Pathologies of the volume function

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X smooth projective variety, $D$ a divisor.

Theorem: There are constants $C_i$ such that for large divisible $m$, $C_1 m^\kappa < h^0(X, mD) < C_2 m^\kappa$.

This $\kappa$ is the Iitaka dimension of $D$. 
Numerical invariance

- This is not a numerical invariant: it can happen that $D \equiv D'$ but different Iitaka dimension.

- On threefold: two numerically equivalent divisors, one rigid and one which moves in a pencil.

- We want a numerically invariant version $\nu$
Fix sufficiently ample $A$.

Look at growth of $h^0([mD] + A)$ as $m$ increases.

How does it behave?
Proposition 3.3.2. Let $X$ be a smooth projective variety and let $D$ be a pseudo-effective $\mathbb{R}$-divisor. Let $B$ be any big $\mathbb{R}$-divisor. If $D$ is not numerically equivalent to $N_\sigma(D)$, then there is a positive integer $k$ and a positive rational number $\beta$ such that

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + kB)) > \beta m, \quad \text{for all} \quad m \gg 0.$$  

Proof. Let $A$ be any integral divisor. Then we may find a positive integer $k$ such that

$$h^0(X, \mathcal{O}_X(kB - A)) \geq 0.$$ 

Thus it suffices to exhibit an ample divisor $A$ and a positive rational number $\beta$ such that

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) > \beta m \quad \text{for all} \quad m \gg 0.$$ 

Replacing $D$ by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply (V.1.12) of [28]. \qed
Background

Question (Nakayama, 2002)

Suppose that $D$ is a pseudoeffective divisor and that $A$ is ample. Then there exist constants $C_1, C_2$ and a positive integer $\nu(D)$ so that:

$$C_1 m^{\nu(D)} \leq h^0(\lceil mD \rceil + A) \leq C_2 m^{\nu(D)}$$
Volume

- Volume of $D$ is $\lim_{m \to \infty} \frac{h^0(mD)}{m^d/d!}$.

- $\text{vol}(D + tA)$ for small $t \leftrightarrow h^0(mD + A)$ for large $m$:

$$\text{vol} \left( D + \frac{1}{m} A \right) = \frac{1}{m^3} \text{vol}(mD + A) \approx \frac{1}{m^3} h^0(mD + A)$$
For each $N \geq 3$ there exists a smooth Calabi–Yau $N$-fold such that for any $\delta \in \left[1, \frac{N}{2}\right]$ one can find a pseudoeffective $\mathbb{R}$-divisor $D$ with:

\[
\limsup_{m \to \infty} \frac{\log h^0(X, \lfloor mD \rfloor + A)}{\log m} = N - \delta
\]
\[
\liminf_{m \to \infty} \frac{\log h^0(X, \lfloor mD \rfloor + A)}{\log m} = \frac{N}{2}
\]
\[
\liminf_{s \to 0^+} \frac{\log \text{vol}(D + sA)}{\log s} = \frac{N}{2}
\]
\[
\limsup_{s \to 0^+} \frac{\log \text{vol}(D + sA)}{\log s} = \delta,
\]
Regularity of volume

- Volume is $C^1$ on the big cone, but not $C^2$ in general.

- But what about the pseudoeffective boundary? Could $s \mapsto \text{vol}(D + sA)$ have extra regularity?

- The example: $s \mapsto \text{vol}(D + sA)$ is $C^1$ but not $C^{1,\alpha}$ on $[0, \epsilon)$ for any $\alpha > 0$.

- (Could it be $C^{1,1}_{loc}$ inside the big cone?)
Warm-up

Let $X$ be $(1, 1), (1, 1), (2, 2)$ complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$.

This is a smooth CY3, Picard rank 2.

Studied by Oguiso in connection with Kawamata–Morrison conj.
The example

- It has some birational automorphisms coming from covering involutions.

- Action on $N^1(X)$ given by

  $$\tau_1^* = \begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}, \quad \tau_2^* = \begin{pmatrix} -1 & 0 \\ 6 & 1 \end{pmatrix}, \quad \phi^* = \begin{pmatrix} 35 & 6 \\ -6 & -1 \end{pmatrix}$$

- Composition has infinite order: $\lambda = 17 + 12\sqrt{2}$.

- Nef cone bounded by $H_1, H_2$.

- Psef cone bounded by $(1 \pm \sqrt{2})H_1 + (1 \mp \sqrt{2})H_2$.

- Let $D_+ = c_1 H_1 + c_2 H_2$ be divisor in this class.
For any line bundle whatsoever on $X$, you can compute $h^0(D)$. Pull it back some number of times, it’s ample, and then compute $h^0$ for ample using Riemann-Roch+Kodaira vanishing!

HRR on CY3:

$$
\chi(D) = \frac{D^3}{6} + \frac{D \cdot c_2(X)}{12}.
$$
Let’s compute

- Suppose our ample is $A = M_1D_+ + M_2D_-$. 
- We need to compute $h^0(\lfloor mD \rfloor + A)$. 
- How many times to pull back? Looks like a mess, but there’s an invariant quadratic form: the product of the coefficients when you work in the eigenbasis.

$$\phi^* = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$
Let’s compute, II

- $mD_+ + A = (m + M_1)D_+ + M_2D_-$. 

- The pullback that’s ample has the two coefficients roughly equal, about $\left(\sqrt{(m + M_1)M_2}\right)D_+ + \left(\sqrt{(m + M_1)M_2}\right)D_-$. 

- Then $h^0([mD_+] + A) \approx Cm^{3/2}$. 
Extensions of the computation

We used the fact that $D_+$ and $D_-$ that span an eigenspace intersecting the ample cone.

In this case, an eigenvector always has $\nu_{vol}(D_+) = \frac{\dim X}{2}$ if $\lambda_1(f) = \lambda_1(f^{-1})$. 
Let $X$ be a $(2, 2, 2, 2)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; key aspects of geometry worked out by Cantat–Oguiso.

It’s a smooth CY3.

$(\mathbb{Z}/2\mathbb{Z})^4 \subset \text{Bir}(X)$ coming from covering involutions.

Kawamata–Morrison conjecture is true, so we can still compute volume of any class very easily in principle...
Two kinds of divisors on the psef boundary

- There are some distinguished classes:
  - eigenvectors, which all have \( \text{vol}(D + tA) \sim Ct^{3/2} \);
  - semiample \( \pi_i^*(\mathcal{O}_\mathbb{P}^1(1)) + \pi_j^*(\mathcal{O}_\mathbb{P}^1(1)) \), plus their orbits under \( \text{Bir}(X) \), which all have \( \text{vol}(D + tA) \sim Ct \).
First picture: the eigenvectors
Second picture: the semiample type
There are some “circles” on the boundary of $\overline{\text{Eff}}(X)$ on which both eigenvectors and semiample type are dense.

The former have $\text{vol}(D + tA) \sim Ct^{3/2}$, the latter have $\text{vol}(D + tA) \sim Ct$.

How is this possible?

Because the volume function is so easy to compute numerically, we can plot it!
Volume near the boundary
A quotient of the movable cone

- Action of Bir($X$) preserves a quadratic form of signature $(1, 3)$ on $N^1(X)$ (Cantat–Oguiso).

- Restricting to classes of norm 1 and taking quotient of movable cone by this action, we obtain action of Bir($X$) on a (non-compact) hyperbolic 3-manifold $\Sigma$.

- Volume function descends to $\overline{\text{vol}}: \Sigma \to \mathbb{R}_{\geq 0}$.
How to imagine these classes

Paths $D + sA$ in $N^1(X)$ determine a geodesic on $\Sigma$ after normalization:

$$\gamma(s) = \frac{D + sA}{\sqrt{Q(D + sA, D + sA)}} \approx \frac{D + sA}{\sqrt{s}}$$

Then

$$\text{vol}(D + sA) = \text{vol}(\sqrt{s}\gamma(s)) = s^{3/2} \text{vol}(\gamma(s)) = s^{3/2} \text{vol}([\gamma(s)])$$

When $[\gamma(s)]$ is near the middle of $\Sigma$, we see $s^{3/2}$ behavior, but volume gets larger when the geodesic goes out a cusp.
Recurrent geodesics

- If geodesic stays in a compact region (typical, e.g. eigenvectors), we see $s^{3/2}$ growth as $s \to 0$.

- But if a geodesic wanders out a cusp, we see the larger volume $s^1$.

- Every geodesic ray is either returns infinitely often to a compact set, or goes into the cusp.

- In particular, we either see $\text{vol}(D + sA) \sim s$ growth, or $\text{vol}(D + sA) \sim Cs^{3/2}$ along an infinite subsequence of $s$. 
Cusp excursions

- Suppose $\ell_i$ is a sequence of (sufficiently large) positive reals. For any $x_0 \in \mathcal{M}^{cc}$ and open $U \subset T_{x_0}M$ three is an infinite geodesic ray $\gamma$ starting
  - initial tangent vector is in $U$;

- $\gamma = \bigcup [x_i, x_{i+1})$ with $\ell_{[x_i, x_{i+1})} = \ell_i + O(1)$ and $d(x_0, x_i) = O(1)$;

- $d(x_0, -)$ on $(x_i, x_{i+1})$ is roughly linearly growth out to $\frac{1}{2}\ell_i$ and then decreasing back.

- The key technical ingredient is “gluing geodesics”: we write down each desired cusp excursion separately, and as long as the endpoint data are very close, there is a nearby geodesic approximating the union.
We get oscillation of \( \text{vol}(D + sA) \), hence \( m^3 \, \text{vol} \left( D + \frac{1}{m}A \right) \) if we make sure oscillations occur when \( s = \frac{1}{m} \).

We also need to bound errors of (a) \( h^0(X, \lfloor mD \rfloor + A) \) vs \( \text{vol}(\lfloor mD \rfloor + A) \) and (b) \( \text{vol}(\lfloor mD \rfloor + A) \) vs \( \text{vol}(mD + A) \).

The first is fairly easy since we can compute \( h^0 \) of the ample pullback using HRR; one term is the volume and we bound the error.

For the second we need to check how far rounding moves in the hyperbolic distance and make sure it doesn’t interfere.

(Both are OK.)
Nakayama’s $\kappa_\sigma$s

\[
\kappa_\sigma^+(D) = \min \left\{ k : \limsup_{m \to \infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} < \infty \right\}
\]

\[
\kappa_\sigma(D) = \max \left\{ k : \limsup_{m \to \infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} > 0 \right\}
\]

\[
\kappa_\sigma^-(D) = \max \left\{ k : \liminf_{m \to \infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} > 0 \right\}
\]

In our example:

\[
\kappa_{\mathbb{R},-}(D) = \left\lfloor \frac{N}{2} \right\rfloor, \quad \kappa_{\mathbb{R}}(D) = \kappa_{\mathbb{R},+}(D) = N - 1
\]

So these things are not the same.
Let $X$ be smooth projective over $\mathbb{C}$ and $D$ pseudoeffective. Then there exist $m_0 \geq 1$, $c > 0$, and ample $A$ so that

$$h^0(X, \lfloor mm_0 D \rfloor + A) \geq cm^{\kappa_\sigma(D)}$$

We show that this fails starting in dimension 5.