Fundamental groups of low-coregularity pairs.

joint w/Lukas Braun
Dimension 1 | Curves. (C. Smooth).
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>genus</td>
<td>( \pi_1(x) )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \geq 2 )</td>
<td>( \infty )</td>
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Higher dimensions | Smooth.

| Fano | \( K_x \text{ anti-ample} \) | \( \pi_1(x) \) is trivial. |
| Calabi-Yau | \( K_x \equiv 0 \) | \( \pi_1(x) \) is virtually abelian |
| Canonically Polarized | \( K_x \text{ ample} \) | \( A \triangleleft \pi_1(x), A \text{ is abelian} \) |
| | | \( 1/\pi_1(x): A \not\triangleleft \pi_1(x) \) |
| | | More complicated. |
What happens when we add singularities?

Singularity of the MMP.

- log canonical.
- klt.

The local topology of the singularities matters.

Theorem [Braun 2021]

Let \((X, x)\) be a klt singularity. Then
\[
\pi_1^{\text{loc}}(X, x) = \pi_1^{\text{loc}}(\overline{B(x) \setminus \{x\}})\]

is always finite.

Theorem [Braun 2021]

Let \(X\) be a Fano klt variety. Then the
\[
\pi_1(X_{\text{reg}})
\]

is finite.

klt Calabi–Yau:

Theorem [Campana–Claudon 2014]

Let \(X\) be a klt Calabi–Yau surface, then
\[
\pi_1(X_{\text{reg}})
\]

is virtually abelian.
Pairs \((X, D)\),

Standard approximation:

\[
D_{st} := \max \sum (1 - \frac{1}{m_i}) D_i \leq D^3
\]

\[m_i \in \mathbb{Z}^+ \text{ or } (1 - \frac{1}{m_i}) = 1.\]

\(G\) The orbifold fundamental group of the smooth locus of a pair.

\[
\pi^\text{reg}_1 (X, D) = \pi^\text{reg}_1 (X \text{ reg} \backslash \text{Sup}(D_{st})) / N
\]

\(N\) is the normal group generated by loops \((y_i^{m_i})\). (Where \(y_i\) is a loop around \(D_i\) with coeff \(D_i(D_{st}) = 1 - \frac{1}{m_i}\), i.e., if coeff is \(1\), \((y_i^{m_i})\) does not appear on \(N\).)
Example: \( (C, 0) \) cell curve
\[
\pi_2^{\text{reg}}(C, 0) = \pi_2(C) = \mathbb{Z} \times \mathbb{Z}.
\]

\( (P^2, \{0\} \cup \{\infty\}) \) is log canonical \( C_{-1} \)
\[
\pi_2(P^2 \setminus \{0, \infty\}) = \mathbb{Z} \quad \text{abelian.}
\]

- \( \pi_2^{\text{reg}}(P^2(1 - \frac{1}{x})\{0\} \cup (1 - \frac{1}{x})\{\infty\}) \leftarrow \text{tlt Fano.} \)
- \( \pi_2(P^2 \setminus \{0, \infty\}) \langle \chi_1^a, \chi_2^d \rangle \)
  \[
  = \mathbb{Z} / \alpha \mathbb{Z}. \quad \text{finite.}
  \]

**Theorem (Braun 2021)**

Let \((X, D)\) be a tlt Fano pair, then
\[
\pi_2^{\text{reg}}(X, D) \text{ is finite.}
\]
Theorem (Campana - Claudon 2014)

Let \((X, D)\) be a \(C\)-Y. surface pair with standard coeff. Then \(\mathcal{H}^{\text{reg}}(X, D)\) is virtually abelian with rank at most 4.
Coregularity - Calabi-Yau type pairs.

A pair \((X, D)\) is of Calabi-Yau type, if \(\exists B \geq D\) s.t. \((X, B)\) is (log canonical) Calabi-Yau.

Dual complex of \((X, D)\).

- Let \((Y, D_Y) \to (X, D)\) be a resolution.
- \(D_Y := E_{2t} + \ldots + E_r\).
- \(\mathcal{D}(Y, D_Y) := \mathcal{D}(X, D)\).
- For \(E_i\), there exists a vertex \(S_{E_i}\).
- For \(I \subseteq \{1, \ldots, r\}\) \(a (|I| - 1)\)-dim. simplex and irr. component \(\bigcap E_i\).
- \(\mathcal{D}(X, D)\) is the homotopy class of \(\mathcal{D}(Y, D_Y)\).

For \((X, B)\) a CY pair.

\[ \text{coreg} (X, B) := (\dim X - 1) - \dim \mathcal{D}(X, B) \]

\[ 0 \leq \text{coreg} (X, B) \leq \dim X. \]
For \((x, D)\) pair \(E \subseteq Y\) type \(B \supseteq D\)

\[
\text{coreg}(x, D) = \inf \{ \text{reg}(x, B) \mid (x, B) \in E \}
\]

\(0 \leq \text{coreg} \leq \dim x\), and \(\text{coreg}(x, B) = \infty\) if not \(E \subseteq Y\) type.
Main Theorems

Theorem 1 | Braun-F. 2024

Let $(X, D)$ be a klt pair with
\[ \hat{\operatorname{coreg}}(X, D) = 0. \]
Then $\Pi_{1, \text{reg}}^{\text{reg}}(X, D)$ is finite.

Theorem 2 | Braun-F. 2024

Let $(C \in \{1, 2\})$
Let $(X, D)$ be a klt pair with
\[ \hat{\operatorname{coreg}}(X, D) = C. \]
Then $\Pi_{1, \text{reg}}^{\text{reg}}(X, D)$ is virtually abelian of
rank at most $2C$.

Proof of Theorems

* They hold if we put $\dim = C$, and
  standard coefficients.
* We reduce to the case of std coeff.
If Thm 1-2 hold for std. $\Rightarrow$
Let \((x, D)\) with \(\text{coreg}(x, D) = C\).

\(\implies (x, D_{\text{st}}), \quad D_{\text{st}} \leq D,\)

\[\text{coreg}(x, D_{\text{st}}) \leq \text{coreg}(x, D) = C.\]

& \quad \text{reg}(x, D) = \text{reg}(x, D_{\text{st}}), \quad \text{the Thm holds (std. coeff.)}.\]

---

1st Case | If \((x, D)\) is Calabi-Yau.

- \(\text{coreg}(x, D) = C\) and \((x, D)\) is klt.
  \(\implies \dim x = C \in \{0, 1, 2\}.\)

D with std. coeff. \(\implies\) The result from Campana-Clucke for surfaces or the case of curves.

---

2nd Case | \((x, D)\) is not Calabi-Yau.

\(\exists B \supseteq D, B \neq D \text{ s.t. (}\(x, B)\text{ is C-Y with coreg}(x, B) = C.\))
Take a $\sigma$-Factorial dlt modification of $(X, B)$.  $\phi : (Y, B_Y) \to (X, B)$.

There exists some $D_Y \leq B_Y < B_Y$.

s.t. $\left( \pi_2^{\text{reg}}(Y, B_Y) \right) \to \pi_2^{\text{reg}}(X, D)$

with $(Y, B_Y)$ klt.

\[ D_Y < \frac{K_Y}{\sigma} < B_Y \]

\[ (Y, B_Y) \quad \text{klt} \]

\[ K_Y + B_Y \gg 0 \]

\[ K_Y + B_Y \gg 0 \]

is not pseff.

We can run $(K_Y + B_Y)$ MMP.

\[ Y \to Z \]

\[ W \]

\[ \pi_2^{\text{reg}}(Z, P_Z) \to \pi_2^{\text{reg}}(Y, B_Y). \]

\[ \text{coreg}(Z, B_Z) = \text{coreg}(Y, B_Y) = \text{wreg}(X, B). \]
By the canonical bundle formula, we obtain \((W, B_w)\) s.t.
\[
F^* (K_W + B_w) \sim_{\mathbb{Q}} K_z + B_z.
\]

We obtain
\[
\begin{align*}
F &\Rightarrow \mathbb{C} \\
\text{Fanotype} &\Rightarrow W
\end{align*}
\]

\[
\begin{align*}
\dim \reg (F, P) &\Rightarrow \dim \reg (Z, P_z) \\
\text{coreg} &\leq c
\end{align*}
\]

\[
\begin{align*}
\dim \reg (w, P_w) &\Rightarrow 1 \\
\text{coreg} &\leq c
\end{align*}
\]

\[
\begin{align*}
\text{dim, } \dim x &< \dim x \\
\text{Virt. abelian (by induction)} &\Rightarrow \text{rank } \leq 2c
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \dim \reg (Z, P_z) &\leq \text{virt. ab. of } \\
\text{rank } &\leq 2c
\end{align*}
\]

\[
\dim \reg (Z, P_z) \Rightarrow \dim \reg (X, P_x)
\]
Theorem (Braun-Filippazzi; Moraga-Svaldi: 2022)
Let $(x, D)$ be a klt Fano pair of dim $= d$, $\Pi_1^{\text{reg}}(x, D)$ is virt. abelian of rank $= d$ with index bounded by $c(d)$.

Theorem (Braun F.: 2024)
Let $(x, D)$ klt pair with coreg$(x, D) = c \leq 2$, and $\dim x = d$.
Then $\Pi_1^{\text{reg}}(x, D)$ is virt. solvable of length $\leq 2d - 3$, and index is bounded by $i(c, d)$. 