

## Teaching Plan, week 7

Math 3c  
5-16-17

- Plan for today:
1. Warm-up
  2. Matrices & vectors
  3. Eigenvalues & eigenvectors
  4. Problems
  5. Quiz (10 minutes, know how to find eigenvalues & eigenvectors)

1. Warm-up: a) Find the unique solution to  $\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$ , if  $ad-bc \neq 0$ .  
 b) Show that there are infinitely many solutions to the above linear system if  $e=f=0$  and  $ad-bc=0$ .

Solution: a)  $\begin{cases} ax+by=e \\ cx+dy=f \end{cases} \Rightarrow \begin{cases} acx+bcy=ec \\ acx+ady=af \end{cases} \Rightarrow$

$$acx-acx+ady-bcy=af-ec, \Rightarrow \underbrace{(ad-bc)}_{\neq 0} y = af - ec,$$

$$\text{so } y = \frac{1}{ad-bc} (af - ec). \text{ Also,}$$

$$\begin{cases} ax+by=e \\ cx+dy=f \end{cases} \Rightarrow \begin{cases} adx+bdy=de \\ bcx+bdy=bf \end{cases} \Rightarrow (ad-bc)x = de - bf \Rightarrow$$

$$x = \frac{1}{ad-bc} (de - bf). \text{ Testing these values yields:}$$

$$ax+by = \frac{ade-abf+abf-bce}{ad-bc} = \frac{e(ad-bc)}{ad-bc} = e, \text{ and}$$

$$cx+dy = \frac{cde-bcf+tadf-cde}{ad-bc} = \frac{(ad-bc)f}{ad-bc} = f. \text{ This shows that}$$

whenever  $ad-bc \neq 0$ , there is one, and only one, solution to the system.

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b) One of the equations will be a multiple of the other. So solve one to get:  $x = -tb$ ,  $y = ta$ , for any  $t \in \mathbb{R}$ . Then  $ax + b = -tab + tab = 0$ , and  $cx + dy = -tbc + tad = t(ad - bc) = 0$ . Note: if  $a$  &  $b$  are 0, then we can instead put  $x = -td$ ,  $y = tc$ , and this will solve our system too. If  $a, b, c, d$  are zero, then any choice of  $x$  &  $y$  will solve our system. So there will be infinitely many solutions in any of these cases.

## 2. Matrices & vectors

vectors - columns of numbers

matrix - a rectangular array of numbers. Only  $2 \times 2$  type

In this class: 2 rows, 2 columns.

vector algebra is intuitive:  $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$ ,  $k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ak \\ bk \end{pmatrix}$ .

matrix algebra is not as intuitive, but you'll see why it is this way shortly:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ak & bk \\ ck & dk \end{pmatrix}, \text{ and the weirdest of all: } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

The determinant of  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is:  $\det(A) = ad - bc$ , and whether it's 0 or not will tell us something about the system.

$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ , by the warm-up. Note: put  $\vec{v} = \begin{pmatrix} e \\ f \end{pmatrix}$ . Then the system can be written as  $A(\vec{x}) = \vec{v}$ . If we put  $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ , then it's  $A\vec{u} = \vec{v}$ . If  $\det(A) \neq 0$ , then we showed that for any  $\vec{v}$ , there's exactly one  $\vec{u}$  that solves  $A\vec{u} = \vec{v}$ . If  $\det(A) = 0$ , then

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for each  $\vec{v}$ , there are either no  $\vec{u}$  that solve  $A\vec{u} = \vec{v}$ , or infinitely many (do a similar computation to the warm-up to show this. It's also in the textbook). One fancy way to prove this is the following: suppose  $\det(A) \neq 0$  and  $A\vec{u}_1 = \vec{v}$  for some  $\vec{u}_1$  vector (using symbols:  $\vec{u}_1 \in \mathbb{R}^2$ ). We already know that  $A\vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  has infinitely many solutions  $\vec{u}$ , and for any of those solutions  $A(\vec{u}_1 + \vec{u}) = A\vec{u}_1 + A\vec{u} = \vec{v} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{v}$ , so  $\vec{u}_1 + \vec{u}$  is a solution

$$A(\vec{u}_1 + \vec{u}) = A\vec{u}_1 + A\vec{u} = \vec{v} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{v}$$

by distributivity

to  $A\vec{z} = \vec{v}$ . So if one solution exists, then infinitely many

do. Now let's discuss how to quickly solve linear systems with matrices. If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $\det(A) = ad - bc \neq 0$ , define  $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ . Then

$$A^{-1}A = AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \text{ the "identity" matrix. } I_2 \text{ is called}$$

the identity matrix because  $I_2\vec{u} = \vec{u}$  and  $I_2B = BI_2 = B$

for all  $\vec{u} \in \mathbb{R}^2$  and 2x2 matrices B. Matrix multiplication

is associative, so for 2x2 matrices A, B, & C,

$$(AB)C = A(BC). \text{ But it is } \underline{\text{not commutative}}! \text{ If } \vec{v} \in \mathbb{R}^2$$

is given and so is A, and  $\det(A) \neq 0$ , then we can solve for  $\vec{u}$  in  $A\vec{u} = \vec{v}$  by left-multiplying both sides by  $A^{-1}$ :  $(A^{-1}A)\vec{u} = A^{-1}\vec{v}$ , so  $I_2\vec{u} = \vec{u} = A^{-1}\vec{v}$ .

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For resource level problems, if  $\vec{u}$  represents resources produced and  $A\vec{u}$  is the cost of producing  $\vec{u}$  resources (where  $A$  is a  $2 \times 2$  matrix), then the remaining resources  $\vec{d}$  that we have are given by:

$$\vec{d} = \vec{u} - A\vec{u} = I_2\vec{u} - A\vec{u} = (I_2 - A)\vec{u}. \text{ If } \det(I_2 - A) \neq 0,$$

then given  $\vec{d}$  we can solve for  $\vec{u}$ . Note that this equation is just a variation on the theme in economics:  $\underbrace{\text{profit}}_{\vec{d}} = \underbrace{\text{revenue}}_{\vec{u}} - \underbrace{\text{cost}}_{A\vec{u}}$ .

### 3. Eigenvalues & Eigenvectors

If  $A\vec{v} = \lambda\vec{v}$  for some scalar  $\lambda \in \mathbb{R}$ , some vector  $\vec{v} \in \mathbb{R}^2$ , and some  $2 \times 2$  matrix  $A$ , then  $\lambda$  is called an eigenvalue of  $A$ , and  $\vec{v}$  an eigenvector of  $A$  corresponding to eigenvalue  $\lambda$ . So each eigenvalue comes with (at least) one eigenvector (since we can multiply by an arbitrary nonzero constant  $k$  to get more eigenvectors).

$kA\vec{v} = k\lambda\vec{v}$ , so  $A(k\vec{v}) = \lambda(k\vec{v})$ , and each eigenvector is associated with exactly one eigenvalue (since  $\lambda\vec{v} = \mu\vec{v}$ ,  $\vec{v} \neq \vec{0}$  implies  $\lambda = \mu$ ).  $A\vec{v} = \lambda\vec{v}$  implies  $A\vec{v} = \lambda I_2 \vec{v}$ , which implies  $A\vec{v} - \lambda I_2 \vec{v} = \vec{0}$ , which implies  $(A - \lambda I_2)\vec{v} = \vec{0}$ .

If  $\det(A - \lambda I_2) \neq 0$ , then we could invert  $(A - \lambda I_2)$  and left-multiply by it to get  $\vec{v} = (A - \lambda I_2)^{-1}\vec{0} = \vec{0}$ . But we want nonzero  $\vec{v}$ , so we must have  $\det(A - \lambda I_2) = 0$ . Then the resulting system  $(A - \lambda I_2)\vec{v} = \vec{0}$ , or  $A\vec{v} = \lambda\vec{v}$ , will have a redundant equation & infinitely many solutions.

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Pick one of those to be your eigenvector (not (0) though).  
 For everything in this class, you'll only be finding eigenvalues/eigenvectors of  $2 \times 2$  matrices, and those matrices will always have 2 distinct eigenvalues.

## 4. Problems

a) Write the linear system  $\begin{cases} -3x+5y=-10 \\ 2x+4y=2 \end{cases}$  in matrix form and solve for  $x, y$ .

Solution:  $A = \begin{pmatrix} -3 & 5 \\ 2 & 4 \end{pmatrix}$ ,  $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} -10 \\ 2 \end{pmatrix}$ . Then  $A\vec{u} = \vec{v}$ , so

$$\vec{u} = A^{-1}\vec{v} = \frac{1}{-12-10} \begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} -10 \\ 2 \end{pmatrix} = \frac{1}{-22} \begin{pmatrix} -50 \\ 14 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 25 \\ -7 \end{pmatrix} = \begin{pmatrix} 25/11 \\ -7/11 \end{pmatrix}, \text{ so}$$

$$\begin{cases} x = 25/11 \\ y = -7/11 \end{cases}$$

b) If  $A = \begin{pmatrix} 2 & 3 \\ 1 & -6 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} 2 \\ -14 \end{pmatrix}$ , solve  $A\vec{u} = \vec{v}$ .

$$\text{Solution: } \vec{u} = A^{-1}\vec{v} = \frac{1}{-12-3} \begin{pmatrix} -6 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -14 \end{pmatrix} = \frac{1}{-15} \begin{pmatrix} 30 \\ -30 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \text{ so}$$

$$\vec{u} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

c) Find the eigenvalues of the matrix  $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$  corresponding to eigenvectors  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ .

$$\text{Solution: } \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

so the corresponding eigenvalues are 3 and 2, respectively.

d) Find eigenvectors corresponding to the eigenvalues 6 and 2 for  $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$ .

$$\text{Solution: } 6: A \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} 5x+3y=6x \\ x+3y=6y \end{cases} \Leftrightarrow \begin{cases} -x+3y=0 \\ x-3y=0 \end{cases} \Leftrightarrow -x+3y=0,$$

so pick  $\begin{cases} x=3 \\ y=1 \end{cases}$

$$2: A \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} 5x+3y=2x \\ x+3y=2y \end{cases} \Leftrightarrow \begin{cases} 3x+3y=0 \\ x+y=0 \end{cases} \Leftrightarrow x+y=0,$$

so pick  $x=1, y=-1$ . Thus our eigenvectors are  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , respectively

corresponding to eigenvalues 6 & 2.

e) Find all eigenvalues & their corresponding eigenvectors for the matrix  $A = \begin{pmatrix} -6 & 1 \\ 0 & -9 \end{pmatrix}$ .

Solution: First find eigenvalues, then eigenvectors.

$$\det(A - \lambda I_2) = \det \begin{pmatrix} -6-\lambda & 1 \\ 0 & -9-\lambda \end{pmatrix} = (-6-\lambda)(-9-\lambda) - (0)(1) = 0$$

$\Leftrightarrow \lambda = -6 \text{ or } -9$ , so these are the eigenvalues.

$$\lambda = -6: A \begin{pmatrix} x \\ y \end{pmatrix} = -6 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{array}{l} -6x + y = -6x \\ 0x - 9y = -6y \end{array} \Leftrightarrow \begin{array}{l} y = 0 \\ -3y = 0 \end{array} \Leftrightarrow y = 0, x \text{ free.}$$

so  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  will do.

$$\lambda = -9: A \begin{pmatrix} x \\ y \end{pmatrix} = -9 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{array}{l} -6x + y = -9x \\ 0x - 9y = -9y \end{array} \Leftrightarrow \begin{array}{l} 3x + y = 0 \\ 0 = 0 \end{array} \checkmark$$

so  $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$  will do. Thus the eigenvalues of  $A$  are  $-6$  and  $-9$ , with corresponding eigenvectors  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ .