On cubic surface bundles

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Birational Geometry Seminar
Online seminar
General question of interest: determine which smooth projective varieties $X$ are rational: is $X$ birational to $\mathbb{P}_k^n$? (or stably rational, or retract rational...)

Methods:
- $X$ is rational: come up with a geometric construction;
- $X$ is not rational: find invariants of $X$; find invariants of a (maybe singular) specialization $X_0$ of $X$.

This motivates: what is the available pool of $X_0$ with invariants?

Goal: add to the pool $X_0 \to \mathbb{P}^2_C$ with fibers cubic surfaces: invariants: use Galois cohomology and geometry of cubics; example: $X$: $x z^2 u^3 + y^2 z v^3 + x y^2 w^3 + f t^3 = 0 \subset \mathbb{P}^2[x:y:z] \times \mathbb{P}^3[u:v:w:t]$ $f = x^3 + y^3 + z^3 + 3 x^2 y + 3 x y^2 + 3 y^2 z + 3 y z^2 + 3 x z^2 + 3 x^2 z$. 


Summary/plan

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2. Methods:
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\[ xz^2u^3 + y^2z^3v^3 + xy^2w^3 + ft^3 = 0 \subset \mathbb{P}^2 \times \mathbb{P}^3 \]
\[ f = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3xz^2. \]
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Example: $X: xz^2u^3 + y^2z^3 + xy^2w^3 + ft^3 = 0 \subset \mathbb{P}^2 \times \mathbb{P}^3$ with $f = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3yz^2 + 3xz^2 + 3x^2z$. 


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INTRODUCTION
Properties of rationality

Let $k$ be a field, $X/k$ a projective integral variety.

- **$X$ is rational**: $X$ is birational to $\mathbb{P}^n_k \iff k(X)/k$ is a purely transcendental extension;
- **$X$ is stably rational**: $X \times \mathbb{P}^m_k$ is rational, for some $m$;
- **$X$ is unirational**: there is a dominant rational map $\mathbb{P}^n_k \dasharrow X$;

We have implications $\implies$.

All notions are equivalent for $X/C$ smooth, of dimension 1 ($X \cong \mathbb{P}^1_C$) or 2 (birational class of $\mathbb{P}^2_C$).
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**Next**: typical examples and counterexamples.
Rationality proofs

Notation:

\[ X_d \subset \mathbb{P}^n_k : f(x_0, \ldots x_n) = 0, \deg f = d \text{ a smooth hypersurface.} \]
Rationality proofs

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\[ X_d \subset \mathbb{P}^n_k : f(x_0, \ldots x_n) = 0, \deg f = d \text{ a smooth hypersurface.} \]

- smooth quadrics \( X_2 \) with \( X_2(k) \neq \emptyset \) are **rational**:

**Rational** parametrization:

\textit{nontangent} lines through \( A \leftrightarrow \) second intersection point with the quadric.
Irrationality proofs over $\mathbb{C}$: classical

Classical methods:

- compute some invariant $i(X)$;
- $i(X) \neq 0 \Rightarrow X$ is not rational.
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Examples of not rational smooth threefolds:
1. $X_3 \subset \mathbb{P}_\mathbb{C}^4$ (Clemens-Griffiths, using intermediate Jacobian);
2. $X_4 \subset \mathbb{P}_\mathbb{C}^4$ (Iskovskikh-Manin, using rigidity);
3. $Z$ a resolution of

$$Y : z_4^2 - f_4(x_0, x_1, x_2, x_3) = 0$$

a double cover of $\mathbb{P}_\mathbb{C}^3$ ramified along some quartic (Artin-Mumford, $H^3(Z, \mathbb{Z})_{tors} = Br Z \neq 0$).

These varieties provide examples of unirational not rational complex threefolds.
Irrationality proofs over $\mathbb{C}$: specialization

(Beauville, Voisin, Colliot-Thélène–Pirutka, Totaro, Schreieder):

- consider a family of varieties:

$$
\begin{array}{c}
\mathcal{X} \leftarrow X_0 \leftarrow \text{reference variety} \\
\downarrow \downarrow \\
B \leftarrow 0 
\end{array}
$$

- compute a suitable invariant $i(X_0)$;
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- compute a suitable invariant $i(X_0)$;

- $i(X_0) \neq 0 + \text{EPSILON} \Rightarrow \text{a very general } X = \mathcal{X}_b \text{ is not (stably) rational};$

- (in some cases, all previously computable $i(X)$ vanish);
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  \end{array}
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(in some cases, all previously computable \( i(X) \) vanish);

\( \chi_b \text{ very general}: b \notin \bigcup_{i \in \mathbb{N}} B_i(\mathbb{C}), B_i \subset B \text{ closed.} \)

EPSILON:

- restriction on singularities of \( X_0 \);
- "restriction to subvarieties" for \( i \) (Schreieder).
$X$ not stably rational by specialization

1. $\dim X_d = 3$: (Colliot-Thélène–Pirutka), $d = 4$;
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(Schreieder) $X_d \subset \mathbb{P}^{n+1}$ with

$$d \geq \log_2 n + 2,$$

this generalizes previous bounds by Kollár, and Totaro, of order $d \sim \geq 2/3n$. 

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   (Kollár) \( d = 6 \);
4. (Schreieder) \( X_d \subset \mathbb{P}^{n+1} \) with
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   this generalizes previous bounds by Kollár, and Totaro, of order
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Other examples:
- cyclic covers,
- complete intersections,
- hypersurfaces in \( \mathbb{P}^m \times \mathbb{P}^n \), and more.
Available reference varieties $X_0$

- $X_0$: a conic of quadric surface bundle over $\mathbb{P}^2$,

\[ i = Br(X'_0)[2] = H^2_{nr}(X_0, \mathbb{Z}/2) \subset H^2(\mathbb{C}(X_0), \mathbb{Z}/2) \]

here $X'_0 \to X_0$ is a resolution of singularities.
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- $X_0$ : a fibration over $\mathbb{P}^n$ in Fermat-Pfister forms, $i = H^m_{nr}(X_0)$. 
Galois cohomology
Assume: $K \supset \mu_n$.

- $H^0(K, \mathbb{Z}/n) \simeq \mathbb{Z}/n$;
- $H^1(K, \mathbb{Z}/n) \simeq K^*/K^{*n}$ (Kummer), for $a \in K^*$, we will still denote by $a$ its class in $H^1(K, \mathbb{Z}/n)$.
- $Br(K)[n] = H^2(K, \mathbb{Z}/n)$ (Kummer);
  symbols: $(a, b) := a \cup b \in H^2(K, \mathbb{Z}/n), a, b \in K^*$.
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- $\nu: K \to \mathbb{Z} \cup \{\infty\}$ a discrete valuation of rank 1:
  Recall: $\nu(x) = \infty \iff x = 0$
  $\nu(xy) = \nu(x) + \nu(y)$
  $\nu(x + y) \geq \min(\nu(x), \nu(y))$
- $A$ be the valuation ring: $A = \{x, \nu(x) \geq 0\}$,
- $\kappa(\nu)$ the residue field: $\kappa(\nu) = A/m$,
  $m = \{x, \nu(x) > 0\} = (\pi_A)$, $\pi_A$ is a uniformizer
Assume: $K \supset \mu_n$.

1. $H^0(K, \mathbb{Z}/n) \cong \mathbb{Z}/n$;
2. $H^1(K, \mathbb{Z}/n) \cong K^*/K^{*n}$ (Kummer),
   for $a \in K^*$, we will still denote by $a$ its class in $H^1(K, \mathbb{Z}/n)$.
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   symbols: $(a, b) := a \cup b \in H^2(K, \mathbb{Z}/n)$, $a, b \in K^*$.

2. $v : K \to \mathbb{Z} \cup \infty$ a discrete valuation of rank 1:
   Recall: $v(x) = \infty \iff x = 0$
   $v(xy) = v(x) + v(y)$
   $v(x + y) \geq \min(v(x), v(y))$

4. $\kappa(v)$ the residue field: $\kappa(v) = A/m$,
5. $m = \{x, v(x) > 0\} = (\pi_A)$, $\pi_A$ is a uniformizer
6. this gives $\partial^i_v : H^i(K, \mathbb{Z}/n) \to H^{i-1}(\kappa(v), \mathbb{Z}/n)$.
7. $\partial^i_v$ factors through the completion $H^i(K_v, \mathbb{Z}/n)$.
Formulas for residus

\[ a, b \in H^1(K, \mathbb{Z}/n) \cong K^*/K^{*n} \]

\[ \partial_1^1(a) = \nu(a) \mod n \in H^0(\kappa(\nu), \mathbb{Z}/n) \cong \mathbb{Z}/n, \]
Formulas for residus

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1. \( \partial_1^1(a) = \nu(a) \mod n \in H^0(\kappa(\nu), \mathbb{Z}/n) \simeq \mathbb{Z}/n, \)

2. \( \partial_2^2(a, b) = (-1)^{\nu(a)\nu(b)} \frac{a^\nu(b)}{b^\nu(a)} \)

where \( \frac{a^\nu(b)}{b^\nu(a)} \) is the image of the unit \( \frac{a^\nu(b)}{b^\nu(a)} \) in \( \kappa(\nu)^*/\kappa(\nu)^*n \).
Formulas for residus

\[ a, b \in H^1(K, \mathbb{Z}/n) \cong K^*/K^{*n} \]

1. \( \partial^1_v(a) = v(a) \mod n \in H^0(\kappa(v), \mathbb{Z}/n) \cong \mathbb{Z}/n, \)

2. \( \partial^2_v(a, b) = (-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}} \)

where \( \frac{a^{v(b)}}{b^{v(a)}} \) is the image of the unit \( \frac{a^{v(b)}}{b^{v(a)}} \) in \( \kappa(v)^*/\kappa(v)^{*n} \).

3. In particular, \( \partial^2_v(a, b) = 0 \) if \( v(a) = v(b) = 0. \)
Example

- $S = \mathbb{P}^2_\mathbb{C}$, $K = \mathbb{C}(x, y)$, $\alpha = (x, y) \in H^2(K, \mathbb{Z}/2)$;
- $\nu_D : K^* \to \mathbb{Z}$ is the order of vanishing at $D = \{x = 0\}$;
- recall: $\partial^2_v(a, b) = (-1)^{\nu(a)\nu(b)} \frac{a^{\nu(b)}}{b^{\nu(a)}}$;
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- then $\partial^2_{v_D}(\alpha) = \partial^2_{v_D}(x, y) = y \in \mathbb{C}(y)^*/\mathbb{C}(y)^*^2$. 
$H^i_{nr}$: definition

- $X/k$ an integral variety, then

$$H^2_{nr}(X) = H^2_{nr}(k(X)/k) = \bigcap_v \ker \partial^2_v$$

where the intersection is over all discrete valuations $v$ on $k(X)$ (of rank one), trivial on the field $k$. 

Birational invariant by definition (Saltman, Bogomolov, Colliot-Thélène-Ojanguren).

If $X/k$ is stably rational, then

$$H^i(k) \cong H^i_{nr}(k(X)/k).$$

Advantage: No need to compute a smooth model of $X/k$.

Fact: if $X$ is smooth and projective, $H^2_{nr}(X, \mathbb{Z}/n) \cong Br(X)[n]$. 
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- One has

$$H^2(k) \to H^2_{nr}(k(X)/k)$$

(recall: if $v(a) = v(b) = 0$, then $\partial(a, b) = 0$.)

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Strategy for fibrations (Colliot-Thélène - Ojanguren)

Set up:

\[ \begin{array}{c}
X_K \\ \downarrow \\
K \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow \\
\end{array} \quad \begin{array}{c}
X \\ \downarrow \pi \\
S = \mathbb{P}^2_{\mathbb{C}} \\
\end{array} \quad \text{fibration in geometrically rational varieties}

where \( K = \mathbb{C}(x, y) \) is the field of functions of \( S \),
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fibration in geometrically rational varieties

where \( K = \mathbb{C}(x, y) \) is the field of functions of \( S \),

note: \( K(X_K) = \mathbb{C}(X) \).
Set up:

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\begin{align*}
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\downarrow & \downarrow \pi \\
K & \to S = \mathbb{P}^2_{\mathbb{C}}
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\[
H^2_{nr}(\mathbb{C}(X)/\mathbb{C}) \leftarrow H^2_{nr}(K(X_K)/K) \leftarrow H^2(\mathbb{C}(X))
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\[H^2(K)\]
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  \[ X_K \rightarrow X \xleftarrow{\text{fibration in geometrically rational varieties}} \]
  \[ K \rightarrow S = \mathbb{P}^2_{\mathbb{C}} \]

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- \( H^2_{nr}(\mathbb{C}(X)/\mathbb{C}) \leftarrow H^2_{nr}(K(X_K)/K) \rightarrow H^2(\mathbb{C}(X)) \)

\[ H^2(K) \]

- \( \alpha \in H^2(K) \) is ramified on \( S \) as \( H^2_{nr}(\mathbb{C}(S)/\mathbb{C}) = H^2(\mathbb{C}) = 0 \).
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\end{align*} \]

where \( K = \mathbb{C}(x, y) \) is the field of functions of \( S \),
note: \( K(X_K) = \mathbb{C}(X) \).

\[ H^2_{nr}(\mathbb{C}(X)/\mathbb{C}) \hookrightarrow H^2_{nr}(K(X_K)/K) \hookrightarrow H^2(\mathbb{C}(X)) \]

\[ H^2(K) \]

\( \alpha \in H^2(K) \) is ramified on \( S \) as \( H^2_{nr}(\mathbb{C}(S)/\mathbb{C}) = H^2(\mathbb{C}) = 0 \).

idea: if \( \partial^2_{v_D}(\alpha) \neq 0 \), then \( \pi \) degenerates along \( D \).
Relative unramified cohomology \( H^i_{nr, \pi}(k(X)/k) \subset H^i(k(X)) \)

Set up: \( X_{K_x} \xrightarrow{\pi} X \xleftarrow{\text{integral}} X_K \xrightarrow{\text{smooth, } k \text{ alg. closed}} S/k \)

Here \( K(X_K) = k(X) \).
Relative unramified cohomology $H_{nr,\pi}^i(k(X)/k) \subset H^i(k(X))$

Set up:

\[
\begin{array}{ccccccccc}
X_{K_x} & X_K & X & \leftarrow & \text{integral} \\
\downarrow & \downarrow & \downarrow \pi & & \\
K_x & K & S/k & \leftarrow & \text{smooth, } k \text{ alg.closed} \\
\end{array}
\]

here $K(X_K) = k(X)$.

Definition

\[
H_{nr,\pi}^i(k(X)/k) = \text{Im}[H^i(K) \to H^i(K(X_K))] \bigcap \bigcap_P \text{Ker}[H^i(K) \to H^i(K_P) \to H^i(K_P(X_{K_P}))],
\]

where

- $P$ runs over all scheme points of $S$ of positive codimension:
  
  $P \in S^{(i)}$ for $i > 0$

- $K_P$ is the field of fractions of the completed local ring $\hat{O}_{S,P}$. 
Properties

- $H^i_{nr, \pi}(k(X)/k) \subset H^i_{nr}(k(X)/k)$. if $\alpha \in H^i_{nr, \pi}(k(X)/k)$ is nonzero, then $X$ is a reference variety (Schreieder).
• $H^i_{nr,\pi}(k(X)/k) \subset H^i_{nr}(k(X)/k)$.

• if $\alpha \in H^i_{nr,\pi}(k(X)/k)$ nonzero, then $X$ is a *reference variety* (Schreieder).
Cubic surface bundles
Computing $H^2_{nr,\pi}$

$$H^2_{nr,\pi}(k(X)/k) = \text{Im}[H^2(K) \to H^2(K(X_K))] \cap$$
$$\cap_P \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))].$$

Let $Y = X_K$. 

Question: when $H^2(F) \to H^2(F(Y))$ is injective ($F = K$) not injective, and what is the kernel ($F = K_P$)?

Known answers for:
- $Y$ a quadric (Arason, Pfister, Kahn-Rost-Sujatha)
- $Y$ a geometrically rational surface (Colliot-Thélène - Karpenko - Merkurjev).
Computing $H^2_{nr, \pi}$

\[ H^2_{nr, \pi}(k(X)/k) = \text{Im}[H^2(K) \to H^2(K(X_K))] \bigcap \]
\[ \cap_P \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))]. \]

Let $Y = X_K$.

Question: when $H^2(F) \to H^2(F(Y))$ is:

- injective ($F = K$)
- not injective, and what is the kernel ($F = K_P$)?
Computing $H_{nr,\pi}^2$

\[ H_{nr,\pi}^2(k(X)/k) = \text{Im}[H^2(K) \to H^2(K(X_K))] \bigcap \]
\[ \cap_P \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))]. \]

Let $Y = X_K$.

Question: when $H^2(F) \to H^2(F(Y))$ is:
- injective ($F = K$)
- not injective, and what is the kernel ($F = K_P$)?

Known answers for:
- $Y$ a quadric (Arason, Pfister, Kahn-Rost-Sujatha)
- $Y$ a geometrically rational surface (Colliot-Thélène - Karpenko - Merkurjev).
Rational surfaces and kernels for $H^2(\cdot, \mathbb{Z}/3)$

(Colliot-Thélène - Karpenko - Merkurjev)

$F$ a field, $Y/F$ geometrically rational surface. Then

- $\text{Ker}[H^2(F, \mathbb{Z}/3) \to H^2(F(Y), \mathbb{Z}/3)] \neq 0$ iff
- $Y$ is $F$-birational to $Y'$ a non-split Severi-Brauer (SB) surface.
(Colliot-Thélène - Karpenko - Merkurjev)

$F$ a field, $Y/F$ geometrically rational surface. Then

- $\text{Ker} [H^2(F, \mathbb{Z}/3) \to H^2(F(Y), \mathbb{Z}/3)] \neq 0$ iff $Y$ is $F$-birational to $Y'$ a non-split Severi-Brauer (SB) surface.
- Then

$$\text{Ker} [H^2(F, \mathbb{Z}/3) \to H^2(F(Y), \mathbb{Z}/3)] \cong \mathbb{Z}/3,$$

generated by the class of $Y'$. 
Example: minimal cubic

\[ Y : au^3 + bv^3 + abw^3 + ft^3 = 0, \ a, b, f \in F \]

- Assume: none of the elements \( a, b, ab, f, af, bf \) is a cube in \( F \).
- (Segre) then the surface is minimal, and

\[ H^2(F, \mathbb{Z}/3\mathbb{Z}) \to H^2(F(Y), \mathbb{Z}/3\mathbb{Z}) \]

is injective.
Example: nonminimal cubic

\[ Y : au^3 + bv^3 + abw^3 + t^3 = 0, \ a, b \in F \]
then \( (a, b) \in \text{Ker}[H^2(F, \mathbb{Z}/3) \rightarrow H^2(F(Y), \mathbb{Z}/3)] : \)
Example: nonminimal cubic

\[ Y : au^3 + bv^3 + abw^3 + t^3 = 0, a, b \in F \]

then \((a, b) \in \text{Ker}[H^2(F, \mathbb{Z}/3) \rightarrow H^2(F(Y), \mathbb{Z}/3)]\):

- if \(a\) is a cube in \(F(Y)\), then \((a, b) = 0\).
Example: nonminimal cubic

\[ Y : au^3 + bv^3 + abw^3 + t^3 = 0, \quad a, b \in F \]

then \((a, b) \in \ker[H^2(F, \mathbb{Z}/3) \to H^2(F(Y), \mathbb{Z}/3)]\):

- if \(a\) is a cube in \(F(Y)\), then \((a, b) = 0\).
- Otherwise, let \(L = F(Y)(\sqrt[3]{a})\). In \(F(Y)\) we have a relation

\[
    b = -\frac{t^3 + au^3}{v^3 + aw^3},
\]

so

\[
    b = N_{L/F(Y)}(\beta)
\]

where

\[
    \beta = -\frac{t + \sqrt[3]{au}}{v + \sqrt[3]{aw}}.
\]
Example: nonminimal cubic

\[ Y : au^3 + bv^3 + abw^3 + t^3 = 0, \ a, b \in F \]

then \((a, b) \in \text{Ker}[H^2(F, \mathbb{Z}/3) \to H^2(F(Y), \mathbb{Z}/3)]:\)

- if \(a\) is a cube in \(F(Y)\), then \((a, b) = 0\).
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so

\[ b = N_{L/F(Y)}(\beta) \]

where

\[ \beta = -\frac{t + \sqrt[3]{au}}{v + \sqrt[3]{aw}}. \]

Hence in \(H^2(F(Y), \mathbb{Z}/3\mathbb{Z})\):

\[(a, b) = (a, N_{L/F(Y)}(\beta)) = N_{L/F(Y)}(a, \beta) = 0.\]
Example

- $k = \mathbb{C}$
  (or $k$ an algebraically closed field of $\text{char } (k) \neq 3$)
- $X \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[u:v:w:t]}$ is a cubic surface bundle over $k$:
  \[ xz^2 u^3 + y^2 z v^3 + xy^2 w^3 + ft^3 = 0, \]
  where
  \[ f = x^3 + y^3 + z^3 + 3x^2 y + 3xy^2 + 3y^2 z + 3yz^2 + 3xz^2 + 3x^2 z, \]
Example

- \( k = \mathbb{C} \)
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  \[
  xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0,
  \]
  where
  \[
  f = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3xz^2 + 3x^2z,
  \]
- Let \( K = \mathbb{C}(\mathbb{P}^2) = \mathbb{C}(x/z, y/z) \), let
  \( \alpha = (x/z, y/z) \in H^2(K, \mathbb{Z}/3) \). Then
  \[
  \alpha \in H^2_{nr, \pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3).
  \]
Sketch of proof: $\alpha$ nonzero in $C(X) = K(X_K)$

the generic fibre $Y = X_K$ of $\pi$ is a minimal cubic surface:

$$xz^2 u^3 + y^2 zv^3 + xy^2 w^3 + ft^3 = 0,$$

where $f = x^3 + y^3 + z^3 + 3x^2 y + 3xy^2 + 3y^2 z + 3yz^2 + 3xz^2 + 3x^2 z.$
Sketch of proof: $\alpha$ nonzero in $C(X) = K(X_K)$

the generic fibre $Y = X_K$ of $\pi$ is a minimal cubic surface:

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0,$$

where

$$f = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3xz^2 + 3x^2z.$$

Recall:

$$au^3 + bv^3 + abw^3 + ft^3 = 0, \ a, b, f \in K$$

if none of the elements $a, b, ab, f, af, bf$ is a cube then

$H^2(K, \mathbb{Z}/3) \to H^2(K(Y), \mathbb{Z}/3)$ is injective.
Sketch of proof: ramification of $\alpha$

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).$$
Sketch of proof: ramification of $\alpha$

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).$$

Question: For which divisors $D \subset \mathbb{P}^2_C$ one has $\partial_D(\alpha) \neq 0$?
Sketch of proof: ramification of $\alpha$

$$xz^2 u^3 + y^2 z v^3 + xy^2 w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).$$

Question: For which divisors $D \subset \mathbb{P}^2_C$ one has $\partial_D(\alpha) \neq 0$?

Answer: $x = 0$ or $y = 0$ or $z = 0$. 
Sketch of proof: $\alpha$ zero in $K_P(Y)$

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).$$
Sketch of proof: $\alpha$ zero in $K_P(Y)$

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).$$

Let $P \in \mathbb{P}_k^2$ be a point of positive codimension. We have three cases:

1. $P$ is the generic point of one of three lines $x = 0$, $y = 0$, or $z = 0$, or an intersection point of two of these lines.
2. $P$ is a closed point lying on only one of the lines $x = 0$, $y = 0$, or $z = 0$.
3. All other cases.
Blackboard
Sketch of proof: $\alpha$ zero in $K_P(Y)$, case 1

$ux^2u^3 + vy^2zv^3 + xxyw^3 + ft^3 = 0$, $\alpha = (x/z, y/z)$, where

$f = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3xz^2 + 3x^2z$.

$P$ is the generic point of one of three lines $x = 0$, $y = 0$, or $z = 0$, or an intersection point of two of these lines.
Sketch of proof: $\alpha$ zero in $K_P(Y)$, case 1

$\alpha = (x/z, y/z)$, where

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$P$ is the generic point of one of three lines $x = 0$, $y = 0$, or $z = 0$, or an intersection point of two of these lines. Then

- $f$ is a nonzero cube in $\kappa(P)$, so that $f$ is a cube in $K_P$ (Hensel)
Sketch of proof: $\alpha$ zero in $K_P(Y)$, case 1

$$xz^2 u^3 + y^2 z v^3 + xy^2 w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z),$$
where

$$f = x^3 + y^3 + z^3 + 3x^2 y + 3xy^2 + 3y^2 z + 3yz^2 + 3xz^2 + 3x^2 z.$$  

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- $f$ is a nonzero cube in $\kappa(P)$, so that $f$ is a cube in $K_P$ (Hensel)
- $Y_{K_P}$ is

$$\frac{x}{z} u^3 + \frac{y^2}{z^2} v^3 + \frac{x}{z} \frac{y^2}{z^2} w^3 + t^3 = 0$$

so that the element $(x/z, y^2/z^2) = 2\alpha$ is in the kernel of the map

$$H^2(K_P, \mathbb{Z}/3) \to H^2(K_P(Y), \mathbb{Z}/3\mathbb{Z}).$$
Sketch of proof: $\alpha$ zero in $K_P(Y)$, case 2

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).$$

$P$ is a closed point lying on only one of the lines $x = 0$, $y = 0$, or $z = 0$. 
Sketch of proof: \( \alpha \) zero in \( K_P(Y) \), case 2

\[
{xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z)}.
\]

\( P \) is a closed point lying on only one of the lines \( x = 0, y = 0, \) or \( z = 0 \).

- enough: \( \alpha = 0 \) over \( K_P \).
Sketch of proof: $\alpha$ zero in $K_P(Y)$, case 2

\[ xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z). \]

$P$ is a closed point lying on only one of the lines $x = 0$, $y = 0$, or $z = 0$.

- enough: $\alpha = 0$ over $K_P$.
- assume: $P$ is on the line $x = 0$:
  - then $y/z$ is a nonzero element in the residue field $\kappa(P) = \mathbb{C}$, hence a cube
  - Hence $y/z$ is a cube in $K_P$. 
Sketch of proof: $\alpha$ zero in $K_P(Y)$, case 3

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).$$

$P$ is not on the lines $x = 0$, $y = 0$, or $z = 0$.

- $x/z$ and $y/z$ are units in the local ring of $P$, so that the image of $\alpha$ in $K_P$ comes from the cohomology group $H^2_{\text{ét}}(\hat{O}_{\mathbb{P}^2,P}, \mathbb{Z}/3)$.
Sketch of proof: \( \alpha \) zero in \( K_P(Y) \), case 3

\[
{xz}^2 u^3 + y^2 z v^3 + xy^2 w^3 + ft^3 = 0, \quad \alpha = (x/z, y/z).
\]

\( P \) is not on the lines \( x = 0, y = 0, \) or \( z = 0 \).

- \( x/z \) and \( y/z \) are units in the local ring of \( P \), so that the image of \( \alpha \) in \( K_P \) comes from the cohomology group
  \[
  H^2_{\text{ét}}(\hat{O}_{\mathbb{P}^2, P}, \mathbb{Z}/3).
  \]

- \( H^2_{\text{ét}}(\hat{O}_{\mathbb{P}^2, P}, \mathbb{Z}/3) = H^2(\kappa(P), \mathbb{Z}/3) = 0 \) by cohomological dimension.
Corollary

We obtained:

$$xz^2 u^3 + y^2 zv^3 + xy^2 w^3 + ft^3 = 0 \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[u:v:w:t]}$$

where

$$f = x^3 + y^3 + z^3 + 3x^2 y + 3xy^2 + 3y^2 z + 3yz^2 + 3xz^2 + 3x^2 z$$

is a reference variety.
Corollary

We obtained:

\[ xz^2 u^3 + y^2 zv^3 + xy^2 w^3 + ft^3 = 0 \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[u:v:w:t]} \]

where

\[ f = x^3 + y^3 + z^3 + 3x^2 y + 3xy^2 + 3y^2 z + 3yz^2 + 3xz^2 + 3x^2 z \]

is a reference variety.

Then:

Theorem (Krylov-Okada, Nicaise-Ottem)

Let \( k \) be an algebraically closed field of char \( (k) \neq 3 \). A very general hypersurface of bidegree \((3, 3)\) in \( \mathbb{P}^2_k \times \mathbb{P}^3_k \) is not stably rational.
\[ \pi : X \to S = \mathbb{P}^2_C \text{ cubic surface bundle, } K = \mathbb{C}(x, y). \]

\[ H^2_{nr, \pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \text{Im}[H^2(K, \mathbb{Z}/3) \to H^2(K(X_K), \mathbb{Z}/3)] \bigcap \]

\[ \cap_{P \in S(1) \cup S(2)} \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))], \]
\[ \pi : X \to S = \mathbb{P}^2_C \text{ cubic surface bundle, } K = \mathbb{C}(x, y). \]

\[ H^2_{nr, \pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \text{Im}[H^2(K, \mathbb{Z}/3) \to H^2(K(X_K), \mathbb{Z}/3)] \cap \]
\[ \cap_{P \in S(1) \cup S(2)} \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))], \]

- \( \alpha \in H^2(K) \) is determined by residues at \( P \in S \) of codimension 1, by Bloch-Ogus:

\[ 0 \to H^2(K, \mathbb{Z}/3) \oplus \partial^2 \oplus_{P \in S(1)} H^1(\kappa(P), \mathbb{Z}/3) \to \]
\[ \oplus \partial^1 \oplus_{p \in S(2)} H^0(\kappa(p), \mathbb{Z}/3) \]

- we need to specify which residues are allowed:
\[ \pi : X \to S = \mathbb{P}^2_C \text{ cubic surface bundle, } K = \mathbb{C}(x, y). \]

\[ H^2_{nr, \pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \text{Im}[H^2(K, \mathbb{Z}/3) \to H^2(K(X_K), \mathbb{Z}/3)] \bigcap \]

\[ \bigcap_{P \in S(1) \cup S(2)} \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))], \]

\[ \alpha \in H^2(K) \text{ is determined by residues at } P \in S \text{ of codimension 1, by Bloch-Ogus:} \]

\[ 0 \to H^2(K, \mathbb{Z}/3) \bigoplus_{P \in S(1)} H^1(\kappa(P), \mathbb{Z}/3) \to \bigoplus_{P \in S(2)} H^0(\kappa(p), \mathbb{Z}/3) \]

we need to specify which residues are allowed: 

\( X_{K_P} \text{ is birational to a SB surface } \Rightarrow \text{ the fiber} \)

\( X_P = \cup 3 \text{ conjugated planes} \)
General formula

\[ \pi : X \to S = \mathbb{P}_\mathbb{C}^2 \] cubic surface bundle, \( K = \mathbb{C}(x, y) \).

\[ H^2_{nr, \pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \text{Im}[H^2(K, \mathbb{Z}/3) \to H^2(K(X_K), \mathbb{Z}/3)] \cap \]
\[ \cap_{P \in S(1) \cup S(2)} \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))], \]

- \( \alpha \in H^2(K) \) is determined by residues at \( P \in S \) of codimension 1, by Bloch-Ogus:

\[ 0 \to H^2(K, \mathbb{Z}/3) \xrightarrow{\oplus \partial^2} \oplus_{P \in S(1)} H^1(\kappa(P), \mathbb{Z}/3) \to \]
\[ \oplus \partial^1 \oplus_{p \in S(2)} H^0(\kappa(p), \mathbb{Z}/3) \]

- we need to specify which residues are allowed:
\( X_{K_P} \) is birational to a SB surface \( \Rightarrow \) the fiber
\( X_P = \cup 3 \) conjugated planes (condition appeared in a joint work with A. Auel and C. Böhning).
Set up: $\pi : X \to S = \mathbb{P}^2_C$ cubic surface bundle, $K = \mathbb{C}(x, y)$. Assume:

- $X_K$ is a smooth minimal cubic surface (so $H^2(K, \mathbb{Z}/3) \hookrightarrow H^2(K(X_K), \mathbb{Z}/3)$);
- fibres of $\pi$ over codimension 1 points of $S$ are reduced.
Set up: $\pi : X \to S = \mathbb{P}^2_C$ cubic surface bundle, $K = \mathbb{C}(x, y)$.

Assume:
- $X_K$ is a smooth minimal cubic surface (so $H^2(K, \mathbb{Z}/3) \hookrightarrow H^2(K(X_K), \mathbb{Z}/3)$);
- fibres of $\pi$ over codimension 1 points of $S$ are reduced.

Determine:
- $C = \bigcup_{i=1}^n C_i \subset S$ a divisor corresponding to the set of codimension 1 points of $S$ over which the fibre of $\pi$ is geometrically a union of three planes permuted by Galois.
- $\gamma_i \in \kappa(C_i)^*/(\kappa(C_i)^*)^3$ the class corresponding to the cyclic extension.

Assume $C$ is snc.
General formula

Set up: $\pi : X \to S = \mathbb{P}^2_C$ cubic surface bundle, $K = \mathbb{C} \langle x, y \rangle$.

Assume:

- $X_K$ is a smooth minimal cubic surface
  (so $H^2(K, \mathbb{Z}/3) \hookrightarrow H^2(K(X_K), \mathbb{Z}/3)$);
- fibres of $\pi$ over codimension 1 points of $S$ are reduced.

Determine:

- $C = \bigcup_{i=1}^n C_i \subset S$ a divisor corresponding to the set of codimension 1 points of $S$ over which the fibre of $\pi$ is geometrically a union of three planes permuted by Galois.
- $\gamma_i \in \kappa(C_i)^*/(\kappa(C_i)^*)^3$ the class corresponding to the cyclic extension.

Assume $C$ is snc. Then (briefly):

- $\alpha \in H^2_{nr, \pi}$ is only allowed to have residues $\gamma_i$ at $C_i$ + condition on $K_P$.
- glue by Bloch-Ogus.
Set up: \( \pi : X \to S = \mathbb{P}^2_C, \ C = \bigcup_{i=1}^n C_i, \ \gamma_i \in \kappa(C_i)^*/(\kappa(C_i)^*)^3. \)

\[
H^2_{nr, \pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \text{Im}[H^2(K, \mathbb{Z}/3) \to H^2(K(X_K), \mathbb{Z}/3)] \bigcap \\
\bigcap_{P \in S(1) \cup S(2)} \text{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))],
\]
Set up: $\pi: X \to S = \mathbb{P}^2_\mathbb{C}$, $C = \bigcup_{i=1}^n C_i$, $\gamma_i \in \kappa(C_i)^*/(\kappa(C_i)^*)^3$.

\[
H^2_{nr,\pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \text{Im}[H^2(K, \mathbb{Z}/3) \to H^2(K(X_K), \mathbb{Z}/3)] \cap \bigcap_{P \in S_1 \cup S_2} \ker[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))],
\]

Then

\[
H^2_{nr,\pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \{a = \{a_i\}_{i=1}^n, a_i \in \{-1, 0, 1\}\} \subset (\mathbb{Z}/3)^n
\]

(i) $a_i \neq 0 \Rightarrow X_{K_{C_i}}$ is birational to SB;

(ii) (Bloch-Ogus)

\[
\sum_{i=1}^n \sum_{P \in S_2} \partial_P(\gamma_i^{a_i}) = 0
\]

(iii) if $P \in C_i \cap C_j$ and if $\partial_P(\gamma_i^{a_i}) = -\partial_P(\gamma_j^{a_j}) \neq 0$, one has that the base change $X_{K_P}$ is birational to SB.
THANK YOU!!!