Fano and cluster geometry

J. Moraga

UCLA.

UC Riverside Colloquium

Outline

Table of Contents

Algebraic varieties

Let $\mathbb{K}$ be a field. We write $\mathbb{K}[x_1, \dots, x_n]$ for the polynomial ring over $\mathbb{K}$ in n variables.

An *affine variety* X is the vanishing set in $\mathbb{K}^n$ of finitely many polynomials $f_1, \dots, f_k \in \mathbb{K}[x_1, \dots, x_n]$.

$$X := \{(x_1, \dots, x_n) \mid f_1(x_1, \dots, x_n) = \cdots = f_k(x_1, \dots, x_n) = 0\} \subset \mathbb{K}^n.$$

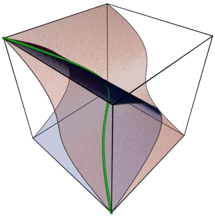

Figure: The twisted cubic.

The *projective space* $\mathbb{P}^n_{\mathbb{K}}$ consists of $(n+1)$-tuples $[x_0 : \cdots : x_n]$ up to rescaling by a non-zero constant in the field.

The projective space $\mathbb{P}^n_{\mathbb{K}}$ can be constructed from the affine space $\mathbb{K}^n$ by adding a copy of $\mathbb{P}^{n-1}_{\mathbb{K}}$ at *infinity*.

A *projective variety* $X \subset \mathbb{P}^n_k$ is the vanishing locus of finitely many homogeneous polynomials in $\mathbb{K}[x_0, \ldots, x_n]$.

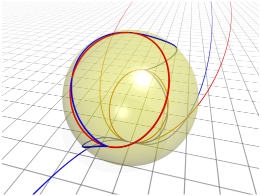

Figure: The projective line.

Projective Geometry

The idea of using projective geometry is not new. Artists from the Renaissance (15th century) already had in mind the idea of a point at infinity.

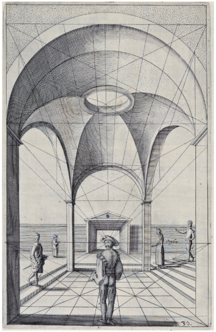

Figure: Painting using perspective.

Hypersurfaces

A *projective hypersurface* is a projective variety $X \subset \mathbb{P}^n$ defined by a single homogeneous polynomial in the variables $x_0, \dots, x_n$.

For instance, the *Fermat curves* are given by

$$C_d := \{[x_0 : x_1 : x_2] \mid x_0^d + x_1^d + x_2^d = 0\} \subset \mathbb{P}^2.$$

The number d, the degree of the polynomial, is called the *degree* of the hypersurface.

Over the field of complex numbers, the Fermat curve of degree d is a Riemann surface of genus $(d-1)(d-2)/2$.

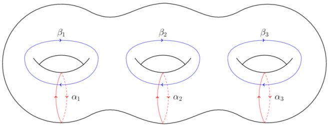

Figure: Fermat curve of degree 4.

Table of Contents

The tangent bundle

Let $X \subseteq \mathbb{P}^N_{\mathbb{K}}$ be a smooth projective variety of dimension $n \leqslant N$. To each point $x \in X$ we can associate an n-dimensional tangent space $T_{X,x}$.

The *tangent bundle* is the disjoint union of these tangent spaces.

We usually write $T_X \to X$ for the tangent bundle.

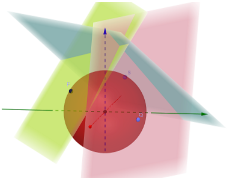

Figure: Some tangents to the sphere.

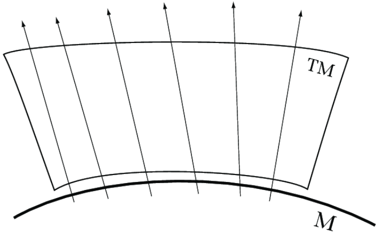

Figure: Tangent bundle of a line.

The canonical line bundle

The *canonical line bundle* of a smooth projective variety X is defined to be

$$\omega_X := \wedge^n T_X^*.$$

This bundle associates a projective line to each point x of the variety X.

We call this line bundle *canonical* because, up to isomorphism, is independent of the chosen projective embedding. Therefore, it gives a canonical way to construct a line bundle on a smooth projective variety.

The trichotomy

Let X be a smooth projective variety.

We say that X is *canonically polarized* if for some positive integer m the sections of ω_X^m defines an embedding of X into a projective space.

We say that X is *Calabi–Yau* if for some positive integer m we have $\omega_X^m \simeq \mathcal{O}_X$.

We say that X is *Fano* if for some negative integer m the sections of ω_X^m defines an embedding of X into a projective space.

Examples

The classic examples of Fano, Calabi–Yau's, and canonically polarized varieties are smooth hypersurfaces $X_d \subset \mathbb{P}^n$ of degree $d < n+1, d = n+1$, and $d > n+1$, respectively.

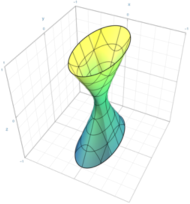

Figure: Smooth quadric.

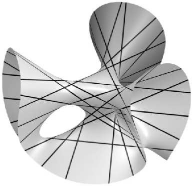

Figure: Smooth cubic.

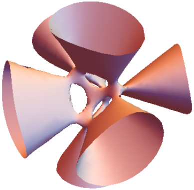

Figure: smooth quartic.

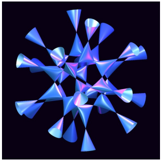

Figure: Singular sextic.

Barth sextic

Figure: Glass version of Barth sextic.

Different aspects of the trichotomy

	$\pi_1(X)$	$\mathrm{Aut}(X)$	$\mathrm{Bir}(X)$	$X(\mathbb{Q})$
Fano	Trivial	Linear Algebraic	Cremona	Dense/Empty
Calabi–Yau	Virtually Abelian	?	?	??
Canonically Polarized	??	Finite	Finite	Contained in a proper closed set[1]

Table: Some aspects of the trichotomy.

[1] Known for curves due to Faltings, open in dimension at least two.

Table of Contents

Gino Fano

Figure: Gino Fano.

Gino Fano was an italian mathematician born in 1871. Among many things, he studied 3-dimensional smooth projective varieties X that admit a dense open subset U isomorphic to a dense open of $\mathbb{P}^3$. He realized that many of these varieties have ω_X *negative*. For instance, positive multiples of ω_X did not have sections in these examples.

More Fano varieties

The projective space $\mathbb{P}^n$ is a Fano variety for every n. In dimension one, the Riemann sphere $\mathbb{P}^1$ is the only smooth Fano curve.

Products of Fano varieties are Fano varieties, hence $\prod_i \mathbb{P}^{n_i}$ is a Fano variety.

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface defined by a homogeneous polynomial of degree d. Then, the variety X_d is Fano if and only if $d < n + 1$.

Corollary

There are only finitely many families of n-dimensional smooth Fano hypersurfaces.

More Fano varieties

Not all Fano varieties are hypersurfaces.
Indeed, we can consider the embedding:

$$\pi \colon \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$$

given by

$$\pi([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = [x_0y_0 : x_0y_1 : \cdots : x_2y_2].$$

The variety $X = \pi(\mathbb{P}^2 \times \mathbb{P}^2)$ is defined by several (more than 5) equations in the 9 variables of $\mathbb{P}^8$.

If we consider instead the similarly defined embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, we obtain the following picture:

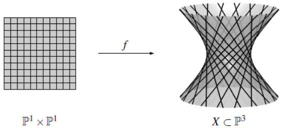

Figure: Quadratic hypersurface.

Smooth Fano surfaces

Smooth Fano surfaces, also known as *del Pezzo surfaces*, were classified by the Italian school of algebraic geometers, circa 1920.

They realized that a smooth cubic hypersurface X_3 in $\mathbb{P}^3$ contains precisely 27 lines. Further, 6 of these lines are pairwise skew. If we *blow down* these lines 6 lines, we get back to $\mathbb{P}^2$. In other words, every smooth cubic surface is the *blow-up* of $\mathbb{P}^2$ at 6 points.

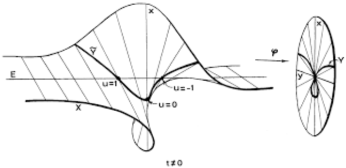

Figure: Blow-up

Boundedness of smooth Fano surfaces

The Italian school of algebraic geometers proved the following theorem.

Theorem

Every smooth Fano surface X is either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of $\mathbb{P}^2$ in at most 8 points in general position. In particular, there are only finitely many families of smooth Fano surfaces.

Smooth Fano 3-folds were classified by the work of many mathematicians, inlcuding Shokurov, Iskovskikh, Prokhorov, Mori, and Mukai.

There are 105 families of smooth Fano threefolds.

FANOGRAPHY

A tool to visually study the geography of Fano 3-folds.

Fano threefolds with $\rho = 1$

ID	$-K_X^3$	g	$h^{1,2}$	index	description	blowups	rational	unirational	moduli	Aut^0
1-1	2	2	52	1	double cover of $\mathbb{P}^3$ with branch locus a divisor of degree 6 **alternative** hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$		no	?	68	0
1-2	4	3	30	1	a) hypersurface of degree 4 in $\mathbb{P}^4$ b) double cover of 1-16 with branch locus a divisor of degree 8		no	some	a) 45 b) 44	0
1-3	6	4	20	1	complete intersection of quadric and cubic in $\mathbb{P}^5$		no	yes	34	0
1-4	8	5	14	1	complete intersection of 3 quadrics in $\mathbb{P}^6$		no	yes	27	0
1-5	10	6	10	1	**Gushel–Mukai 3-fold** a) section of Plücker embedding of $\mathrm{Gr}(2,5)$ by codimension 2 subspace and a quadric b) double cover of 1-15 with branch locus an anticanonical divisor		generically non-rational	yes	a) 22 b) 19	0
1-6	12	7	7	1	section of half-spinor embedding of a connected component of $\mathrm{OGr}_+(5,10)$ by codimension 7 subspace		yes	yes	18	0
1-7	14	8	5	1	section of Plücker embedding of $\mathrm{Gr}(2,6)$ by codimension 5 subspace		no	yes	15	0
1-8	16	9	3	1	section of Plücker embedding of $\mathrm{SGr}(3,6)$ by codimension 3 subspace		yes	yes	12	0
1-9	18	10	2	1	section of the adjoint G_2-Grassmannian $G_2\mathrm{Gr}(2,7)$ by codimension 2 subspace		yes	yes	10	0

Figure: Fanography webpage.

Boundedness of smooth Fano varieties

In 1992, Kollár, Miyaoka, and Mori proved the following theorem.

Theorem

Let n be a postive integer. There are only finitely many families of n-dimensional smooth Fano varieties.

There are still many open problems regarding smooth Fano varieties. For instance, we do not understand how many families are in each dimension.

Table of Contents

Cluster algebras

Cluster algebras are a special kind of rings introduced by Fomin and Zelevinsky in the early 2000's. They introduced these rings to unify some major areas in Lie theory. Any cluster algebra R can be written as

$$R = L_1 \cap L_2 \cap \cdots \cap L_k \subset \mathbb{K}(x_1, \ldots, x_n),$$

where each L_i is a Laurent polynomial ring.

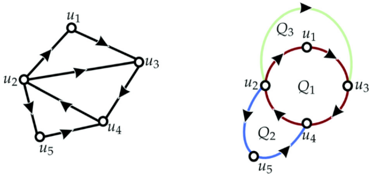

Figure: Quivers representing cluster algebras.

Binomial Mutations

We can transform L_i into L_{i+1} via a *binomial mutation*. This means that the generators of L_{i+1} have the form

$$\left(x_1, \ldots, \frac{b(x_1, \ldots, \hat{x}_i, \ldots, x_n)}{x_i}, \ldots, x_n\right)$$

for a suitable binomial b, where the x_i's are the generators of L_i.

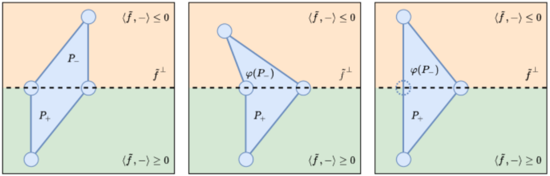

Figure: Combinatorial representation of a Binomial Mutation.

Geometry of cluster algebras

Given a cluster algebra R, its spectrum $U = \mathrm{Spec}(R)$ satisfies two beautiful geometric properties:

(i) U is covered by copies of $\mathbb{G}_m^n$, up to a subset of codimension at least two, and

(ii) the volume form $\Omega := \frac{dx_1 \wedge \cdots \wedge dx_n}{x_1 \cdots x_n}$ of $\mathbb{G}_m^n$ has no poles or zeros on U.

Cluster type varieties

We say that a variety X is of *cluster type* if there exists an embedding in codimension one $\mathbb{G}_m^n \dashrightarrow X$ such that Ω has no zeros on X.

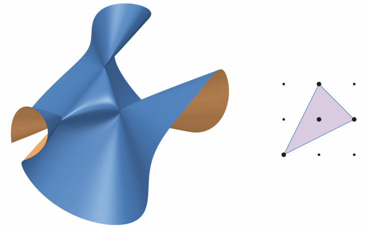

Figure: A toric variety and its polytope.

For a cluster type variety X, the open set $X \setminus \mathrm{Poles}(\Omega)$ is covered by copies of $\mathbb{G}_m^n$ up to a subset of codimension at least two. This motivates the name *cluster type*.

Toric varieties are the quintessential cluster type varieties. Many Fano varieties are cluster type. For instance, any del Pezzo surfaces of degree $\geqslant 2$ are cluster type. Many smooth Fano 3-folds as well.

In work in progress, together with Enwright, Francone, and Spink, we prove that most varieties from Lie theory are cluster type. This includes Grassmannians, Flags, Richardson, Bott-Samelson, etc.

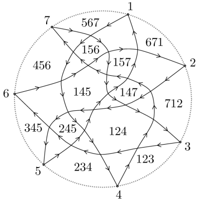

Figure: Combinatorics of a Grassmanian.

Boundedness of smooth Fano cluster type varieties

The following theorem is a consequence of the *constructibility* of the cluster type property. It was proved by Ji (UIUC) and the speaker:

Theorem

In each dimension n, there are only finitely many families of smooth Fano varieties which are cluster type.

Compactifying cluster algebras

The following theorem is work in progress by Enwright (UCLA), Francone (Tor Vergata), M, and Spink (U Toronto):

Theorem

Let R be a locally acyclic finitely generated cluster algebra. Let $U = \mathrm{Spec}(R)$. Then, the variety U admits:

1. *A klt type Fano compactification $U \hookrightarrow X$,*
2. *A canonical Fano type compactification $U \hookrightarrow \tilde{X}$.*

Summary

- Cluster algebras (more generally, mutation algebras) can be compactified into Fano varieties. Thus, tools from Fano geometry may be used to study these algebras.
- Many interesting classes of Fano varieties are cluster type. Thus, we may use tools from cluster algebras to study and classify these varieties. The boundedness theorem implies that this classification is plausible.

Thanks for your attention!