Cluster type varieties

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January 7th, 2026 — Joint Mathematical Meetings 2026

Embeddings in codimension one

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Definition

A variety X is said to be *cluster type* if there is a embedding in codimension one $\mathbb{G}_m^n \dashrightarrow X$ and Ω_n has no zeros on X. The divisor B given by the poles of Ω_n is called a *cluster type boundary*. We say that (X, B) is a *cluster type pair*.

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Theorem (Corti 23, Enwright-Figueroa-M 24)

Let (X, B) be a cluster type pair. Then, a big open subset of $U := X \setminus B$ is covered by images of embedding in codimension ones $\iota_j \colon \mathbb{G}_m^n \dashrightarrow X$. Furthermore, given any two such birational maps ι_1 and ι_2, we have $(\iota_1 \circ \iota_2^{-1})^* \Omega_n = c\Omega_n$ where $c \in \mathbb{K}^*$.

First examples

Example: Toric varieties

A toric variety is cluster type. Indeed, toric pairs can be characterized as cluster type pairs for which there is a unique embedding in codimension one $\mathbb{G}_m^n \dashrightarrow X \setminus B$.

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Let R be a finitely generated cluster algebra. Then $U := \mathrm{Spec}(R)$ is an affine cluster type variety. In this case, Ω_n has no poles or zeros on U.

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Let R be a finitely generated cluster algebra. Then $U := \mathrm{Spec}(R)$ is an affine cluster type variety. In this case, Ω_n has no poles or zeros on U.

Smooth surfaces

Theorem (Gross-Hacking-Keel 05)

A smooth projective surface X is cluster type if and only if $|-K_X|$ admits a nodal curve.

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Corollary

A smooth del Pezzo surface of degree at least two is cluster type.
A general smooth del Pezzo surface of degree one is cluster type.

Singular surfaces

Theorem (EFM25)

Let X be a klt surface and (X, B) be a cluster type pair. Then X has toric singularities and $X \setminus B$ has A_n singularities.

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Let X be a Gorenstein del Pezzo surface of rank one. Then X is cluster type if and only if the following two conditions hold:

1. X only has A_n singularities, and
2. either $\mathrm{vol}(X) \geq 2$ or $|X^{\mathrm{sing}}| < 4$.

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Varieties from Lie theory

The following theorem follows from the work of Knutson-Lam-Speyer and Brion-Kumar.

Theorem

The following classes of varieties are cluster type: Flag varieties, Schubert varieties, Bott-Samelson varieties, Richardson varieties, and Brick manifold compactifications of Richardson varieties.

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Algebraic mutation datum

Algebraic mutation datum

Let N be a free finitely generated abelian group and M its dual. An *algebraic mutation datum* is a pair $(u, h = g^k)$, where $u \in N$ is a primitive vector, $g \in \mathbb{K}[u^\perp \cap M]$ is an irreducible Laurent polynomial, and k is a positive integer. For a polyhedral cone $\sigma \subset N$, we say that $(u, h = g^k)$ is σ-*admissible* if $u \notin \sigma$.

Algebraic mutation

Algebraic mutation

Let $\sigma_1, \sigma_2 \subset N_{\mathbb{Q}}$ be two rational polyhedral cones and $U_{\sigma_1}, U_{\sigma_2}$ be the corresponding affine toric varieties. A birational map $\mu\colon U_{\sigma_1} \dashrightarrow U_{\sigma_2}$ is a *mutation* if the two following conditions are satisfied:

(i) the induced isomorphism $\mu^*\colon \mathbb{K}(M) \to \mathbb{K}(M)$ is given on monomials by $\mu^*(x^m) = x^m h^{-\langle u,m\rangle}$ for some σ_2-admissible algebraic mutation datum (u, h), and

(ii) strict transform via μ induce a bijection between the prime torus invariant divisors of U_{σ_1} and U_{σ_2}.

Embedded semigroup algebras

Embedded semigroup algebras

Let $\sigma \subset N_{\mathbb{Q}}$ be a rational polyhedral cone and $\mathbb{K} \hookrightarrow F$ be a field extension. A $\mathbb{K}$-algebra homomorphism $\iota \colon \mathbb{K}[\sigma^\vee \cap M] \hookrightarrow F$ is an *embedded semigroup algebra* if it induces an isomorphism $\mathbb{K}(M) \simeq F$.

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Two embedded semigroup algebras $\iota_i\colon \mathbb{K}[\sigma_1^\vee \cap M] \hookrightarrow F$ with $i \in \{1, 2\}$ are said to *differ by a mutation* if $\iota_1^{-1} \circ \iota_2$ is a mutation.

Mutation semigroup algebra

Mutation semigroup algebra

A *mutation semigroup algebra* is a finitely generated ring R over $\mathbb{K}$ which can be expressed as

$$R = R_0 \cap \cdots \cap R_k$$

where the following conditions hold for $i \in \{1, \ldots, k\}$:

(i) there is an embedded semigroup algebra $\iota_i \colon \mathbb{K}[\sigma_i^\vee \cap M] \hookrightarrow \mathrm{Frac}(R)$ with image R_i;

(ii) the embedded semigroup algebras ι_0 and ι_i differ by a mutation; and

(iii) for each prime ideal $\mathfrak{p} \subset R_i$ of height one, the prime ideal $\mathfrak{p} \cap R_i$ has height one in R_i.

We abbreviate mutation semigroup algebra by MSA. We say that R is just a *mutation algebra* if each $\sigma_i = \{0\}$.

MSA vs cluster type

Theorem (Enwright-Francone-M-Spink 25)

Let R be a finitely generated commutative ring over $\mathbb{K}$. Assume that $U = \mathrm{Spec}(R)$ has klt singularities. Then, the following two conditions are equivalent:

- R is a mutation semigroup algebra,
- there exists a normal projective X, a cluster type pair (X, B), an ample divisor $A \leqslant B$, and an isomorphism
$$R \simeq \mathcal{O}(X \setminus A).$$

Applications

Theorem (EFMS25)

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Theorem (EFMS25)

Let X be a $\mathbb{Q}$-factorial Fano variety. The variety X is cluster type if and only if $\mathrm{Cox}(X)$ is a $\mathrm{Cl}(X)$-graded MSA.

Smooth Fano threefolds

The previous theorem hints that *many* rational smooth Fano threefolds are cluster type. Derenthal, Hausen, Heim, Keicher, and Laface has described the Cox rings of smooth Fano threefold. One example is the following.

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(12) † The smooth Fano threefold X_{12} has the $\mathbb{Z}^2$-graded Cox ring $\mathbb{K}[T_1, \ldots, T_{10}]/I$ with generators for I and the degree matrix given by

$$
\begin{array}{ll}
T_1T_7 - T_2T_8 + T_4T_6, & -T_2T_9 + T_5^2T_6 - T_5T_6T_8 - T_5T_7T_8 - \\
-T_1T_6 + T_2T_7 + T_3T_5 - T_3T_8, & T_6^2T_7 + T_7T_8^2, \\
T_1T_5 - T_2T_6 - T_3T_7 + T_4T_8, & T_1^2T_8^2 - T_1T_2T_6T_7 - T_1T_2T_6T_8 + T_1T_3T_6T_8 + \\
T_4T_9 - T_5^2T_8 + T_5T_6T_7 + T_5T_8^2 - T_7^3, & T_1T_3T_7^2 - 2T_1T_3T_7T_8 + T_1T_4T_8^2 + T_2^2T_5T_6 \\
T_1^3 - T_1T_2T_3 + T_1T_2T_4 + T_3^2T_4 - T_8T_{10}, & -T_2^2T_6T_8 + T_2T_3T_6T_7 - T_2T_3T_7T_8 - T_2T_4T_5T_8 \\
T_1T_9 - T_5T_6^2 + T_5T_8^2 + T_6^2T_8 + T_6T_7^2 - T_8^3, & +T_2T_4T_8^2 - T_3^2T_6T_7 + T_3^2T_7^2 + T_3T_4T_7^2 - \\
-T_1^2T_4 + T_2^3 - T_2T_4^2 + T_3^2T_4 - T_5T_{10}, & T_3T_4T_7T_8 - T_9T_{10}, \\
-T_1^2T_3 + T_1T_2^2 - T_1T_3T_4 + T_2T_3^2 - T_6T_{10}, & -T_3T_9 + T_4T_9 - T_5^2T_8 + T_5T_8^2 + T_6^3 - T_6T_8^2 \\
T_1^2T_2 + T_1T_3T_4 - T_2^2T_3 + T_3T_4^2 - T_7T_{10}, & -T_7^3 + T_7^2T_8
\end{array}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 8 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 3 & 1
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Thanks for your attention!