL} = \mathsf{IA}_1 + \mathsf{QF\text{-}AC}_{00}$.
2. $\mathsf{IRA}^{+g} = \mathsf{IRA} + \Sigma^0_1\text{-}\mathsf{DNS}_0$ where $\mathsf{IRA} = \mathsf{IA}_1$ + Rec Comp.

Scedrov and Vesley proved that $\mathsf{B} \nvdash \Sigma^0_1\text{-}\mathsf{DNS}_0$.

Other restricted versions of $\mathsf{DNS}_0$ include

$$\mathsf{DNS}_{00}^-: \quad \forall x \neg\neg \exists y A(x,y) \rightarrow \neg\neg \forall x \exists y A(x,y),$$

$$\mathsf{DNS}_{01}^-: \quad \forall x \neg\neg \exists \alpha A(x,\alpha) \rightarrow \neg\neg \forall x \exists \alpha A(x,\alpha)$$

for $A(x,y)$ *negative* (no $\vee$ or $\exists$), and $\mathsf{DNS}_{0\sigma}^-$ for finite types $\sigma$.

**Proposition 2.** Minimum classical extensions of systems with countable choice $\mathsf{AC}_{00}$, $\mathsf{AC}_{01}$ or $\mathsf{AC}_{0\sigma}$ for all finite types $\sigma$:

1. $(\mathsf{EL} + \mathsf{AC}_{0i})^{+g} = \mathsf{EL} + \mathsf{AC}_{0i} + \mathsf{DNS}_{0i}^-$ for i = 0,1.
2. $(\mathsf{IRA} + \mathsf{AC}_{0i})^{+g} = \mathsf{IA}_1 + \mathsf{AC}_{0i} + \mathsf{DNS}_{0i}^-$ for i = 0,1.
3. $(\mathsf{HA}^\omega + \mathsf{AC}_{0\infty})^{+g} = \mathsf{HA}^\omega + \mathsf{AC}_{0\infty} + \mathsf{DNS}_{0\infty}^-$.

Refinements include e.g. Fujiwara's observation that (in effect)

4. $(\mathsf{EL} + \Pi_1^0\text{-}\mathsf{AC}_{00})^{+g} = \mathsf{EL} + \Pi_1^0\text{-}\mathsf{AC}_{00} + \Sigma_2^0\text{-}\mathsf{DNS}_0$.

# Doubly Negated Characteristic Function Principles

Over EL or IRA, **if** $\mathrm{A(x)}$ has a characteristic function **then** $\forall \mathrm{x}(\mathrm{A(x)} \vee \neg \mathrm{A(x)})$ holds. Vafeiadou observed that *unique choice*

$$\mathsf{AC_{00}!}: \forall \mathrm{x} \exists ! \mathrm{y} \mathrm{A(x,y)} \to \exists \alpha \forall \mathrm{x} \mathrm{A(x}, \alpha(\mathrm{x}))$$

is equivalent over EL or IRA to the converse implication:

$$\mathsf{CF_d}: \forall \mathrm{x}(\mathrm{A(x)} \vee \neg \mathrm{A(x)}) \to \exists \chi_{\mathrm{B}(\chi)} \forall \mathrm{x}(\chi(\mathrm{x}) = 0 \leftrightarrow \mathrm{A(x)}).$$

The schema $\neg\neg\mathsf{CF_0}$: $\neg\neg\exists\chi\forall \mathrm{x}(\chi(\mathrm{x}) = 0 \leftrightarrow \mathrm{A(x)})$

says it is *consistent* to assume $\mathrm{A(x)}$ has a characteristic function.

$\neg\neg\Pi^0_1\text{-}\mathsf{CF_0}$ is $\forall\alpha[\neg\neg\exists\chi\forall \mathrm{x}(\chi(\mathrm{x}) = 0 \leftrightarrow \forall \mathrm{y}\alpha(\langle \mathrm{x},\mathrm{y}\rangle) = 0)]$.

$\neg\neg\mathsf{CF}_0^-$ is the restriction of $\neg\neg\mathsf{CF_0}$ to *negative* $\mathrm{A(x)}$.

**Proposition 3.** Over $\mathsf{IA_1}$ or EL, $(\mathsf{CF_d})^g$ is equivalent to $\neg\neg\mathsf{CF}_0^-$ and $(\Pi^0_1\text{-}\mathsf{CF_0})^g$ is equivalent to $\neg\neg\Pi^0_1\text{-}\mathsf{CF_0}$.

Now we can improve on Proposition 2.

Let $\mathsf{AC}_{00}^{Ar}$ be the restriction of numerical countable choice $\mathsf{AC}_{00}$ to *arithmetic* predicates (no sequence quantifiers allowed).

**Theorem 1.**

1. $(\mathsf{IA}_1 + \mathsf{AC}_{00}^{Ar})^{+g} = \mathsf{IA}_1 + \mathsf{AC}_{00}^{Ar} + \Sigma_1^0\text{-}\mathsf{DNS}_0 + \neg\neg\Pi_1^0\text{-}\mathsf{CF}_0$.
2. $(\mathsf{EL} + \mathsf{AC}_{00}!)^{+g} = \mathsf{EL} + \mathsf{CF}_\mathrm{d} + \Sigma_1^0\text{-}\mathsf{DNS}_0 + \neg\neg\mathsf{CF}_0^-$.
3. $(\mathsf{IA}_1 + \mathsf{AC}_{00}!)^{+g} = \mathsf{IRA} + \mathsf{CF}_\mathrm{d} + \Sigma_1^0\text{-}\mathsf{DNS}_0 + \neg\neg\mathsf{CF}_0^-$.
4. $(\mathsf{IA}_1 + \mathsf{AC}_{00})^{+g} = \mathsf{IA}_1 + \mathsf{AC}_{00} + \Sigma_1^0\text{-}\mathsf{DNS}_0 + \neg\neg\mathsf{CF}_0^-$.

The proof of (1) uses formula induction and the proof of (2) uses $\mathsf{EL} + \mathsf{CF}_\mathrm{d} = \mathsf{EL} + \mathsf{AC}_{00}!$ with Propositions 1 and 3. The proof of (3) is similar using $\mathsf{IRA} + \mathsf{CF}_\mathrm{d} = \mathsf{IA}_1 + \mathsf{AC}_{00}!$. (4) holds because $(\mathsf{AC}_{00}!)^g$ and $(\mathsf{AC}_{00})^g$ are equivalent over EL or IRA.

Kleene's classically sound basic system B $\equiv_{Def}$ $IA_1 + AC_{01} + BI_d$ where $BI_d$ is bar induction with a *decidable* bar predicate $R(w)$:

$$BI_d : \forall\alpha\exists x R(\overline{\alpha}(x)) \ \& \ \forall w(R(w) \vee \neg R(w)) \ \& \ \forall w(R(w) \rightarrow A(w)) \\ \& \ \forall w(\forall x A(w * \langle x+1\rangle) \rightarrow A(w)) \rightarrow A(1).$$

(*Notation:* $\overline{\alpha}(x+1)$ codes the sequence $(\alpha(0), \ldots, \alpha(x))$ and 1 codes the empty sequence. $\langle x+1\rangle$ codes the sequence $(x)$. $w$ varies over sequence codes, and $*$ denotes concatenation.)

*Classical* bar induction $BI^\circ$ drops the premise $\forall w(R(w) \vee \neg R(w))$. Obviously $IA_1 \vdash (BI_d)^g \leftrightarrow (BI^\circ)^g$ since $\vdash (\forall w(R(w) \vee \neg R(w)))^g$.

Weaker than $BI_d$ over $IA_1$ (although $IA_1 + AC_{00}! + BI_1 \vdash BI_d$) is

$$BI_1 : \forall\alpha\exists x \rho(\overline{\alpha}(x)) = 0 \ \& \ \forall w(\rho(w) = 0 \rightarrow A(w)) \\ \& \ \forall w(\forall s A(w * \langle s+1\rangle) \rightarrow A(w)) \rightarrow A(1).$$

The schema $\mathsf{DNS}_1^-$: $\forall\alpha\neg\neg\exists x R(\overline{\alpha}(x)) \rightarrow \neg\neg\forall\alpha\exists x R(\overline{\alpha}(x))$
for negative formulas $R(w)$ of $\mathcal{L}(\mathsf{An})$ has the special case
$\Sigma_1^0$-$\mathsf{DNS}_1$: $\forall\alpha\neg\neg\exists x\rho(\overline{\alpha}(x)) = 0 \rightarrow \neg\neg\forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0$.

**Proposition 4.** $\mathsf{IA}_1 + \mathsf{DNS}_1^- + \mathsf{BI}_d \vdash (\mathsf{BI}_d)^g$

**Theorem 2.**

1. $(\mathsf{IA}_1 + \mathsf{BI}_d)^{+g} = \mathsf{IA}_1 + \mathsf{BI}_d + (\mathsf{BI}^\circ)^g \subseteq \mathsf{IA}_1 + \mathsf{BI}_d + \mathsf{DNS}_1^-$.
2. $(\mathsf{IA}_1 + \mathsf{BI}_1)^{+g} \subseteq \mathsf{IA}_1 + \mathsf{BI}_1 + \Sigma_1^0$-$\mathsf{DNS}_1$, and Solovay proved $(\mathsf{IA}_1 + \mathsf{AC}_{00}^{Ar} + \mathsf{BI}_1)^g \subseteq \mathsf{IRA} + \mathsf{BI}_1 + \Sigma_1^0$-$\mathsf{DNS}_1$.
3. $(\mathsf{IRA} + \mathsf{BI}_d)^{+g} = \mathsf{IRA} + \mathsf{BI}_d + (\mathsf{BI}^\circ)^g + \Sigma_1^0$-$\mathsf{DNS}_0$ $\subseteq \mathsf{IRA} + \mathsf{BI}_d + \mathsf{DNS}_1^-$.
   Kleene proved $\mathsf{IA}_1 + \mathsf{AC}_{00} \vdash^\circ \mathsf{BI}^\circ$, and $(\mathsf{BI}_d)^g = (\mathsf{BI}^\circ)^g$, so
4. $\mathsf{B}^{+g} \equiv (\mathsf{IA}_1 + \mathsf{AC}_{01} + \mathsf{BI}_d)^{+g} = \mathsf{B} + (\mathsf{AC}_{01})^g = \mathsf{B} + \mathsf{DNS}_{01}^-$.
5. $(\mathsf{IA}_1 + \mathsf{AC}_{00} + \mathsf{BI}_d)^{+g} = \mathsf{IA}_1 + \mathsf{AC}_{00} + \mathsf{BI}_d + \mathsf{DNS}_{00}^-$.

To extend CRM to $\mathcal{L}$(An) one might choose as a base theory MRA $\equiv$ IRA + $\mathsf{MP}_1$ + $\forall\alpha$GR($\alpha$), where

- $\mathsf{MP}_1$ is $\forall\alpha(\neg\forall x\alpha(x) = 0 \rightarrow \exists x\alpha(x) \neq 0)$.
- $\forall\alpha$GR($\alpha$) expresses "every $\alpha$ is recursive" and can be abbreviated by $\forall\alpha\exists e\forall x(\alpha(x) = \{e\}(x))$.

MRA is classically sound. It describes the $\omega$-model in which the type-1 objects are recursive sequences, so conflicts with Kleene's B.

**Proposition 5.** (jrm)

1. $\mathsf{MRA}^g \subseteq \mathsf{IRA} + \Sigma^0_1\text{-}\mathsf{DNS}_0 + \forall\alpha\neg\neg\mathsf{GR}(\alpha) \subseteq \mathsf{MRA}$, so MRA contains its classical content, so $\mathsf{MRA}^{+g} = \mathsf{MRA}$.
2. $\mathsf{MRA}^g$ is consistent with $\mathsf{I} + \neg\mathsf{MP}_1$.

Next we apply some constructive decomposition theorems.

*Monotone* bar induction $\mathsf{BI}_{\mathrm{mon}}$, provable in **I** but not in **B**, is

$$\forall\alpha\exists x R(\overline{\alpha}(x)) \ \&\ \forall w(R(w) \to \forall u R(w * u)) \ \&\ \forall w(R(w) \to A(w)) \\ \&\ \forall w(\forall x A(w * \langle x + 1\rangle) \to A(w)) \to A(1).$$

Kleene proved in 1965 that $\mathsf{IA}_1 + \mathsf{AC}_{00} + \mathsf{BI}_{\mathrm{mon}} \vdash \mathsf{BI}_{\mathrm{d}}$, so $\mathsf{BI}_{\mathrm{mon}}$ lies between $\mathsf{BI}_{\mathrm{d}}$ and $\mathsf{BI}^{\circ}$ in strength over $\mathsf{IA}_1 + \mathsf{AC}_{00}$.

He also proved $\mathsf{IRA} + \mathsf{BI}^{\circ} \vdash \mathsf{WLPO}$ so $\mathsf{BI}^{\circ}$ is inconsistent with I.

Fujiwara proved in 2019 that $\mathsf{BI}^{\circ}$ is equivalent to $\mathsf{BI}_{\mathrm{mon}} + \mathsf{CD}$ over $\mathsf{EL}_0$, where CD is $\forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$ (x not free in B).

**Proposition 6.** $(\mathsf{IRA} + \mathsf{BI}_{\mathrm{d}})^{g} = (\mathsf{IRA} + \mathsf{BI}_{\mathrm{mon}})^{g} = (\mathsf{IRA} + \mathsf{BI}^{\circ})^{g}$.

**Corollary.** The neutral subsystem B of Kleene and Vesley's I has the same classical content as the variant B′ with $\mathsf{BI}_{\mathrm{mon}}$ replacing $\mathsf{BI}_{\mathrm{d}}$, and so $(\mathsf{B}')^{+g} \equiv (\mathsf{IA}_1 + \mathsf{AC}_{01} + \mathsf{BI}_{\mathrm{mon}})^{+g} = \mathsf{B}' + \mathsf{DNS}_{01}^{-}$.

Over a constructive base theory $\mathsf{EL}' \equiv_{Def} \mathsf{EL} + \Pi^0_1\text{-}\mathsf{AC}_{00}$, Ishihara and Schuster decomposed a restricted version

$$\begin{aligned} \text{WC-N}': \quad & \forall\alpha\exists n\forall k\,\sigma(\langle\overline{\alpha}(k), n\rangle) = 0 \\ & \quad \&\ \forall w\forall m\forall n(\sigma(\langle w, m\rangle) = 0\ \&\ m \leq n \to \sigma(\langle w, n\rangle) = 0) \\ & \qquad \to \forall\alpha\exists n\exists m\forall\beta \in \overline{\alpha}(m)\forall k\,\sigma(\langle\overline{\beta}(k), n\rangle) = 0 \end{aligned}$$

of weak continuity into a classically correct mathematical principle

$$\text{BD-N}: \quad \forall\alpha\exists m\forall n \geq m\,\beta(\alpha(n)) < n \to \exists m\forall n\,\beta(n) \leq m$$

and the classically false ¬LPO: $\neg\forall\alpha(\exists x\,\alpha(x) \neq 0 \vee \forall x\,\alpha(x) = 0)$.

**Proposition 7.**

1. $(\mathsf{EL}')^{+g} \equiv (\mathsf{EL} + \Pi^0_1\text{-}\mathsf{AC}_{00})^{+g} = \mathsf{EL}' + \Sigma^0_2\text{-}\mathsf{DNS}_0$.
2. $(\mathsf{EL}' + \text{BD-N})^{+g} = \mathsf{EL}' + \text{BD-N} + \Sigma^0_2\text{-}\mathsf{DNS}_0$.

(1) is by Proposition 2(4). (2) holds because $\mathsf{EL}^{+g}$ proves the contrapositive of $(\text{BD-N})^g$ (equivalent to $(\text{BD-N})^g$ over EL).

# Classical Content of a Classically Unsound Theory?

Ishihara and Schuster's $\mathsf{EL}' + \mathsf{WC\text{-}N}'$ proves BD-N (which is classically correct) and $\neg$LPO (which is not).

**Question.** Does such a system S in $\mathcal{L}(\mathsf{An})$ have a classical content, and if so, what is it? Consider this possibility:

The **classical subtheory** cls(S) of S consists of all theorems of S that hold in classical Baire space. The **classical content** $\mathsf{S}^g$ of S is $(\mathsf{cls}(\mathsf{S}))^g$ and $\mathsf{S}^{+g}$ is the closure under intuitionistic logic of $\mathsf{S} \cup \mathsf{S}^g$.

**Theorem 3.** (gvf) $(\mathsf{EL}' + \mathsf{WC\text{-}N}')^{+g} = \mathsf{EL}' + \mathsf{WC\text{-}N}' + (\Gamma^\circ)^g$ where $\Gamma^\circ$ is the set of *all classically true sentences* in $\mathcal{L}(\mathsf{EL}')$. The same result holds for I and its subsystem $\mathsf{IA}_1 + \Pi^0_1\text{-}\mathsf{AC}_{00} + \mathsf{WC\text{-}N}'$.

Kleene proved all true negative sentences of $\mathcal{L}(\mathsf{An})$ are realized by primitive recursive functions, so $\mathsf{I}^{+g} = \mathsf{I} + (\Gamma^\circ)^g$ is consistent.

Can this appeal to *truth in the preferred classical model* be avoided?

Apparently not. If $\mathcal{Y}$ is the collection of all subsystems $\mathsf{S}$ of $\mathsf{I}$ which extend $\mathsf{B}$ *and are consistent with classical logic*, then $\mathsf{I}^{+g}$ *cannot* usefully be identified with $\mathsf{I} + \bigcup\{\mathsf{S}^{+g} \colon \mathsf{S} \in \mathcal{Y}\}$.

**Proposition 8.** Consider systems $\mathsf{S}_1 = \mathsf{B} + (\mathsf{WLPO} \to \mathsf{Con}(\mathsf{B}))$ and $\mathsf{S}_2 = \mathsf{B} + (\mathsf{WLPO} \to \neg\mathsf{Con}(\mathsf{B}))$.

1. $\mathsf{S}_1$ and $\mathsf{S}_2$ belong to $\mathcal{Y}$ (Gödel's 2nd incompleteness theorem).
2. $\mathsf{S}_1 \vdash^{\circ} \mathsf{Con}(\mathsf{B})$ and $\mathsf{S}_2 \vdash^{\circ} \neg\mathsf{Con}(\mathsf{B})$.
3. $(\mathsf{S}_1)^g \vdash (\mathsf{Con}(\mathsf{B}))^g$ and $(\mathsf{S}_2)^g \vdash \neg(\mathsf{Con}(\mathsf{B}))^g$ so $\bigcup\{\mathsf{S}^{+g} \colon \mathsf{S} \in \mathcal{Y}\}$ is inconsistent.

(Inspired by Vafeiadou's idea for the proof of Theorem 3.)

We have suggested a way to compute and compare the precise constructive cost of accepting the classical interpretations of constructive systems S which are classically sound, or which consistently extend systems with preferred classical models.

There are other applications, e.g.

- The fan theorem $\mathsf{FT}_1$ is conservative over HA (Troelstra). $(\mathsf{IRA} + \mathsf{FT}_!)^{+g}$ proves intuitionistic predicate logic is weakly complete for Beth's interpretation (Gödel, Dyson, Kreisel).
- **BISH** (Bishop, Bridges, Ishihara): Informal constructive analysis, which is classically sound, is now being formalized. Resulting decomposition theorems help to compare classical contents of constructive and semi-constructive theories.
- Constructive algebra or IZF or CZF?

1. Moschovakis, J. R. and Vafeiadou, G., *Minimum classical extensions of constructive theories*, in the volume for this conference.
2. Moschovakis J. R., *Calibrating the negative interpretation*, arXiv 2101.10313 [math.LO].

and the bibliographies of these two papers.

Thank you for listening!