Birational geometry of Calabi-Yau pairs

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Birational geometry of Calabi-Yau pairs

Joint with Alessio Corti and Alex Massarenti

(We always work over \(\mathbb{C}\))
**Motivation:** Automorphisms of Smooth Hypersurfaces

\[ X = X_d \subset \mathbb{P}^{n+1} \text{ smooth hypersurface of degree } d \]

**Theorem (Matsumura-Monsky 1964)**

If \((n, d) \neq (1, 3), (2, 4)\), then

\[ \text{Aut}(\mathbb{P}^{n+1}, X) \twoheadrightarrow \text{Aut}(X). \]

- \( C = X_3 \subset \mathbb{P}^2 \) genus 1 curve (\( \text{Aut}(C) \cong C \rtimes \mathbb{Z}/d\mathbb{Z} \))
- \( S = X_4 \subset \mathbb{P}^3 \) K3 surface (\( \text{Aut}(S) \) discrete and possibly infinite)

In both cases, the image of \( \text{Aut}(\mathbb{P}^{n+1}, X) \rightarrow \text{Aut}(X) \) is finite.
Theorem

- Every automorphism of $C$ is induced by a Cremona transformation of the ambient $\mathbb{P}^2$.

$$1 \rightarrow \text{Ine}(\mathbb{P}^2, C) \rightarrow \text{Dec}(\mathbb{P}^2, C) \rightarrow \text{Aut}(C) \rightarrow 1$$

- (Pan 2007) Generators for decomposition group $\text{Dec}(\mathbb{P}^2, C)$
- (Blanc 2008) Generators for inertia group $\text{Ine}(\mathbb{P}^2, C)$
$S = X_4 \subset \mathbb{P}^3$ K3 surface

**Question (Gizatullin)**

Is every automorphism of $S$ induced by a Cremona transformation of the ambient space $\mathbb{P}^3$?

**Examples (Oguiso 2012)**

- $\text{Aut}(S) \cong \mathbb{Z}$, and no nontrivial automorphism of $S$ is induced by a Cremona transformation of $\mathbb{P}^3$.
- $\text{Aut}(S) \cong (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z})$, and every automorphism of $S$ is induced by a Cremona transformation of $\mathbb{P}^3$.

**Example (Paiva-Quedo 2022)**

$\text{Aut}(S) \cong (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z})$, and no nontrivial automorphism of $S$ is induced by a Cremona transformation of $\mathbb{P}^3$. 
$S = X_4 \subset \mathbb{P}^3$ K3 surface

**Problem**

To describe the decomposition group of $S \subset \mathbb{P}^3$

$$\text{Dec}(\mathbb{P}^3, S) = \left\{ \varphi \in \text{Bir}(\mathbb{P}^3) \mid \varphi_* S = S \right\}$$

and its image in $\text{Aut}(S)$

$(\mathbb{P}^3, S)$ is a Calabi-Yau pair
Calabi-Yau pairs

**Definition (Calabi-Yau pair \((X, D)\))**

- \(X\) terminal projective variety
- \(D\) is a hypersurface \(\sim -K_X\)
- \((X, D)\) is log canonical

**Example**

\((\mathbb{P}^n, D)\) where \(D \subset \mathbb{P}^n\) is a smooth hypersurface of degree \(n + 1\)
Calabi-Yau pairs

**Definition (Calabi-Yau pair \((X, D)\))**

- \(X\) terminal projective variety
- \(D\) is a hypersurface \(\sim -K_X\)
- \((X, D)\) is log canonical

**Remark**

\((X, D)\) Calabi-Yau pair \(\leadsto \exists\ \omega_D\) (unique up to scaling)

\[\text{div}(\omega_D) = -D\]
**Calabi-Yau pairs**

**Definition (Calabi-Yau pair \((X, D)\))**
- \(X\) terminal projective variety
- \(D\) is a hypersurface \(\sim -K_X\) \(\quad (D = -\text{div}(\omega_D))\)
- \((X, D)\) is log canonical

**Definition (volume preserving map \((X, D_X) \to (Y, D_Y)\))**

\[ f : X \to Y \text{ birational map} \implies f_* : \Omega^n_{\mathbb{C}(X)/\mathbb{C}} \to \Omega^n_{\mathbb{C}(Y)/\mathbb{C}} \]

If \(f_*\omega_{D_X} = \omega_{D_Y}\) (up to scaling) then we say that

\[ f : (X, D_X) \to (Y, D_Y) \text{ is volume preserving} \]
CALABI-YAU PAIRS

Remark (Valuative interpretation)

\[ W \]

\[ X \xrightarrow{f} Y \]

\[ \forall E \subset W, \quad a(E, K_X + D_X) = a(E, K_Y + D_Y) \]

Example

If \( D \subset \mathbb{P}^n \) is a smooth hypersurface of degree \( n + 1 \), and \( f : X \to \mathbb{P}^n \) is a volume preserving blowup along a smooth center \( Z \), then

\[ Z \subset D \quad \text{and} \quad \text{codim}_{\mathbb{P}^n} (Z) = 2. \]
Problem

Given a Calabi-Yau pair \( (X, D) \), to determine

\[
\text{Bir}(X, D) := \left\{ \varphi \in \text{Bir}(X) \mid \varphi : (X, D) \to (X, D) \text{ is volume preserving} \right\}
\]

Example

\( D = D_4 \subset \mathbb{P}^3 \) smooth K3 surface

\[
\text{Dec}(\mathbb{P}^3, D) = \left\{ \varphi \in \text{Bir}(\mathbb{P}^3) \mid \varphi_* D = D \right\} = \text{Bir}(\mathbb{P}^3, D)
\]
**Remark**

If \((X, D)\) is a Calabi-Yau pair with **canonical** singularities, then

\[
\text{Dec}(X, D) = \left\{ \varphi \in \text{Bir}(X) \mid \varphi_* D = D \right\} = \text{Bir}(X, D)
\]

**Example (Canonicity is necessary)**

\[(X, D) = \left( \mathbb{P}^2, \sum_{i=0}^{2} H_i \right) \quad \left( \omega_D = \frac{dx}{x} \wedge \frac{dy}{y} \right)\]
Remark
If \((X, D)\) is a Calabi-Yau pair with \textit{canonical} singularities, then
\[
\text{Dec}(X, D) = \left\{ \varphi \in \text{Bir}(X) \mid \varphi_* D = D \right\} = \text{Bir}(X, D)
\]

Theorem (Blanc 2013)
\[
\text{Bir} \left( \mathbb{P}^2, \sum_{i=0}^{2} H_i \right) = \left\langle \begin{pmatrix} 1 & 1 + y \\ x & x \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (x, y) \mapsto (x^a y^b, x^c y^d)
\]

\[
\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \text{ preserve the torus } (\mathbb{C}^*)^2
\]
**Problem**

Given a Calabi-Yau pair $(X, D)$, to determine $\text{Bir}(X, D)$.

**Theorem A**

If $(X, D)$ is terminal with $\text{Pic}(X) = \mathbb{Z} \cdot H$ and $\text{Pic}(D) = \mathbb{Z} \cdot (H|_D)$, then

$$\text{Bir}(X, D) = \text{Aut}(X, D).$$

**Corollary**

If $D \subset \mathbb{P}^n$ is a general hypersurface of degree $n + 1$ ($n \geq 3$), then

$$\text{Bir}(\mathbb{P}^n, D) = \text{Aut}(\mathbb{P}^n, D).$$
**Theorem B**

If $D \subset \mathbb{P}^3$ is a general quartic surface with one singular point, then

\[
\text{Bir}(\mathbb{P}^3, D) \cong G \times \mathbb{Z}/2\mathbb{Z}
\]

$G$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$

\[
x_0^2 A_2(x_1, x_2, x_3) + x_0 B_3(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0
\]

$G = \left\{ \left[ (AG - BF)x_0 - CF : A(Fx_0 + G)x_1 : A(Fx_0 + G)x_2 : A(Fx_0 + G)x_3 \right] \mid F, G \in \mathbb{C}[x_1, x_2, x_3] \text{ homogeneous with } \deg(G) = \deg(F) + 1 \right\}$
$D \subset \mathbb{P}^3$ general quartic hypersurface with one singular point $P$

$$x_0^2 A_2(x_1, x_2, x_3) + x_0 B_3(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0$$

Bir($\mathbb{P}^3, D$) $\xrightarrow{r}$ Bir($D$) $\cong$ Aut($\tilde{D}$) $= \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}$

**Example**

$\varphi : (x_0 : x_1 : x_2 : x_3) \mapsto (-Ax_0 - B : Ax_1 : Ax_2 : Ax_3) \sim \tau$

$$1 \rightarrow G \rightarrow \text{Bir}(\mathbb{P}^3, D) \overset{\sim}{\rightarrow} \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$
$D \subset \mathbb{P}^3$ general quartic hypersurface with 1 singular point $P$

$$1 \rightarrow G \rightarrow \text{Bir}(\mathbb{P}^3, D) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

**Key point:** Given $\psi \in \text{Bir}(\mathbb{P}^3, D)$ there is a commutative diagram:

\[\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{\psi} & \mathbb{P}^3 \\
\mathbb{P}^2 & \xrightarrow{\tilde{\psi}} & \mathbb{P}^2 \\
\end{array}\]

$G$ is the group of birational self-maps of $X$ over $\mathbb{P}^2$ fixing $\tilde{D}$ pointwise

View $X$ as a model of $\mathbb{P}^1$ over $\mathbb{C}(x, y)$

$G$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$
The Cremona Group

\[ \text{Bir}(\mathbb{P}^n) := \{ \varphi : \mathbb{P}^n \to \mathbb{P}^n \text{ birational self-map} \} \]

Example (The standard quadratic transformation)

\[ \tau : \mathbb{P}^2 \to \mathbb{P}^2 \]
\[ (x : y : z) \mapsto \left( \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right) = (yz : xz : xy) \]

Theorem (Noether-Castelnuovo 1870-1901)

\[ \text{Bir}(\mathbb{P}^2) = \langle \text{Aut}(\mathbb{P}^2), \tau \rangle \]

Theorem (Hilda Hudson 1927)

For \( n \geq 3 \), \( \text{Bir}(\mathbb{P}^n) \) cannot be generated by elements of bounded degree.
The Sarkisov program (Corti 1995, Hacon-McKernan 2013)

\[ \mathbb{P}^n = X_0 \xrightarrow{\psi_1} X_1 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{k-1}} X_{k-1} \xrightarrow{\psi_k} X_k = \mathbb{P}^n \]
The Sarkisov program (Corti 1995, Hacon-McKernan 2013)

\[ \mathbb{P}^n = X_0 \longrightarrow_{\psi_1} X_1 \longrightarrow_{\psi_2} \cdots \longrightarrow_{\psi_{k-1}} X_{k-1} \longrightarrow_{\psi_k} X_k = \mathbb{P}^n \]

The \( X_i \rightarrow Y_i \)'s are Mori fiber spaces

- \( X_i \) has terminal singularities
- \( \rho(X_i/Y_i) = 1 \)
- \(-K_{X_i}\) is relatively ample

The \( \psi_i \)'s are elementary links
The surface case

The Mori fiber spaces are:

- $\mathbb{P}^2 \to \text{pt}$
- $F_m \to \mathbb{P}^1$ (\(\mathbb{P}^1\)-bundle)

\((F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1\) and \(F_1 \cong Bl_P \mathbb{P}^2\))

The elementary links are

![Diagram](image-url)
Elementary links in higher dimensions

Type 1

Surfaces

$\mathbb{P}^2 \leftarrow \text{Bl}_P \mathbb{F}_1 \rightarrow \mathbb{P}^1$

pt $\dashrightarrow \mathbb{P}^1$

Higher dimensions

$Z \rightarrow^{\varphi} X'$

$\downarrow f$

$X \rightarrow S' \leftarrow S$

$\downarrow$

$\downarrow$
ELEMENTARY LINKS IN HIGHER DIMENSIONS

Type 2

Surfaces

\[ F_m \quad \rightarrow \quad F_{m\pm 1} \]

\[ \mathbb{P}^1 \quad \rightarrow \quad \mathbb{P}^1 \]

Higher dimensions

\[ Z \rightarrow Z' \]

\[ X \quad \rightarrow \quad X' \]

\[ S \quad \rightarrow \quad S \]
Volume Preserving Sarkisov Program

Theorem (Corti-Kaloghiros 2016)

A volume preserving birational map between Mori fibered Calabi-Yau pairs is a composition of volume preserving Sarkisov links.
**Volume Preserving Sarkisov Program**

**Theorem (Corti-Kaloghiros 2016)**

A volume preserving birational map between Mori fibered Calabi-Yau pairs is a composition of volume preserving Sarkisov links.

\[ Z \rightarrow X' \]
\[ X \rightarrow S' \]
\[ (X, D) \rightarrow S' \]

\[ (Z, D_Z) \rightarrow (X', D') \]
\[ X \rightarrow S' \]
\[ (X, D) \rightarrow S \]
**Theorem A**

If \( n \geq 3 \) and \( D \) is a general hypersurface of degree \( n + 1 \), then

\[
\operatorname{Bir}(\mathbb{P}^n, D) = \operatorname{Aut}(\mathbb{P}^n, D).
\]

(\( D \) is smooth and \( \operatorname{Pic}(D) = \mathbb{Z} \cdot (H|_D) \))
**Theorem A**

If $n \geq 3$ and $D$ is a general hypersurface of degree $n + 1$, then

$$\text{Bir}(\mathbb{P}^n, D) = \text{Aut}(\mathbb{P}^n, D).$$

(D is smooth and $\text{Pic}(D) = \mathbb{Z} \cdot (H|_D)$)

\[\begin{array}{cccccc}
(\mathbb{P}^n, D) & \xrightarrow{\psi_1} & (X_1, D_1) & \xrightarrow{\psi_2} & \cdots & \xrightarrow{\psi_{k-1}} (X_{k-1}, D_{k-1}) & \xrightarrow{\psi_k} (\mathbb{P}^n, D) \\
\downarrow & & \downarrow & & & \downarrow & \\
\text{pt} & & Y_1 & & \cdots & & \text{pt} \\
\end{array}\]

\[\psi\]

$X_1$ has worst than terminal singularities
**Theorem B**

If $D \subset \mathbb{P}^3$ is a general quartic hypersurface with 1 singular point $P$, then

$$\text{Bir}(\mathbb{P}^3, D) \cong G \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $G$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$.
**Theorem B**

If $D \subset \mathbb{P}^3$ is a general quartic hypersurface with 1 singular point $P$, then

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Theorem B

If $D \subset \mathbb{P}^3$ is a general quartic hypersurface with 1 singular point $P$, then

$$\text{Bir}(\mathbb{P}^3, D) \cong G \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $G$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$. 

\[ \begin{array}{c}
(\mathbb{P}^3, D) \xleftarrow{Bl_P} (X, \tilde{D}) \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{k-1}} (X_{k-1}, D_{k-1}) \xrightarrow{\psi_k^*} (\mathbb{P}^3, D) \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{pt} \quad \mathbb{P}^2 \quad Y_{k-1} \quad \text{pt}
\end{array} \]
**Theorem B**

If \( D \subset \mathbb{P}^3 \) is a general quartic hypersurface with 1 singular point \( P \), then

\[
\text{Bir}(\mathbb{P}^3, D) \cong G \rtimes \mathbb{Z}/2\mathbb{Z},
\]

where \( G \) is a form of \( \mathbb{G}_m \) over \( \mathbb{C}(x, y) \).
**Theorem B**

If $D \subset \mathbb{P}^3$ is a general quartic hypersurface with 1 singular point $P$, then

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**Theorem B**

If  $D \subset \mathbb{P}^3$  is a general quartic hypersurface with 1 singular point  $P$, then

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where  $G$  is a form of  $\mathbb{G}_m$  over  $\mathbb{C}(x, y)$. 

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![Diagram](image-url)
Theorem B

If $D \subset P^3$ is a general quartic hypersurface with 1 singular point $P$, then

$$\text{Bir}(P^3, D) \cong G \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $G$ is a form of $G_m$ over $\mathbb{C}(x, y)$. 

\[ \begin{array}{c}
(\mathbb{P}^3, D) \\
\text{pt}
\end{array} \quad \xymatrix{ \mathbb{P}^2 & \cdots & \mathbb{P}^2 \\
\text{pt} & \cdots & \text{pt} } \quad \xymatrix{ \mathbb{P}^2 & \cdots & \mathbb{P}^2 \\
\text{pt} & \cdots & \text{pt} } \]
**Definition (Pliability)**

$(X, D)$ Mori fibered Calabi-Yau pair

\[ \mathcal{P}(X, D) \coloneqq \left\{ (X', D') \text{ Mf CY pair} \mid \exists (X, D) \xrightarrow{\text{vol preserving}} (X', D') \right\} / \sim \]

**Example (Square equivalence)**

\[(\mathbb{P}^3, D) \xrightarrow{Bl_P} (X, \tilde{D}) \xrightarrow{\psi_2} (X', D') \xrightarrow{\psi_3} \cdots \xrightarrow{Bl_P} (\mathbb{P}^3, D)\]

\[(\mathbb{P}^2, \mathbb{P}^2) \xrightarrow{\psi} \mathbb{P}^3\]
**Definition (Pliability)**

$(X, D)$ Mori fibered Calabi-Yau pair

$$
\mathcal{P}(X, D) \:= \left\{ (X', D') \text{ Mf CY pair} \mid \exists (X, D) \xrightarrow{\text{vol preserving}} (X', D') \right\} / \sim
$$

**Theorem C**

If $D \subset \mathbb{P}^3$ general quartic hypersurface with one $A_2$ singularity $P$, then we determine the pliability of $(\mathbb{P}^3, D)$:

- $(\mathbb{P}^3, D)$
- $(\text{Bl}_P \mathbb{P}^3, \tilde{D}) \rightarrow \mathbb{P}^2$
- $(\mathbb{P}(1^3, 2), D_5)$
- $(\mathbb{P}(1^3, 2), D'_5)$
- 3-parameter family $(X_4, D_{3,4})$, with $X_4 \subset \mathbb{P}(1^3, 2^2)$
- 6-parameter family $(X_4, D_{2,4})$, with $X_4 \subset \mathbb{P}(1^4, 2)$
Thank you!